Maker-Breaker Games on Random Geometric Graphs

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Outline

1. Introduction
   - Maker-Breaker Games
   - Random Geometric Graphs

2. Structure of the RGG
   - Dissection of $[0, 1]^2$ into Tiny Cells
   - Structural Lemmas
   - Obstructions

3. Maker-Breaker Games
   - Connectivity Game
   - Hamilton Game
   - Perfect Matching Game

4. Conclusion
   - Summary
Introduction

- Maker-Breaker Games
- Random Geometric Graphs

Structure of the RGG

- Dissection of $[0, 1]^2$ into Tiny Cells
- Structural Lemmas
- Obstructions

Maker-Breaker Games

- Connectivity Game
- Hamilton Game
- Perfect Matching Game

Conclusion

- Summary
Two player, complete information game
Collection of winning subsets $\mathcal{F} \subset 2^{E(G)}$
Breaker and Maker alternately claim edges of $G$
Maker wins if he claims some subset in $\mathcal{F}$. Otherwise Breaker wins.
Typically,

$$\mathcal{F} = \{ F \subset E \mid G[F] \text{ has property } \mathcal{P} \}$$

where $\mathcal{P}$ is an increasing graph property (e.g. has spanning tree, Hamilton cycle, or perfect matching)
Two Classic Results


The Book on Combinatorial Games


A Recent Break-Through

- M. Krivelevich, *The Critical Bias for the Hamiltonicity Game is* \((1 + o(1))n/\ln n\), 2011.
Some additional recent results

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Random Geometric Graph $G(n, r_n)$

- Pick random points $x_1, \ldots, x_n \in [0, 1]^2$
- Connectivity radius $r_n$
- $x_ix_j \in E \iff \|x_i - x_j\| \leq r_n$
- Study expected behavior as $n \to \infty$
- $A_n$ holds whp means $\Pr(A_n) = 1 - o(1)$
Connectivity of RGG


Let \( x \in \mathbb{R} \) be a constant. If

\[
    r_n^2 = \frac{\ln n + \omega(1)}{\pi n}
\]

then

\[
    \lim_{n \to \infty} \mathbb{P}[G(n, r_n) \text{ connected}] = 1.
\]

Key idea:

\[
    E(\deg(v)) = n \cdot \text{area}(B(v, r_n)) \approx \ln n
\]

and this is enough to guarantee connectivity.
Hitting Radius of an Increasing Property

The hitting radius of increasing graph property $\mathcal{P}$ is

$$
\rho_n(\mathcal{P}) = \inf\{r \geq 0 : G(n, r) \text{satisfies } \mathcal{P}\}
$$

Example:

The hitting radius for connectivity is

$$
\rho_n(G \text{ is connected}) = \sqrt{\frac{\ln n}{\pi n}}.
$$
Let $x \in \mathbb{R}$ be a constant.

- Hitting radius for minimum degree 2 is
  \[ \rho_n(\delta(G) \geq 2) = \sqrt{\frac{\ln n + \ln \ln n}{\pi n}} \]

- Hitting radius for minimum degree 4 is
  \[ \rho_n(\delta(G) \geq 4) = \sqrt{\frac{\ln n + 5 \ln \ln n}{\pi n}} \]
Theorem (BDFMS 13+)

The hitting radius for the random geometric graph $G(n, r)$ to be Maker’s win corresponds to a simple minimum degree condition as follows:

- **Connectivity game** $\iff \delta(G(n, r)) \geq 2$
- **Hamilton Cycle game** $\iff \delta(G(n, r)) \geq 4$
- **Perfect Matching game** $\iff \delta(G(n, r)) = 2$ and minimum edge degree $\geq 3$. 

**Random Geometric Graphs**
Why are these minimum degree conditions necessary?

<table>
<thead>
<tr>
<th>when $\delta(G)$ is...</th>
<th>then <strong>Breaker</strong> wins...</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Connectivity game</td>
</tr>
<tr>
<td>3</td>
<td>Hamilton Cycle game</td>
</tr>
</tbody>
</table>

because **Breaker** goes first!
## Maker-Breaker Hitting Radii for RGG

<table>
<thead>
<tr>
<th>Game</th>
<th>Minimum Degree Condition</th>
<th>Hitting Radius (essentially)</th>
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</thead>
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<td>( \delta(G) \geq 2 )</td>
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   - Random Geometric Graphs

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4. Conclusion
   - Summary
The Problem: vertices with very low degree

If we had $\delta(G) = \omega(1)$, then our games would be easy Maker-win. Must deal with vertices of constant degree.

- **We dissect** the square $[0, 1]$ into **very small cells** (squares).
- **The good news**: most points have lots of neighbors in nearby dense cells.
- **The not-so-bad news**: the rest are in clusters of well-separated sparse cells.

The dense cells provide the backbone of our strategy. We use them to handle the sprinkling of sparse cells.
Dissection $\mathcal{D}$ of unit square $[0, 1]^2$ into cells

Given

$$r^2 = \frac{\ln n + \Theta(\ln \ln n)}{\pi n}.$$

Let $\eta > 0$ be a small constant. Choose $q = q(n)$ such that

$$q = \eta r$$

This ensures that you need

$$\approx \frac{1}{\eta^2} < \infty$$

$q \times q$ squares to cover $B(v, r)$. 
The Structure of $\Gamma$

- Fix a large constant $T > 0$.
- A cell $c$ is **good** if $|V \cap c| \geq T$. Otherwise, $c$ is **bad**.

Define graph $\Gamma$ using good cells of dissection $\mathcal{D}$.
- $V(\Gamma) =$ all good cells
- $E(\Gamma) = \{cc' : \text{dist}(c, c') \leq r\}$

Gives rise to connected components $\Gamma_{\text{max}}$ and other smaller components $\Gamma_2, \Gamma_3, \ldots$
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Gives rise to connected components $\Gamma_{\text{max}}$ and other smaller components $\Gamma_2, \Gamma_3, \ldots$
Cells of $\Gamma$ are Good or Bad
Vertices of $G$ are Safe, Risky or Dangerous

Components are $\Gamma_{\text{max}}$ and the smaller $\Gamma_2, \Gamma_3, \ldots$

Categorize each $v \in V$ as follows:

- $v$ is **safe**: has $\geq T$ neighbors in a good cell $c$ of $\Gamma_{\text{max}}$
- $v$ is **risky**: has $\geq T$ neighbors in a good cell $c$ of $\Gamma_i$, for $i \geq 2$
- $v$ is **dangerous**: otherwise

Vertices in good cells are safe or risky.

Vertices in bad cells can be safe, risky or dangerous.
The Giant and the Obstructions
The Giant and the Obstructions

Partition $G$ into the unique Giant and a collection of two types of Obstructions.

The Giant
- $\Gamma_{\max}^+ = \Gamma_{\max}$ and its nearby safe points

The Obstructions
- $\Gamma_i^+ = \Gamma_i$ and its nearby risky points
- Dangerous Cluster: a maximal clique of dangerous points
1 Introduction
   • Maker-Breaker Games
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   • Summary
Global Structure of RGG $G(n, r)$

The Dissection Lemma

The largest component $\Gamma^+_{\text{max}}$ is giant.

$$\Gamma_{\text{max}} \text{ contains } \geq 0.99 \cdot |\mathcal{D}| \text{ cells whp.}$$

The Obstructions are small and very far apart.

\textit{Whp}, for obstructions $\mathcal{O}_i \neq \mathcal{O}_j$

- $\text{diam}(\mathcal{O}_i) < r/100$
- $\text{dist}(\mathcal{O}_i, \mathcal{O}_j) > r \cdot 10^{10}$

Obstructions = small components and dangerous clusters
Lemma (The Giant)

$\Gamma_{\text{max}}$ contains $\geq 0.99 \cdot |\mathcal{D}|$ cells whp.

Recall: cell $c$ has side length $q = \eta r$

Set $K > \frac{1}{\eta^2} > 0$.

Pick any $\mathcal{B} = K \times K$ block of cells.
\( \Gamma_{\text{max}} \) contains \( \geq 0.99 \cdot |D| \) cells

**Lemma (The Giant)**

\( \Gamma_{\text{max}} \) contains \( \geq 0.99 \cdot |D| \) cells whp.

Recall: cell \( c \) has side length \( q = \eta r \)

Set \( K > \frac{1}{\eta^2} > 0 \).

Pick any \( B = K \times K \) block of cells.

\[
\text{Area}(B) = \frac{K^2}{\eta^2} B(v, r)
\]
\[ \Gamma_{\text{max}} \text{ contains } \geq 0.99 \cdot |D| \text{ cells} \]

Recall: cell \( c \) has side length \( q = \eta r \)

Set \( K > \frac{1}{\eta^2} > 0. \)

Pick any \( B = K \times K \) block of cells.

- 0.99% rows/columns have no bad cells, because
  \[ E(|V \cap c|) = \Theta(\log n) \gg T. \]
\[ \Gamma_{\text{max}} \text{ contains } \geq 0.99 \cdot |\mathcal{D}| \text{ cells} \]

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- Creates largest component in \( \mathcal{B} \)
\( \Gamma_{\text{max}} \) contains \( \geq 0.99 \cdot |\mathcal{D}| \) cells

Recall: cell \( c \) has side length \( q = \eta r \)

Set \( K > \frac{1}{\eta^2} > 0 \).

Pick any \( B = K \times K \) block of cells.

- 0.99% rows/columns have no bad cells, because
  \( E(|V \cap c|) = \Theta(\log n) \gg T \).
- Creates largest component in \( B \)
- Take overlapping blocks to get
  \( \Gamma_{\text{max}} \)
$$\text{diam}(\Gamma_i^+) < \frac{r}{100} \text{ when } i \geq 2$$

**Lemma**

*Whp, $\text{diam}(\Gamma_i^+) < \frac{r}{100}$ for $i \geq 2$.*

- No good cells in surrounding half-disks of radius $r$
- If $\text{diam}(\Gamma_i^+) \geq \frac{r}{100}$ then there are too many bad cells in a small area

Similar proofs that other obstructions are small & that pairs of obstructions are well-separated
1. **Introduction**
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4. **Conclusion**
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Assign vertices to help with obstruction $\mathcal{O}$

Point $v \in V$ is **crucial** for $\mathcal{O}$ if
- $v$ is **safe**, and
- $\mathcal{O} \subset B(v; r)$, and

Recall: $v$ safe $\Rightarrow \exists c \in \Gamma_{\text{max}}$ with $|B(v; r) \cap c \cap V| \geq T$

The $T$ vertices in $c$ are **important** for $v$ and for $\mathcal{O}$. 
Obstructions have Crucial Vertices

The Obstruction Lemma

Consider $G(n, r)$ where

$$\pi r^2 = \ln n + (2k - 3) \ln \ln n + O(1),$$

with $k \geq 2$ fixed. \textit{Whp} the following holds for all obstructions $\mathcal{O}$. Let $|\mathcal{O}| = s$

- If $2 \leq s \leq T$ then $\mathcal{O}$ has $\geq k + s - 2$ crucial vertices;
- If $s \geq T$, then $\mathcal{O}$ has $\geq k$ crucial vertices.

\textbf{Note:} Obstructions far apart $\Rightarrow$ crucial vertices for $\mathcal{O}_i \neq \mathcal{O}_j$ are distinct.
Obstructions have Crucial Vertices

- \(|\mathcal{O}| \leq T\)
- Must be a finite number of vertices in outer ring
- Forces existence of vertices in middle ring
  - These vertices adjacent to \(\mathcal{O}\)
  - Not part of \(\mathcal{O} \Rightarrow \text{safe or risky}\)
  - Must be adjacent to good cells in \(\Gamma_{\text{max}}\)
Summary: Structure of RGGs

- There is a giant component $\Gamma_{\text{max}}$ of dense cells.
- Obstructions are small and far from one another.
- Obstructions have enough crucial vertices to help connect them to $\Gamma_{\text{max}}$. 
1 Introduction
   - Maker-Breaker Games
   - Random Geometric Graphs

2 Structure of the RGG
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   - Obstructions

3 Maker-Breaker Games
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4 Conclusion
   - Summary
Terminology Reminders

**Minimum Degree** \( \delta(G) = \min_{v \in V} \deg(v) \)

**With High Probability (whp)**
Event \( A = A_n \) holds \( \text{whp} \) if \( \lim_{n \to \infty} \Pr(A_n) = 1 \).

**Hitting Radius**
The *hitting radius* of increasing graph property \( \mathcal{P} \) is

\[
\rho_n(\mathcal{P}) = \inf\{r \geq 0 : G(n, r) \text{ satisfies } \mathcal{P}\}
\]

- If \( r < \rho_n \) then \( G(n, r) \) **DOES NOT** have property \( \mathcal{P} \) \( \text{whp} \).
- If \( r \geq \rho_n \) then \( G(n, r) \) **DOES** have property \( \mathcal{P} \) \( \text{whp} \).
Hitting Radius for the Connectivity Game

Theorem (BDFMS 2013+)

Whp, the RGG process $G(n, r)$ satisfies

$$\rho_n(\text{Maker wins connectivity game}) = \rho_n(\delta(G(n, r)) \geq 2).$$

In particular, if

$$\pi nr^2 = \ln n + \ln \ln n + x_n$$

then

$$\lim_{n \to \infty} \mathbb{P}(\text{Maker wins}) = \begin{cases} 1 & \text{if } x_n \to +\infty, \\ e^{-(e^{-x} + \sqrt{\pi e^{-x}})} & \text{if } x_n \to x \in \mathbb{R}, \\ 0 & \text{if } x_n \to -\infty. \end{cases}$$
Breaker wins when $\delta(G) \leq 1$

- Breaker makes an isolated vertex on the very first move

When $\delta(G) \geq 2$

- We use the Shannon Switching Game result

**Theorem (A. Lehman, 1964)**

The connectivity game is Maker-win if and only if $G$ admits two disjoint spanning trees.
Two Disjoint Spanning Trees in $G(n, r)$
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   - Maker-Breaker Games
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4. Conclusion
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Hitting Radius for the Hamilton Game

Theorem (BDFMS 2013+)

\( Whp, \) the RGG process \( G(n, r) \) satisfies

\[
\rho_n(\text{Maker wins Hamilton game}) = \rho_n(\delta(G(n, r)) \geq 4).
\]

In particular, if

\[
\pi n r^2 = \ln n + 5 \ln \ln n - 2 \ln 6 + x_n
\]

then

\[
\lim_{n \to \infty} \mathbb{P}(\text{Maker wins}) = \begin{cases} 
1 & \text{if } x_n \to +\infty, \\
e^{-e^{-x}} & \text{if } x_n \to x \in \mathbb{R}, \\
0 & \text{if } x_n \to -\infty.
\end{cases}
\]
Before the Game Begins:

Pick a spanning tree $T$ of $\Gamma_{\text{max}}$ with maximum degree $\leq 5$

- Such a tree $T$ exists because $\Gamma_{\text{max}}$ is a geometric graph

Every good cell $c \in \Gamma_{\text{max}}$

- At most $T = O(1)$ vertices are marked. They will be used to (a) connect with vertices in bad cells, and (b) create matchings between cells adjacent in $T$.
- The remaining vertices in $c$ are unmarked. These will become the bulk of the Hamilton cycle. We make a soup of flexible blob cycles.
Maker’s Hamilton Strategy Overview

During the Game, Maker plays lots of mini-games:

1. Create a path through each obstruction and each safe cluster, ending in marked vertices in the same cell.
2. Claim two edges between cells adjacent in $T$.
3. Create soup of flexible **blob cycles** in the unmarked vertices.
4. Claim half the edges from each marked to vertex to the set of unmarked vertices.

After the Game, Maker stitches together the Hamilton Cycle.
Blob Cycles

Let $k \geq s$. An $s$-blob cycle on $k$ vertices is the union of:

- A $k$-cycle on $v_1, \ldots, v_k$
- A complete graph on $v_1, \ldots, v_s$
Let $k \geq s$. An \textbf{s-blob cycle} on $k$ vertices is the union of

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Maker’s Hamilton Strategy for each Good Cell $c$

Mark $T = O(1)$ vertices for connecting to nearby cells, obstructions and safe vertices. Make blob cycle soup in the rest.
Maker’s Hamilton Strategy for each Good Cell $c$

Mark $T = O(1)$ vertices for connecting to nearby cells, obstructions and safe vertices. Make blob cycle soup in the rest.
Maker’s Hamilton Strategy for each Good Cell $c$

Claim half the edges from each vertex to lower level

Diagram showing the strategy with arrows pointing down from each vertex to the lower level, indicating the half-claim of edges.
Maker's Hamilton Strategy for each Good Cell \( c \)
Maker’s Hamilton Strategy for each Good Cell $c$

The blob absorption
Maker’s Hamilton Strategy for each Good Cell $c$

Final Step: Absorb unused points (not shown)
1. Introduction
   - Maker-Breaker Games
   - Random Geometric Graphs

2. Structure of the RGG
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3. Maker-Breaker Games
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4. Conclusion
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Hitting Radius for Perfect Matching Game

**Theorem (BDFMS 2013+)**

*Whp*, the random geometric graph process satisfies, for $n$ even:

$$
\rho_n(\text{Maker wins p. m. game}) = \rho_n(\delta(G) \geq 2 \text{ and } \delta_e \geq 3)
$$

where $\delta_e(G) = \min_{uv \in E(G)} |N(\{u, v\})|$. In particular, if

$$
\pi nr^2 = \ln n + \ln \ln n + x_n
$$

then

$$
\lim_{n \to \infty, \ n \text{ even}} P(\text{Maker wins}) = \begin{cases} 
1 & \text{if } x_n \to +\infty, \\
\frac{1}{e^{-(1+\pi^2/8)e^{-x}+\sqrt{\pi}(1+\pi)e^{-x/2}}} & \text{if } x_n \to x \in \mathbb{R}, \\
0 & \text{if } x_n \to -\infty.
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- Maker-Breaker Games
- Random Geometric Graphs

2 Structure of the RGG
- Dissection of $[0, 1]^2$ into Tiny Cells
- Structural Lemmas
- Obstructions

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- Connectivity Game
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- Perfect Matching Game

4 Conclusion
- Summary
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Future Directions

Biased Games

- What happens when Breaker claims $b$ edges on every turn, while Maker only claims 1?
- Our results should extend to constant $b$, but what about when $b = b(n) = \omega(1)$?

Higher Dimensions

- What is the critical radius for each of these games for a 3D (and higher) random geometric graph?

Thank you!
Mini-game: the \((a, b)\) Path Game

The \((a, b)\) Path Game:
- Played on \(K_{a+b}\) partitioned into sets \(A, B\) of sizes \(a, b\).
- Maker Goal: create a path between any two \(B\)-vertices that contains all \(A\)-vertices.

Lemma

The \((a, b)\) Path Game is Maker-win when
- \(b \geq 6\), or;
- \(a = 3\) and \(b \geq 5\), or;
- \(a \in \{1, 2\}\) and \(b \geq 4\).
Mini-game: the \((a, b)\) Path Game

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Mini-game: Blob Cycle Game

s-Blob Cycle Game
- Played on $K_m$
- Maker tries to make an $s$-blob on $m$ vertices

Lemma

For $s \geq 4$, there is a constant $N = N(s)$ such that the $s$-Blob Game is Maker-win on $K_m$ for $m \geq N(s)$.

Fun fact: the proof uses Krivelevich’s result on the critical bias of the Hamilton cycle game on $K_n$. 
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The \((a, b)\) Matching Game

- Played on \(K_{a+b}\) partitioned into sets \(A, B\) of sizes \(a, b\).
- Maker Goal: create a matching that saturates \(A\)

**Lemma**

The \((a, b)\) matching game is Maker-win when

- \(b \geq 4\), or;
- \(a \in \{2, 3\}\) and \(b \geq 3\), or;
- \(a = 1\) and \(b \geq 2\).
Mini-game: the \((a, b)\) Matching Game

The \((a, b)\) Matching Game

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