The price of anarchy: The case of exponential multiserver

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The prisoner's dilemma:

Penalty matrix:

$$\begin{pmatrix} (x,x) & (0,x+1) \\ (x+1,0) & (1,1) \end{pmatrix}$$

Nash equilibrium: (x,x) when $x > 1$.

Social optimization: $(1,1)$.

Price of anarchy (PoA): $(x+x)/(1+1) = x$.

The PoA is unbounded.

Nash equilibrium is not always unique:

The options:

- take the socially worst
- take the socially best
- consider only pure equilibria (if exist)
- consider only mixed equilibria (if exist)
- consider only symmetric equilibria (for symmetric games)

Bounding the PoA for special cases:

1. Routing in networks for non-atomic players with **affine** edge latency functions:

$$\Downarrow$$

$$\text{PoA} \leq 4/3$$

(Roughgarden and Tardos).

More on routing

2. Routing in networks for non-atomic players and degree d polynomial edge latency functions:

$$\Downarrow$$

$$\text{PoA} \leq O(d/\log d)$$

(Roughgarden). For atomic players:

$$\Downarrow$$

$$\text{PoA} \leq O(2^d d^{d+1})$$

(Awerbuch, Azar and Apstein).

More on routing

3. For M/M/1 latency functions, $1/(\mu - x)$:

$$\Downarrow$$

$$\text{PoA} \leq \infty$$

(Friedman).

A single exponential multi-server

1. n exponential servers, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. $\mu_{n+1} = 0$.
2. Poisson arrivals. Rate λ, $\lambda < \Sigma_{i=1}^{n} \mu_i$.
3. FCFS.

The routing problem: Which server to join?

The unobservable version.

Servers as routes

A network with one pair of source and destination, and with n edges.

Routing λ_i along edge i costs each one of them

$$1/(\mu_i - \lambda_i)$$

and it comes with a social cost of

$$\Sigma_{i=1}^{n} \lambda_i/(\mu_i - \lambda_i)$$

Bounds on the PoA in terms of n

Result: The PoA is bounded by the number of servers utilized under the socially optimal routing.

$$\Downarrow$$

$$\text{PoA} \leq n$$

The bound is sharp.

Equilibrium (Bell and Stidham)

Servers 1 through i_{eq} are utilized, where

$$i_{eq} = \min\left\{ i \geq 1 | \mu_{i+1} \leq \frac{\sum_{j=1}^{i} \mu_j - \lambda}{i} \right\}$$

Only servers 1 through m are utilized, if and only if

$$\lambda \in [\lambda_{eq}^{\min}(m), \lambda_{eq}^{\min}(m+1)]$$

where

$$\lambda_{eq}^{\min}(m) = \sum_{j=1}^{m-1} \mu_j - \mu_m(m-1)$$

More on equilibrium behavior

The arrival rate to server i,

$$\lambda_i^{eq} = \mu_i - \frac{\sum_{j=1}^{i_{eq}} \mu_j - \lambda}{i_{eq}}, \;\; 1 \le i \le i_{eq} \, .$$

$$W^{eq} = \frac{i_{eq}}{\sum_{j=1}^{i_{eq}} \mu_j - \lambda}$$

$$L^{eq} = \lambda W^{eq} = \lambda \frac{i_{eq}}{\sum_{j=1}^{i_{eq}} \mu_j - \lambda}$$

The social cost under equilibrium (L^{eq}) is a function of the service rates of the utilized servers only through their sum.

Social optimality (Bell and Stidham):

Servers 1 through i_{so} are utilized

$$i_{so} = \min\left\{ i \geq 1 | \mu_{i+1} \leq \frac{(\sum_{j=1}^{i} \mu_j - \lambda)^2}{(\sum_{j=1}^{i} \sqrt{\mu_j})^2} \right\}.$$

Only servers 1 through m are utilized if and only if $\lambda \in [\lambda_{so}^{\min}(m), \lambda_{so}^{\min}(m+1)]$ where

$$\lambda_{so}^{\min}(m) = \sum_{j=1}^{m-1} \mu_j - \sqrt{\mu_m} \sum_{j=1}^{m-1} \sqrt{\mu_j}$$

More on social behavior

$$L^{so} = \frac{(\sum_{i=1}^{i_{so}} \sqrt{\mu_i})^2}{\sum_{j=1}^{i_{so}} \mu_j - \lambda} - i_s$$

The social cost (L^{so}) is a function of the service rates of the i_{so} utilized servers $\mu_1, \ldots, \mu_{i_{so}}$ only through

$$\sum_{i=1}^{i_{so}} \mu_i \quad \text{and} \quad \sum_{i=1}^{i_{so}} \sqrt{\mu_i}$$

More on social behavior

1. Faster server $\Rightarrow$ more customers.
2. Faster server $\Rightarrow$ higher utilization.
3. Faster server $\Rightarrow$ less wait.

Comparing the criteria

$$i_{eq} \leq i_{so} \quad \text{and} \quad \lambda_{eq}^{min}(m) \geq \lambda_{so}^{min}(m)$$

The price of anarchy:

$$\frac{L^{eq}}{L^{so}} = \lambda i_{eq} \frac{\mu_{(i_{so})} - \lambda}{\mu_{(i_{eq})} - \lambda} [(\sum_{i=1}^{i_{so}} \sqrt{\mu_i})^2 - i_{so}(\mu_{(i_{so})} - \lambda)]^{-1}$$

If $i_{eq} = i_{so} = m$,

$$= \frac{\lambda m}{(\sum_{i=1}^{m} \sqrt{\mu_i})^2 - m(\mu_{(m)} - \lambda)}$$

Lemma: $i_{eq} = i_{so} = m \Rightarrow \text{PoA} \leq m$.

Proof. PoA with m is applicable for (possibly empty)

$$\lambda \in [\lambda_{eq}^{\min}(m), \lambda_{so}^{\min}(m+1)].$$

$$\text{PoA} \downarrow \lambda$$

$$\Downarrow$$

$$\text{PoA for } \lambda_{eq}^{\min}(m),\ 1 \leq m \leq n.$$

$$\Sigma_{i=1}^{m} \mu_i = 1 \text{ (wlog)},$$

$$\text{PoA} = \frac{m(1 - m\mu_m)}{(\sum_{i=1}^{m} \sqrt{\mu_i})^2 - m^2 \mu_m}$$

Proof (continue).

The PoA is maximized when $\mu_i = \mu_m, i = 2, \ldots, m$ and $\mu_1 = 1 - (m-1)\mu_m$. The maximal value is smaller than or equal to m.

Result: PoA $\leq i_{so}$

Proof: Suppose $i_{eq} < i_{so}$. Increase $\mu_{i_{eq}+1}, \ldots, \mu_{i_{so}}$ up to almost the same level in which all i_{so} of them are utilized under equilibrium. The PoA increases. i_{so} stays but now $i_{eq} = i_{so}$.

The bound is sharp:

$\lambda > n, \epsilon > 0$
$\mu_1 = \lambda + 1 + \epsilon$
$\mu_i = 1 + \epsilon, 2 \le i \le n.$

$$\Downarrow$$

$i_{eq} = 1, L^e = \lambda/(1+\epsilon).$
Try $\lambda_1 = \lambda - n + 1$ and $\lambda_i = 1, 2 \le i \le n.$

$$L = \frac{\lambda - n + 1}{n + \epsilon} + (n-1)\frac{1}{\epsilon} \ge L^{so}$$

$\lambda \to \infty \Rightarrow L_{eq}/L = (n+\epsilon)/(1+\epsilon) < n.$

THE END