

# Maximum-Likelihood-Estimation of Lévy driven Ornstein-Uhlenbeck Processes

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## Ornstein-Uhlenbeck (OU) Process

Let  $(L_t, t \geq 0)$  be a Lévy process on  $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ . For every  $a \in \mathbb{R}$

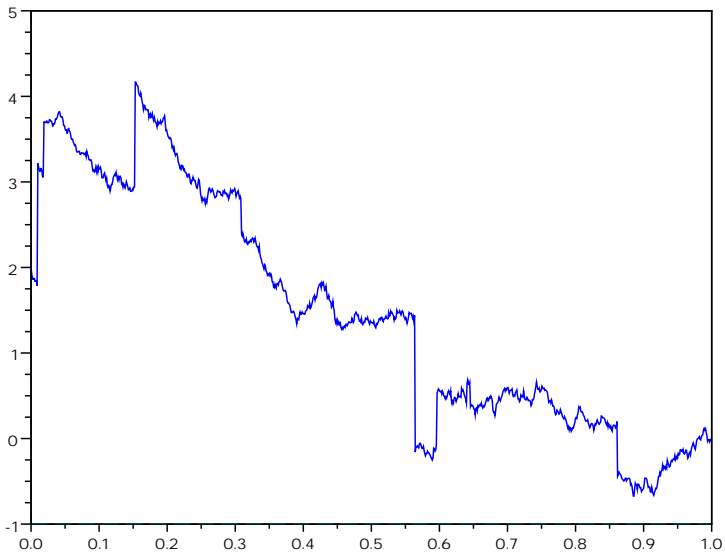
$$dX_t = -aX_t dt + dL_t, \quad t \in \mathbb{R}_+, \quad X_0 = x, \quad (1)$$

defines an Ornstein-Uhlenbeck process driven by the Lévy process  $L$  with initial distribution  $\pi = \mathcal{L}(X_0)$ .

Equivalently,

$$X_t = e^{-at} X_0 + \int_0^t e^{-a(t-s)} dL_s. \quad (2)$$

# Sample path from compound Poisson plus Wiener process driver



## Recent Literature on Lévy OU Inference

- Stochastic volatility modelling: Barndorff-Nielsen and Shephard [2001]
- (Non-)Parameteric estimation for driving subordinators Jongbloed, van der Meulen and van der Vaart [2005]
- Maximum-likelihood-estimation from discrete observations Valdivieso, Schoutens and Tuerlinckx [2009]
- Least squares estimation from discrete observations for  $\alpha$ -stable driver Hu and Long [2009]

## Setting

Problem: Estimation of  $a$  from continuous observations  $X_t$ ,  $0 \leq t \leq T$  and know Lévy-Khintchine triplet of  $L$ .

We work throughout in the canonical setting:

- $\Omega = D(\mathbb{R}_+) = \{f : \mathbb{R}_+ \rightarrow \mathbb{R}; f \text{ càdlàg}\}$
- $X(\omega, t) = \omega(t)$  for all  $\omega \in \Omega$  coordinate process
- Filtration generated by  $X$

$$\mathcal{F}_t = \bigcap_{s>t} \sigma(X_u : u \leq s) \text{ and } \mathcal{F} = \bigvee_t \mathcal{F}_t$$

For every  $a \in \mathbb{R}$  we obtain a solution measure  $P^a$  of the OU equation on  $D(\mathbb{R}_+)$ .

# Absolute Continuity/Singularity (ACS) Problem

$P_t^a := P^a|_{\mathcal{F}_t}$  denotes the restriction of  $P^a$  to  $\mathcal{F}_t$ .

Local absolute continuity:

$$P^{a'} \ll^{loc} P^a \iff P_t^{a'} \ll P_t^a \quad \forall t \in \mathbb{R}_+$$

In order to define an MLE for the statistical experiment  $(\Omega, \mathcal{F}, (\mathcal{F}_t), (P^a)_{a \in \mathbb{R}})$  we need:

① Does

$$P^{a'} \ll^{loc} P^a \text{ hold for all } a, a' \in \mathbb{R}?$$

② Can we derive  $Z_t = \frac{dP_t^{a'}}{dP_t^a}$  explicitly?

# Hellinger Process and ACS Problems

Let  $P, P'$  be two probability measures on  $(\Omega, \mathcal{F}, (\mathcal{F}_t))$ .

## Theorem (Jacod and Mémin (1979))

Let  $h(\alpha)$ ,  $\alpha \in (0, 1)$  be a version of the Hellinger process  $h(\alpha; P, P')$ . Then for every stopping time  $T$  there is equivalence between

- 1  $P'_T \ll P_T$ ;
- 2  $\exists \alpha \in (0, 1)$  such that  $P'(h(\alpha)_T < \infty) = 1$  and  $P'_0 \ll P_0$  and  $P'(h(0)_T = 0) = 1$ .

## Semimartingale characteristics of $X$

Let  $(b, \sigma^2, \mu)$  denote the Lévy-Khintchine triplet of  $L$ .

Then the semimartingale characteristics  $(B, C, \nu)$  of  $X$  are given by

$$\begin{aligned}B(\omega, t) &= bt - a \int_0^t X_{s-}(\omega) ds, \\C(\omega, t) &= \sigma^2 t, \\ \nu(\omega, dt, dx) &= \mu(dx) \lambda(dt),\end{aligned}$$

where  $\lambda$  denotes the Lebesgue measure on  $\mathbb{R}$ .

# The Hellinger Process

## Proposition

*A version of the Hellinger process of two solution measures  $P^a, P^{a'}$  is*

$$h_t(\alpha; a, a') = \frac{\alpha(1 - \alpha)}{2\sigma^2} \int_0^t \left[ \int_0^u \left( a' e^{-a'(u-s)} - a e^{-a(u-s)} \right) L(ds) \right]^2 du.$$

*for  $a, a' \in \mathbb{R}$ .*

## Theorem

Let  $P^a, P^{a'}$  be two solution measures of the OU equation for the driving Lévy process  $L$  with characteristic triplet  $(b, \sigma^2, \rho)$  and initial distributions  $\pi$  and  $\pi'$ . Suppose that  $\sigma^2 > 0$  and  $\pi' \ll \pi$ , then we have

$$P^{a'} \ll^{loc} P^a.$$

## Proposition

*There exists a  $P$ -local martingale  $N : \Delta \rightarrow \mathbb{R}$  on a random interval  $\Delta \subset \Omega \times \mathbb{R}_+$  such that the density process is given by*

$$Z_t = \frac{dP_t^{a'}}{dP_t^a} = Z_0 \exp \left( N_t - \frac{(a' - a)^2}{2\sigma^2} \int_0^t X_{s-}^2 ds \right).$$

*Furthermore, for every stopping time  $S$  such that  $[0, S] \subset \Delta$  the stopped process  $N^S$  is of the form*

$$N^S = \left( \frac{(a' - a)}{\sigma^2} X_{t-1_{[0,S]}} \right) \cdot X^c,$$

*where  $X^c$  denotes the continuous martingale part of  $X$  under  $P^a$ .*

## Maximum-Likelihood-Estimator (MLE)

For continuous observations of the Ornstein-Uhlenbeck process  $X$  the likelihood function  $\mathcal{L}$  for the statistical experiment  $(\Omega, \mathcal{F}, (\mathcal{F}_t), (P^a)_{a \in \mathbb{R}})$  takes the form

$$\mathcal{L}(a, X^T) = \frac{dP_t^a}{dP_t^0} = \exp \left( \frac{a}{\sigma^2} \int_0^T X_{s-} dX_s^c - \frac{a^2}{2\sigma^2} \int_0^T X_{s-}^2 ds \right),$$

Hence, the MLE for  $a$  is explicitly given by

$$\hat{a}_T = \frac{\int_0^T X_{s-} dX_s^c}{\int_0^T X_{s-}^2 ds}.$$

## The continuous martingale part $X^c$

By the Lévy-Itô decomposition of  $L$  we can write  $X$  as

$$X_t = X_0 - a \int_0^t X_{s-} ds + W_t + J_t, \quad t \geq 0,$$

where  $W$  is a Wiener Process and  $J$  a quadratic pure jump process given by

$$J_t = \int_{\{|x|<1\}} x(N_t(dx) - t\mu(dx)) + bt + \sum_{0 \leq s \leq t} \Delta X_s 1_{\{|\Delta X_s| \geq 1\}}.$$

$N$  is the jump measure of  $L$  with compensator  $\mu$ .

Under  $P^0$  it follows that  $X^c = W$ , but under  $P^a$

$$\tilde{W}_t = W_t - a \int_0^t X_{s-} ds$$

defines a Wiener process such that  $X^c = \tilde{W}$  under  $P^a$ .  
Hence, given observations  $(X_t(\omega), t \in [0, T])$

$$X_t^c = X_t - \int_{\{|x| < 1\}} x(N_t(dx) - t\mu(dx)) - bt - \sum_{0 \leq s \leq t} \Delta X_s 1_{\{|\Delta X_s| \geq 1\}}.$$

which can be reconstructed from continuous observations.  
Hence, the MLE can be rewritten as

$$\hat{a}_T = \frac{\int_0^T X_{s-} (dW_s - aX_{s-} ds)}{\int_0^T X_{s-}^2 ds} = \frac{\int_0^T X_{s-} dW_s}{\int_0^T X_{s-}^2 ds} - a.$$

under  $P^a$ .

## Curved Exponential Families

Let  $\{P^\theta, \theta \in \Theta\}$  be a family of measures on  $(\Omega, \mathcal{F}, (\mathcal{F}_t))$ .

**Definition (Kühler and Sørensen (1997))**

*We say that a statistical experiment  $\{P^\theta, \theta \in \Theta\}$  forms a **curved exponential family** if the likelihood function exists and is of the form*

$$\frac{dP_t^\theta}{dP_t^{\theta_0}} = \exp(\theta' A_t - \kappa(\theta) S_t)$$

*where  $\kappa : \Theta \rightarrow \mathbb{R}$ , for  $\theta_0 \in \Theta$  arbitrary but fixed and  $A : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}^d$  is a càdlàg process. Moreover,  $S : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}$  is assumed to be a non-decreasing continuous process with  $S_0 = 0$  and  $S_t \xrightarrow{t \rightarrow \infty} \infty$ .*

## Strong Consistency

### Theorem

*Under the condition  $\sigma^2 > 0$  the MLE  $\hat{a}_T$  for any  $a \in \mathbb{R}$  based on continuous observations  $X_t$ ,  $t \in [0, T]$  exists and is given by*

$$\hat{a}_T = \frac{\int_0^T X_{s-} dX_s^c}{\int_0^T X_{s-}^2 ds}.$$

*Furthermore, under  $P_a$  the MLE is unique for  $T$  sufficiently large and*

$$\hat{a}_T \rightarrow a \quad \text{almost surely}$$

*as  $T \rightarrow \infty$ .*

# Asymptotic Normality

## Theorem

Let  $X$  be a stationary OU process and  $a > 0$  such that  $c = E[X^2] < \infty$ . Then under  $P^a$

$$\sqrt{T}(\hat{a}_T - a) \rightarrow N\left(0, \frac{\sigma^2}{c}\right) \quad \text{weakly}$$

as  $T \rightarrow \infty$ .

## Summary and Outlook

- Under  $\sigma^2 > 0$  the solution measures  $\{P^a, a \in \mathbb{R}\}$  are locally equivalent.
- The MLE takes an explicit form and is consistent and asymptotically normal as well as efficient.
- Computation from discrete observations is straight forward.
- Asymptotics for  $a < 0$  and without second moments of  $X$ ?
- Delay estimation

## Bibliography

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