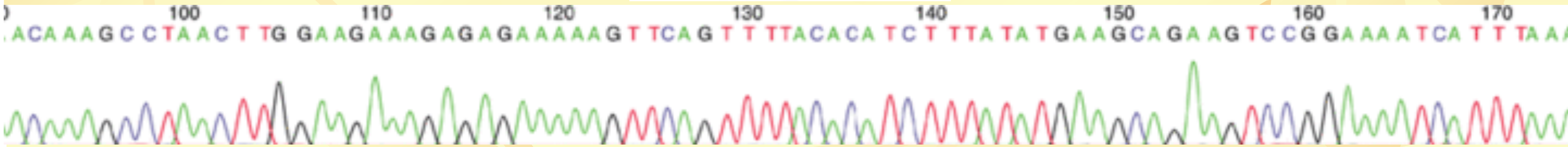
A generalized Verhulst approach of population evolution



Content

- Introduction
- Main aspects
- Conclusions

Introduction

- « Population » « evolution »
  - Malthus (1798)
  - Gompertz (1825)
  - Verhulst (1836)
  - Lotka (1925) - Volterra (1931)
  - Avrami (1939)
  - plenty plenty of others

Growth equations

- Malthus

$$\frac{dx}{dt} = r\,x$$

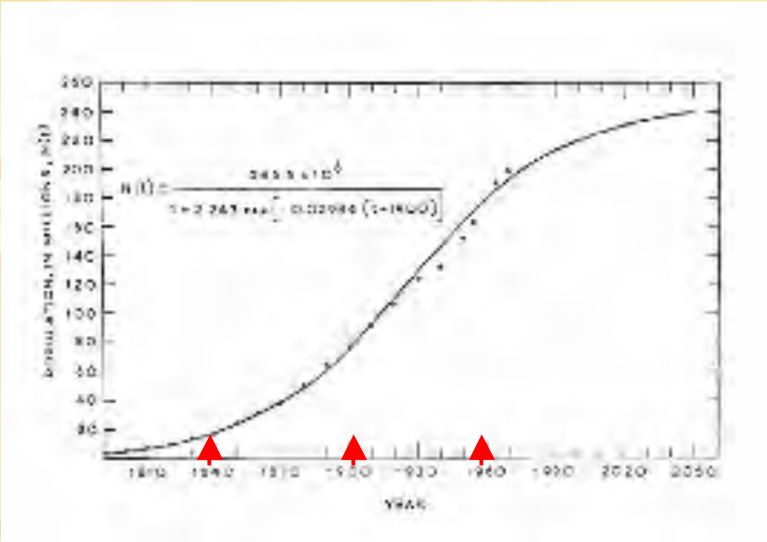
$$x(t) = e^{rt}$$

- Verhulst

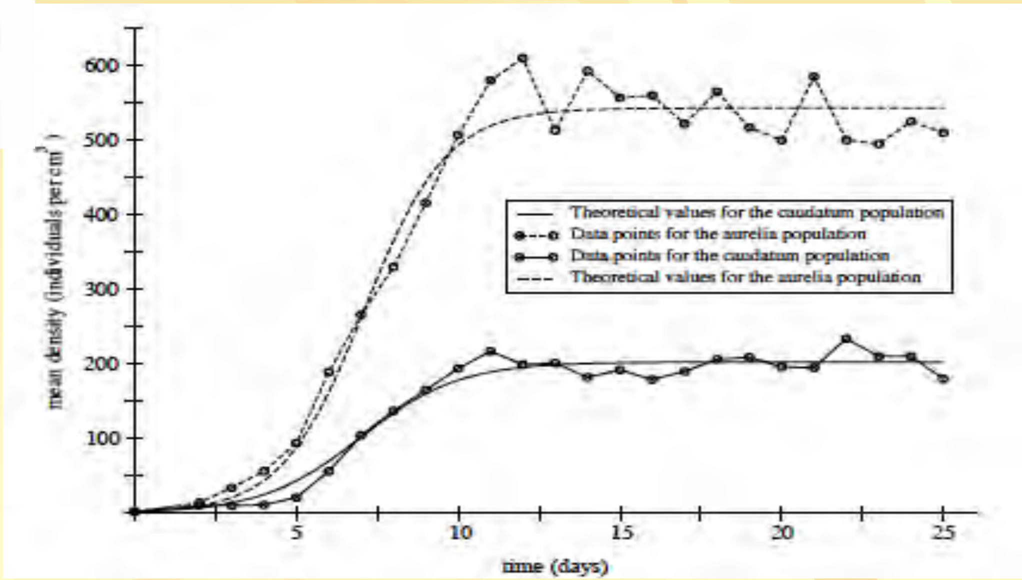
$$\frac{dx}{dt} = r\,x\,[1 - x]\,.$$

$$x(t) = \frac{e^{rt}}{1 + e^{rt}}$$

Data compared to Verhulst logistic map



P. Caudatum and P. Aurelia
De Vries et al. (2003)



USA population (1810-1970)
Montroll & Badger (1974)

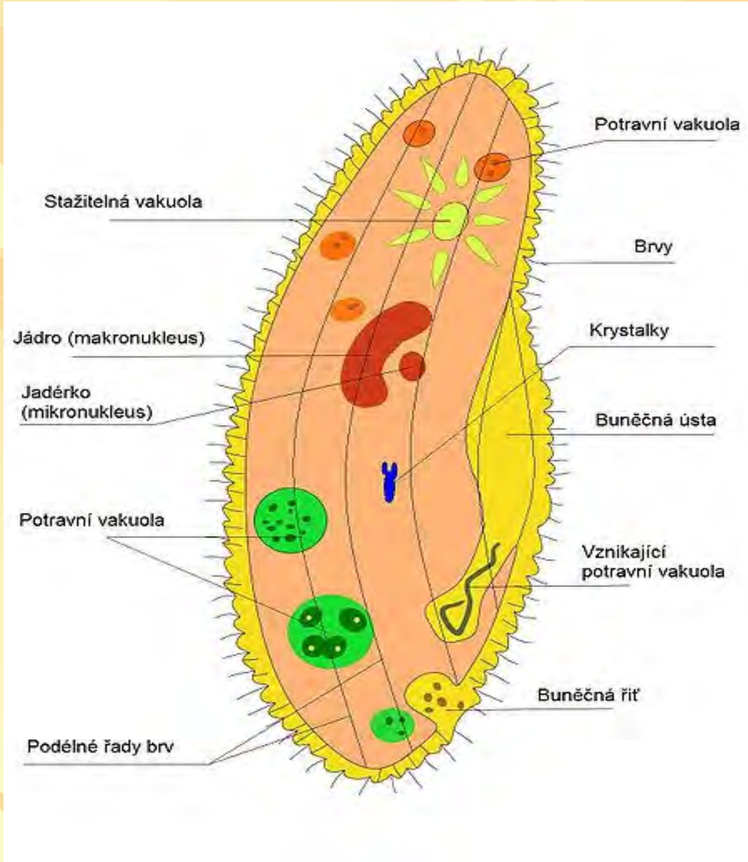
r = 0.02984

Paramecium

paramecium is a group of unicellular ciliate protozoa
... from about 50 to 350 µm in length

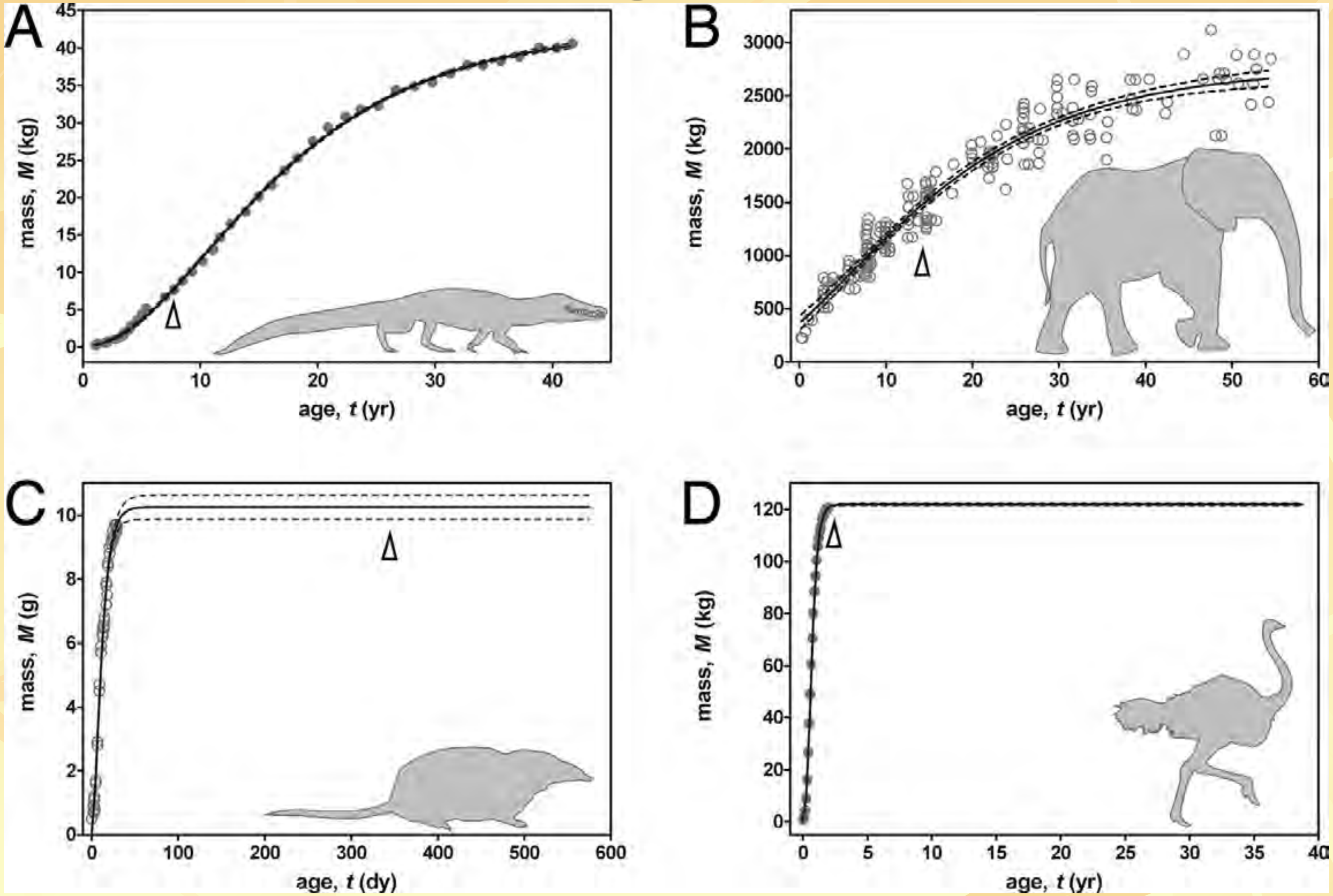
aurelia

caudatum



http://www.ask.com/wiki/Paramecium?qsrc=3044

Mass growth



(A) Alligator (B) Elephant
(C) Shrew (D) Ostrich

Lee & Werning (2008)

Length growth

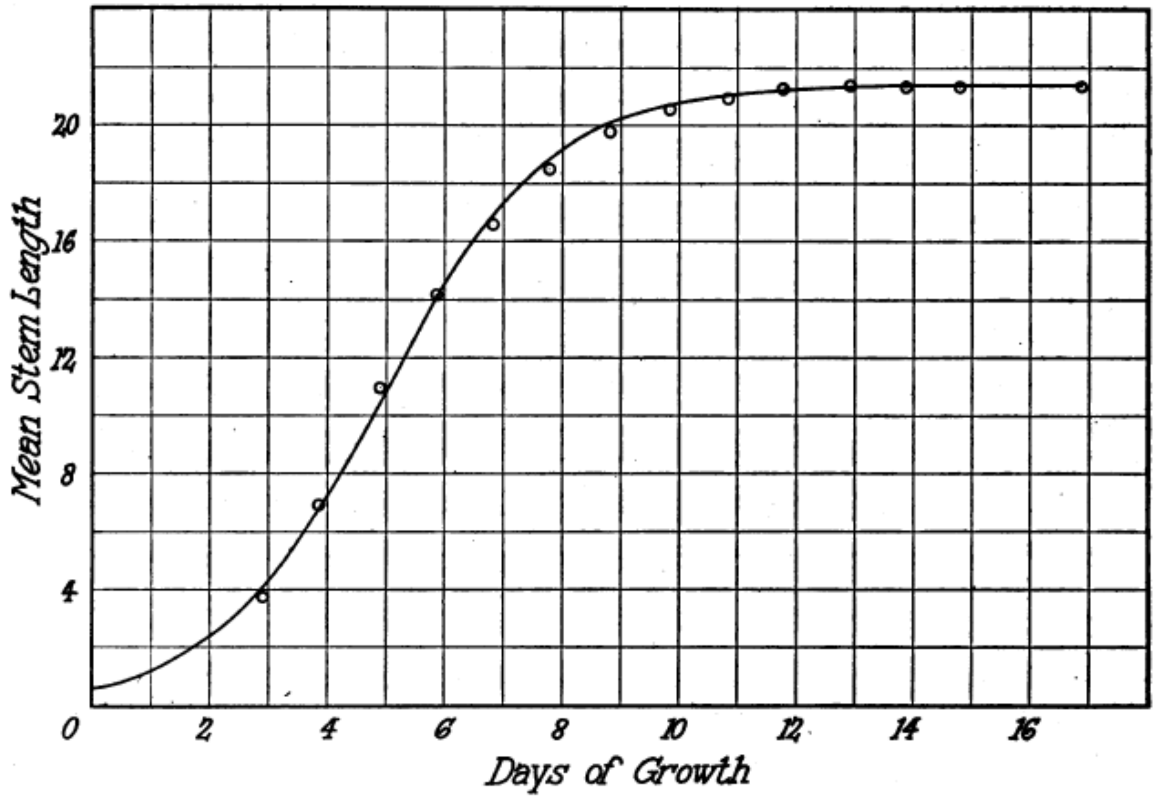


FIGURE 1

Observed and calculated mean stem length of seedlings of *Cucumis melo* grown in the complete absence of exogenous food and light.

Cucumis melo stem length

Pearl et al. (1928)

$r = 0.704$

Skewed logistic

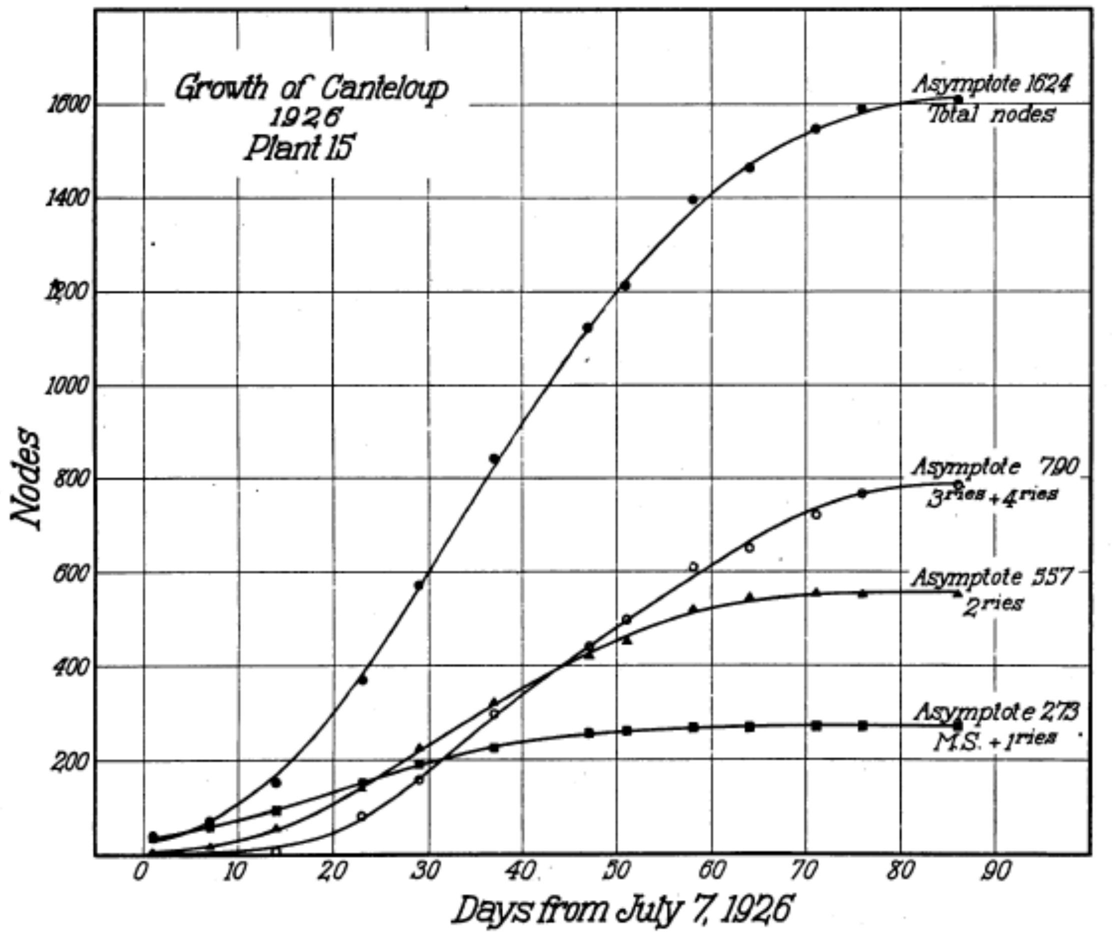


FIGURE 1

The growth of a canteloup plant under normal field conditions.

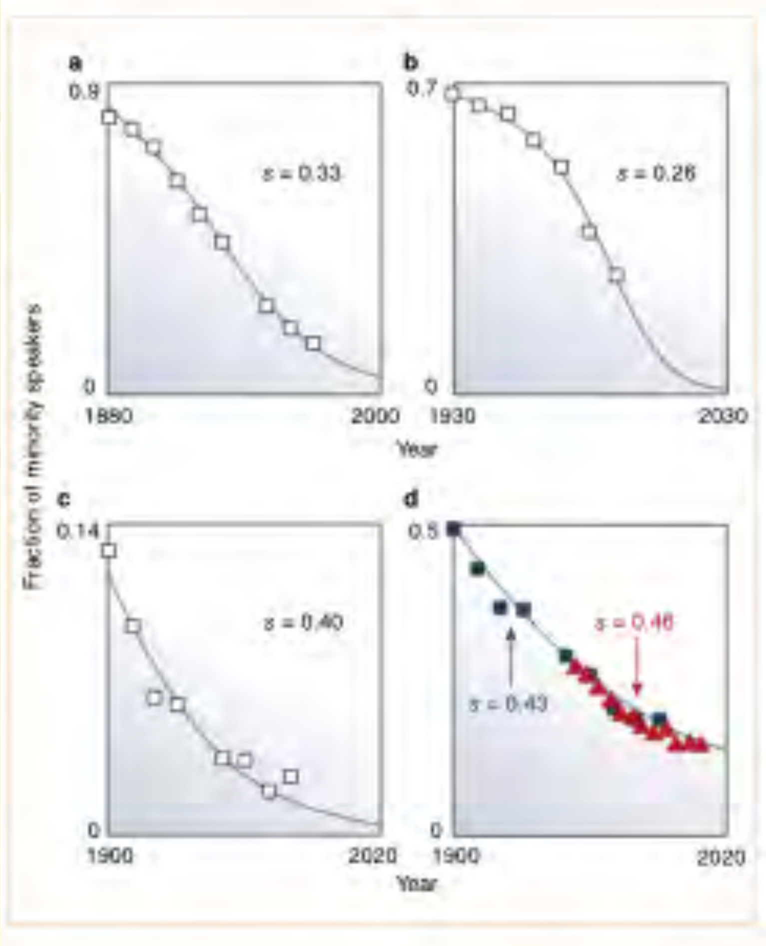
$$y(t) = \frac{k}{1 + e^{a_0 + a_1\, t + a_2\, t^2 + \ldots}}$$

Number of nodes

Canteloup

Pearl et al. (1928)

Language death



Evolution equation à la Verhulst

(a,b): Sc.Gael in Scotl., Quechua in Peru,
(c,d): Welsh in W, W in « W »
Abrams & Strogatz (2003)

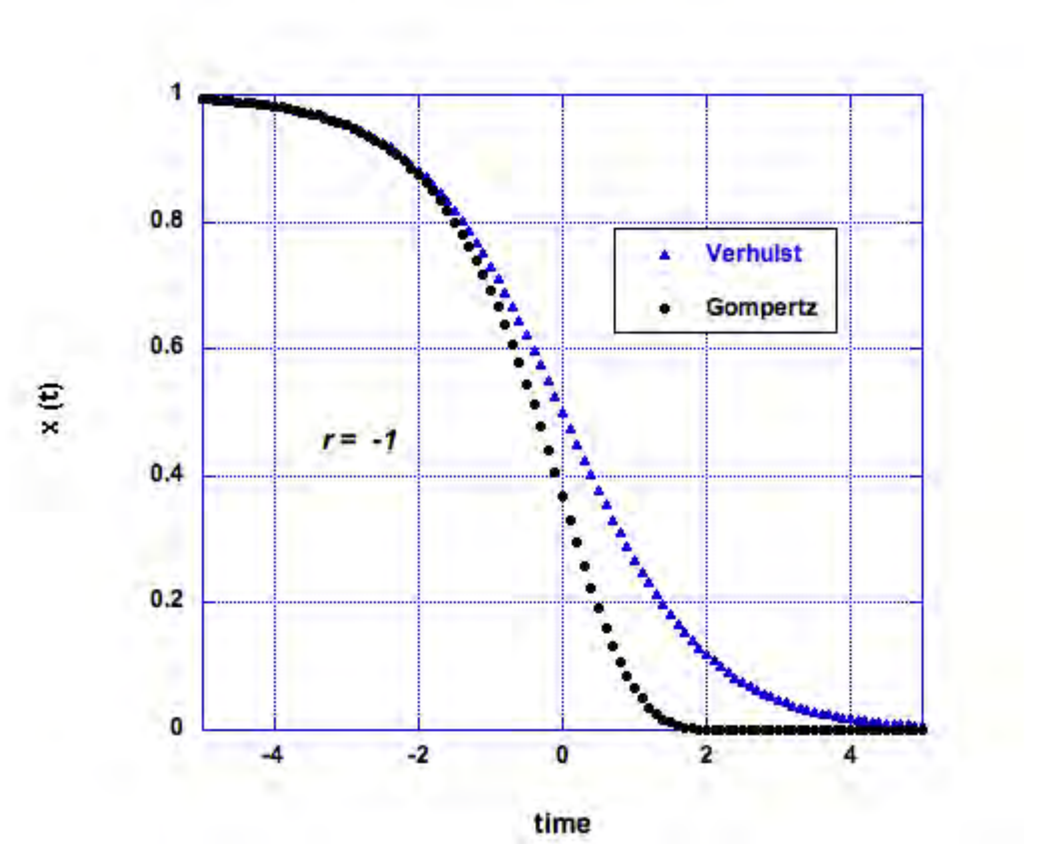
Death law

- Gompertz

$$x(t) = e^{-e^{-rt}}$$

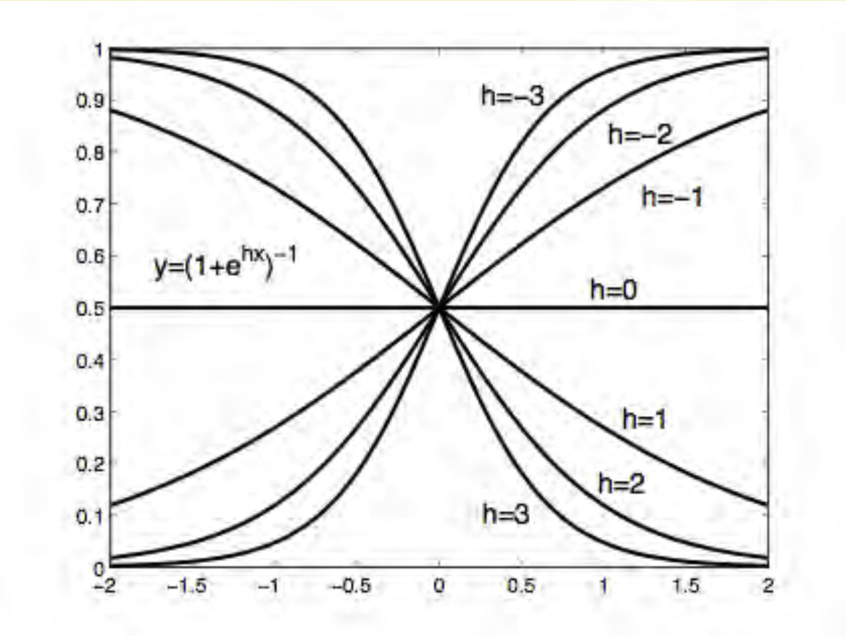
$$\frac{dx}{dt} = r\ x\ log\left[\frac{k}{x}\right]$$

$$k \sim\ M$$



Logistic map/function

$$\frac{dN(t)}{dt} = k\ N(t)\left[1 - \frac{N(t)}{M}\right]$$

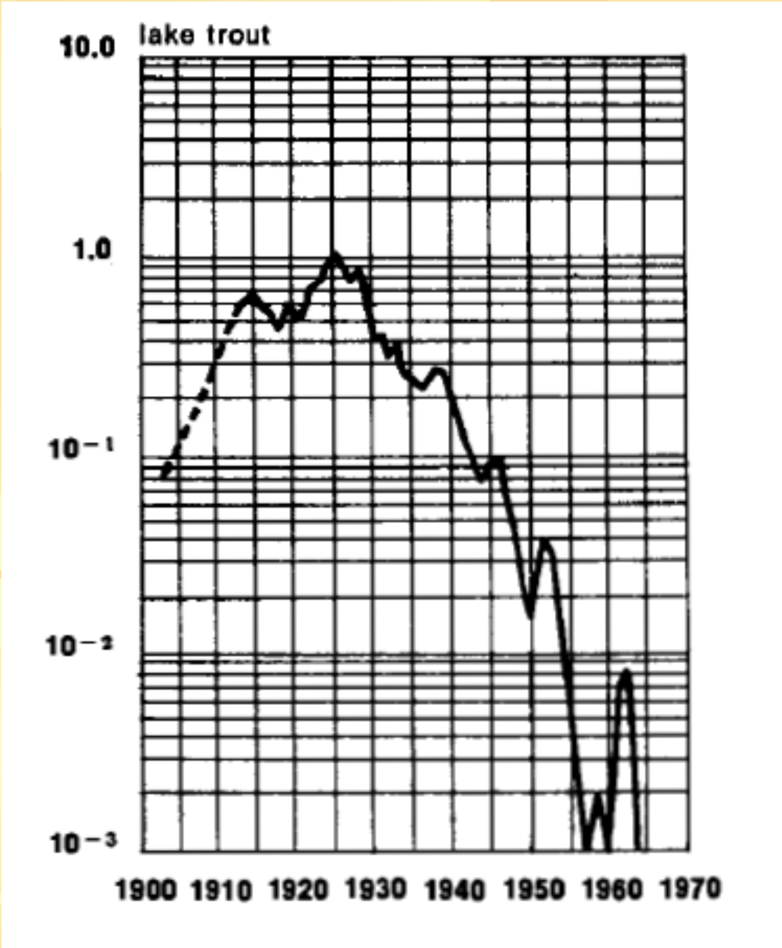


- Sigmoid curve
- Constant growth rate *r*
- Constant carrying capacity, *M* = 1

$$\frac{N(t)}{M} \equiv x\ .$$

$$x = \frac{e^{rt}}{1+e^{rt}} = \frac{1}{1+e^{-rt}}$$

Exogenous causes

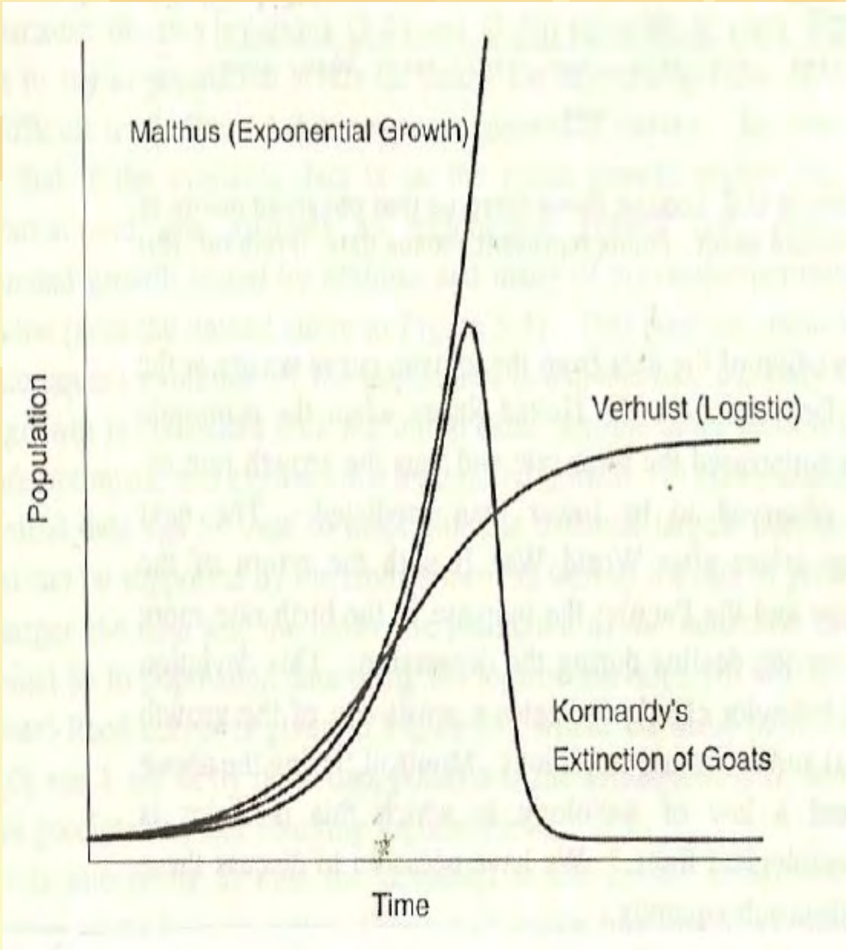


Lake trouts
Beeton (1970);
Meadows et al. (1972)

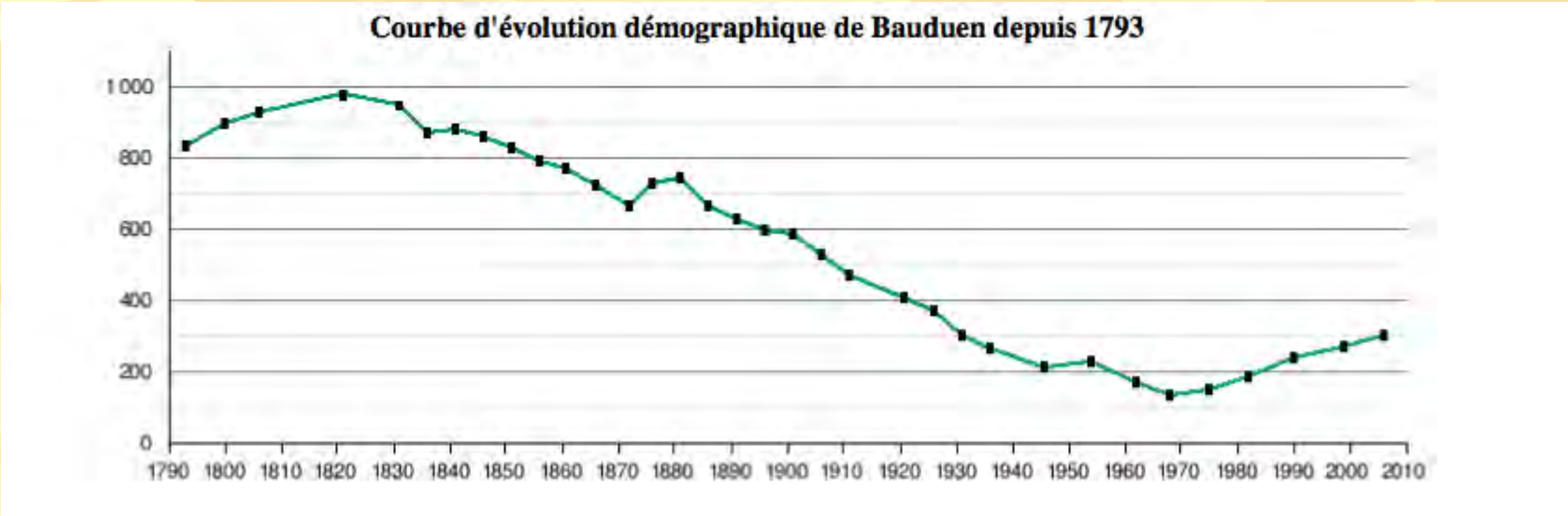
''Self-control''

''deer or goats,
when natural enemies
are absent,
often overgraze their range
and cause erosion
or destruction
of the vegetation,''
and consequently .... die

Kormandy (1969)

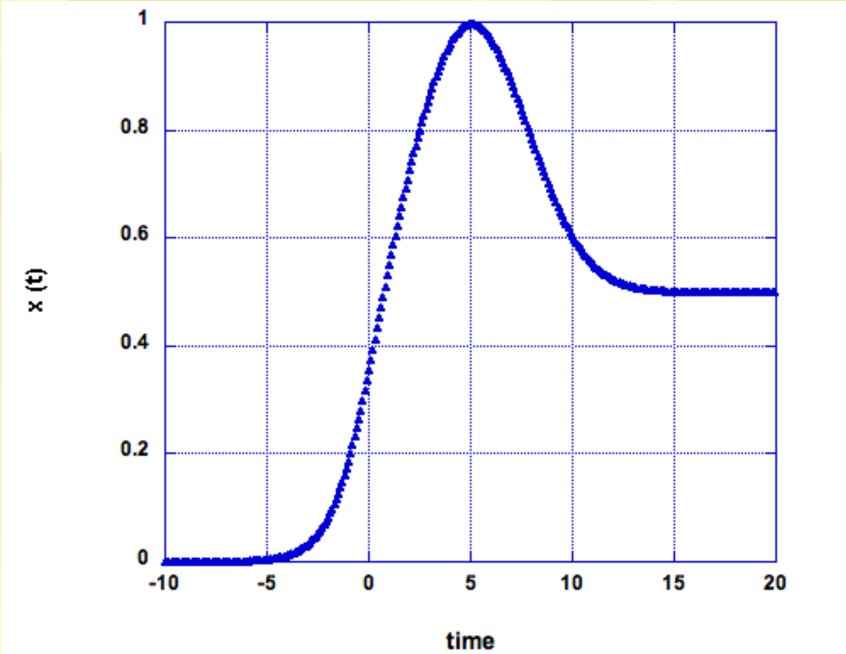


Bauduen, Var



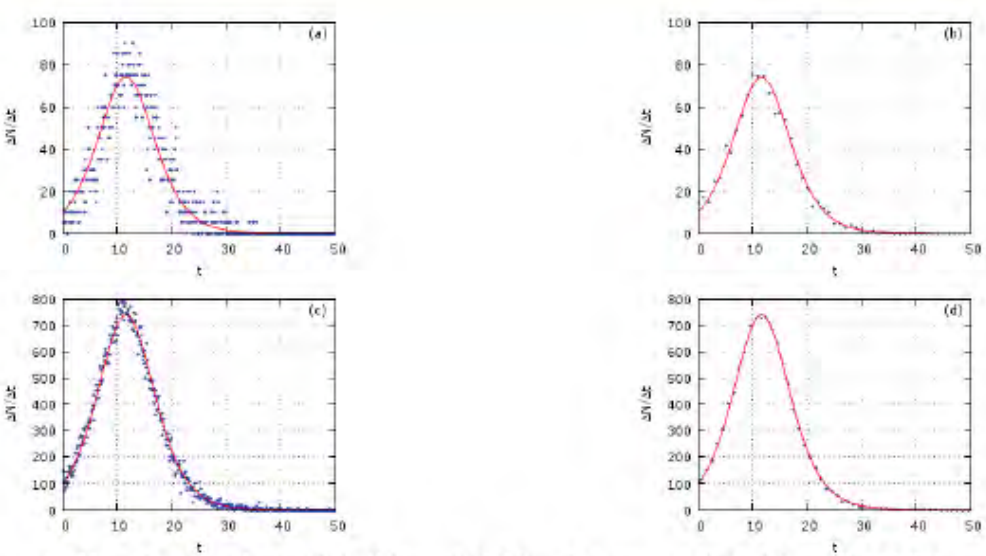
http://fr.wikipedia.org/wiki/Bauduen

What I want !



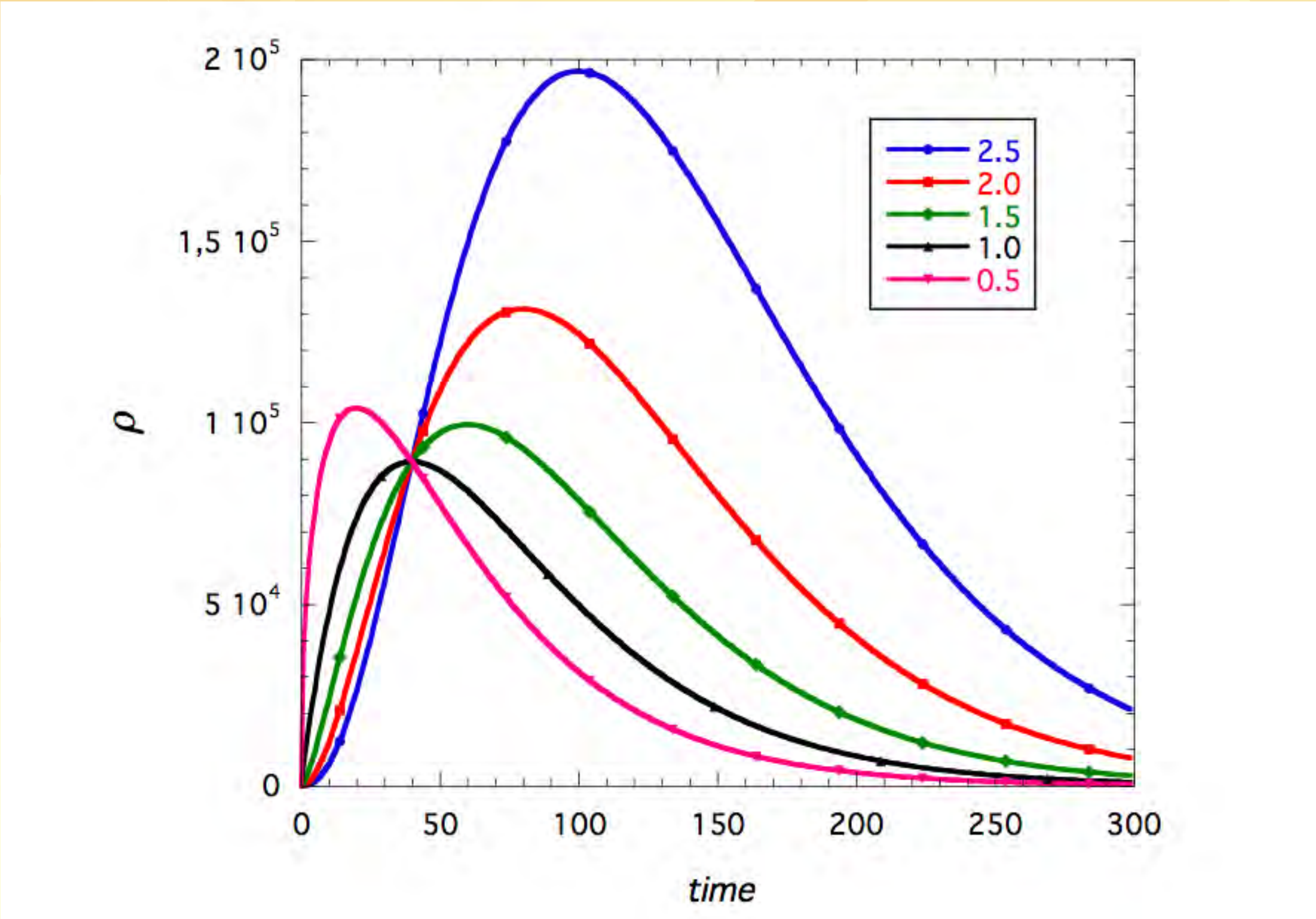
Bass model

Comparison of macroscopic and microscopic Bass diffusion descriptions



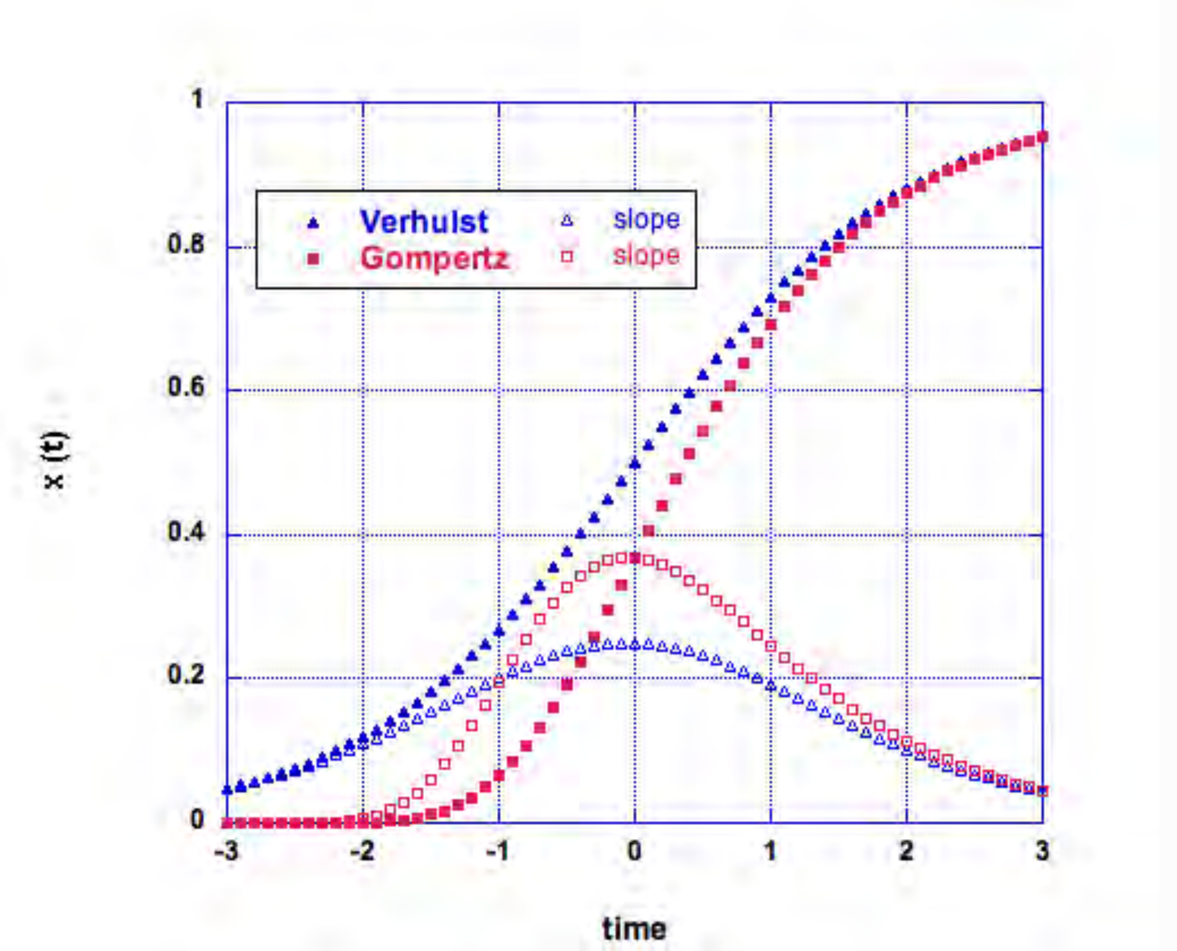
(a) $N = 1000$, $\Delta t = 0.1$; (b) $N = 1000$, $\Delta t = 1$; (c) $N = 10000$, $\Delta t = 0.1$; (d) $N = 10000$, $\Delta t = 1$.

With a maximum ?



Vitanov et al.

V & G V'& G'



Verhulst : *dx/dt*

« size eq. »

$$\frac{dx}{dt} = r\ x\ [1-x]$$

$$x = \frac{e^{rt}}{1+e^{rt}} = \frac{1}{1+e^{-rt}}$$

$$1-x = \frac{1}{1+e^{rt}}$$

« time eq. »

$$\frac{dx}{dt} = \frac{r}{1+e^{rt}}\ \frac{e^{rt}}{1+e^{rt}} = \frac{re^{rt}}{(1+e^{rt})^2}$$

$$= \frac{re^{-rt}}{(1+e^{-rt})^2}\ . = \frac{1}{1+2\ cosh(rt)}\ .$$

time-Verhulst : d^2x/dt^2

$$\frac{d^2x}{dt^2} = \frac{r^2\ e^{rt}\ (1-e^{rt})}{(1+e^{rt})^3}$$

$$\frac{d^2x}{dt^2} = \left[r^2 \frac{1-e^{rt}}{1+e^{rt}}\right]\ x\ (1-x)$$

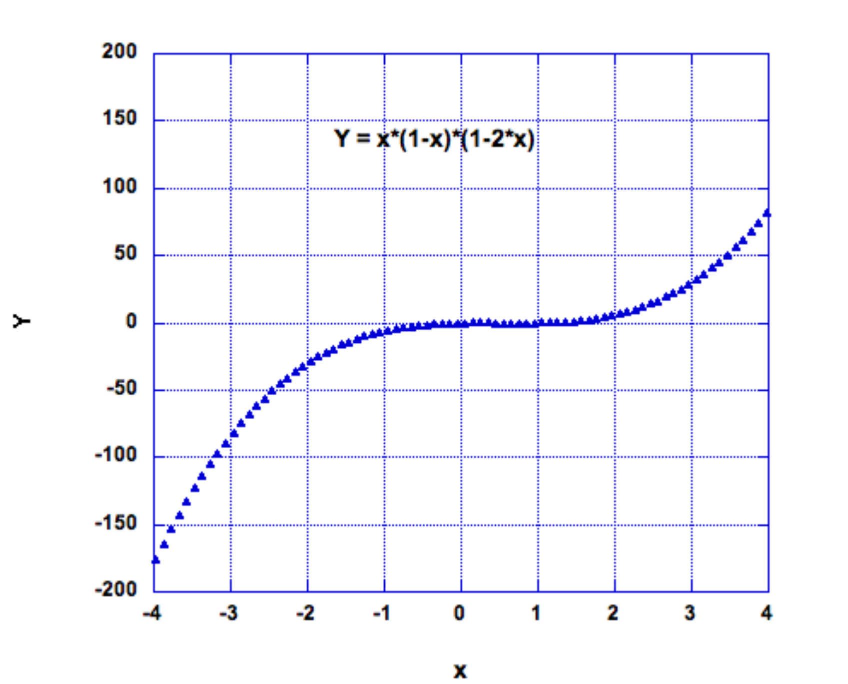
$$R = \left[r^2 \frac{1-e^{rt}}{1+e^{rt}}\right]$$

$$R = -r^2\ tanh \frac{rt}{2}$$

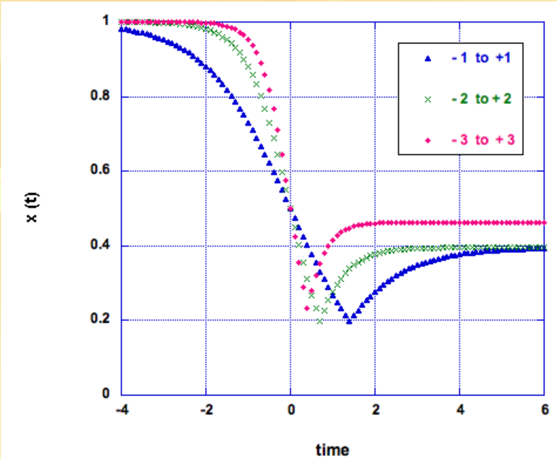
size-Verhulst : d^2x/dt^2

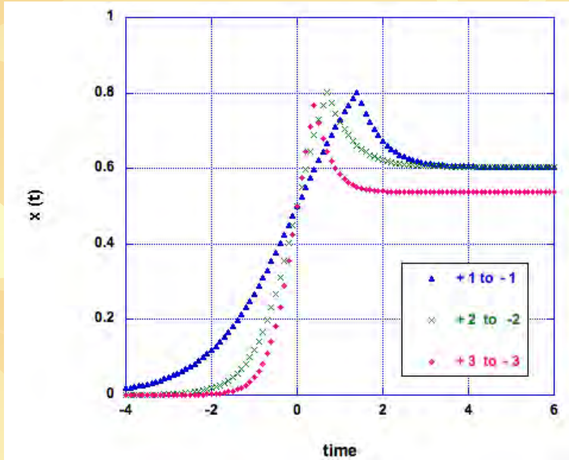
$$\frac{d^2x}{dt^2} = r^2\, x\,(1-x)\,(1-2x),$$

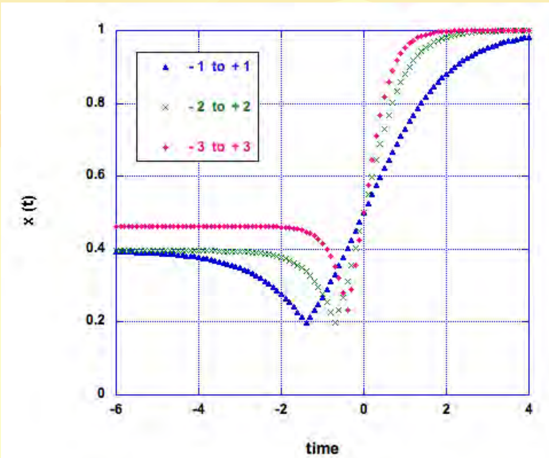
$$\frac{d^2x}{dt^2} = r\,[1-2x]\,\frac{dx}{dt}$$

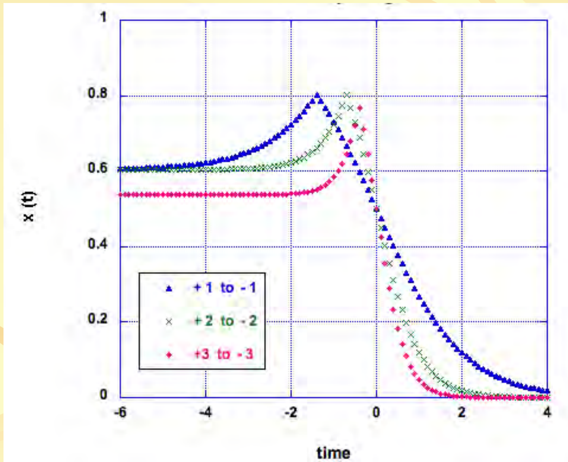


Sharp turn-overs

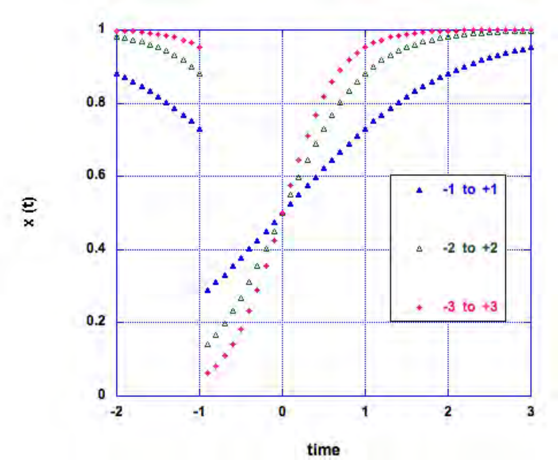


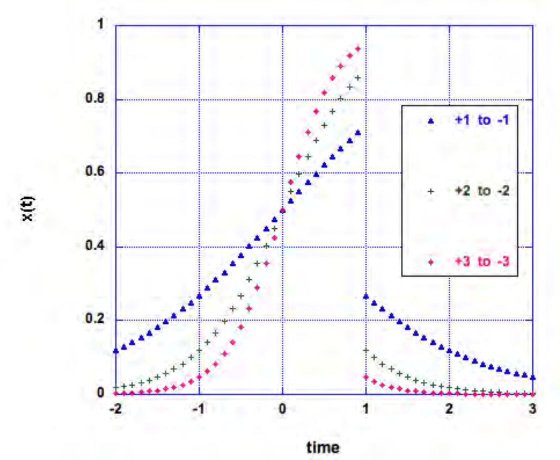


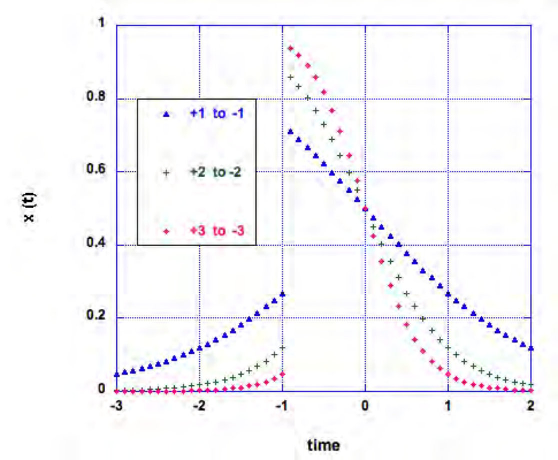


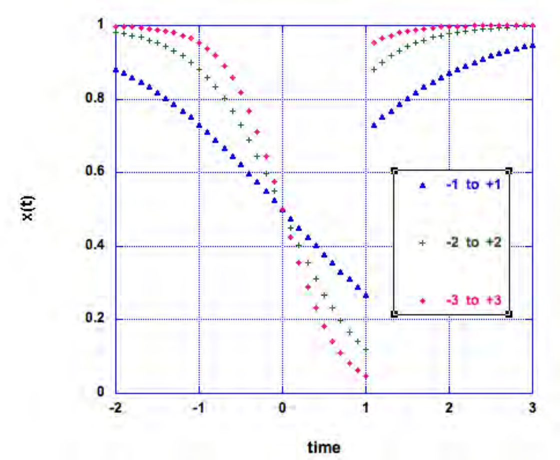


drop or jump transitions

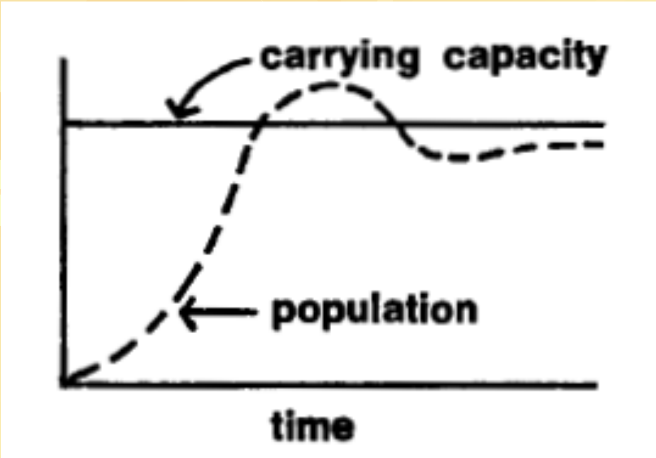






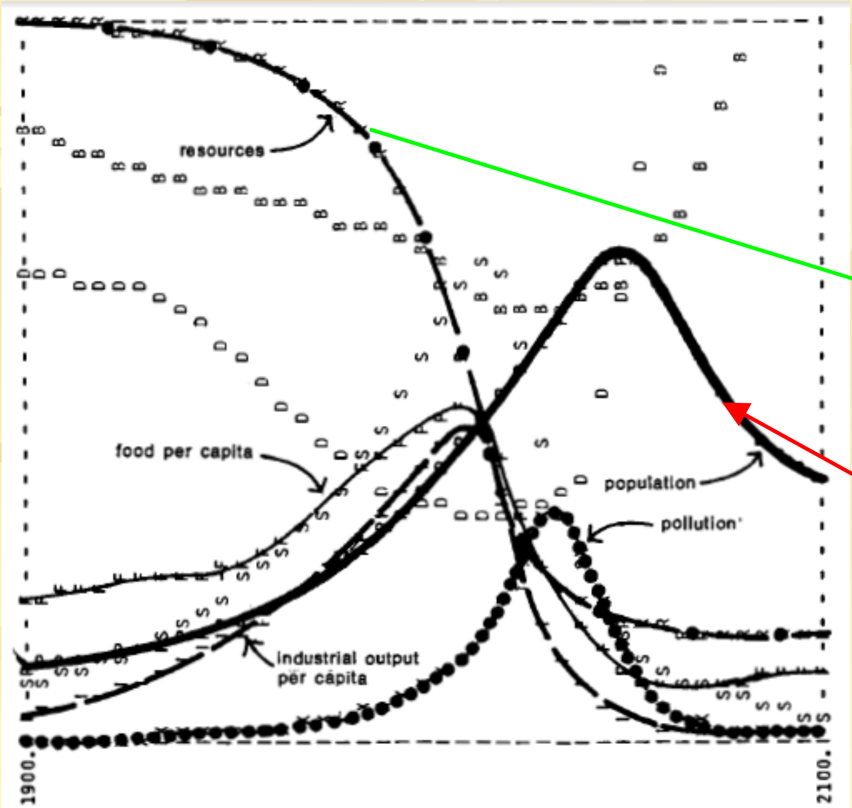


Carrying capacity



- A population growing in a limited environment can approach the ultimate carrying capacity of that environment in several possible ways.
- It can adjust smoothly to an equilibrium.
- It can overshoot the limit and then die back again in either a smooth
- or an oscillatory way.
- It can overshoot carrying capacity and in the process decrease the ultimate carrying capacity by consuming some necessary nonrenewable resource:
- *e.g., deer or goats, when natural enemies are absent, often overgraze their range and cause erosion or destruction of the vegetation*

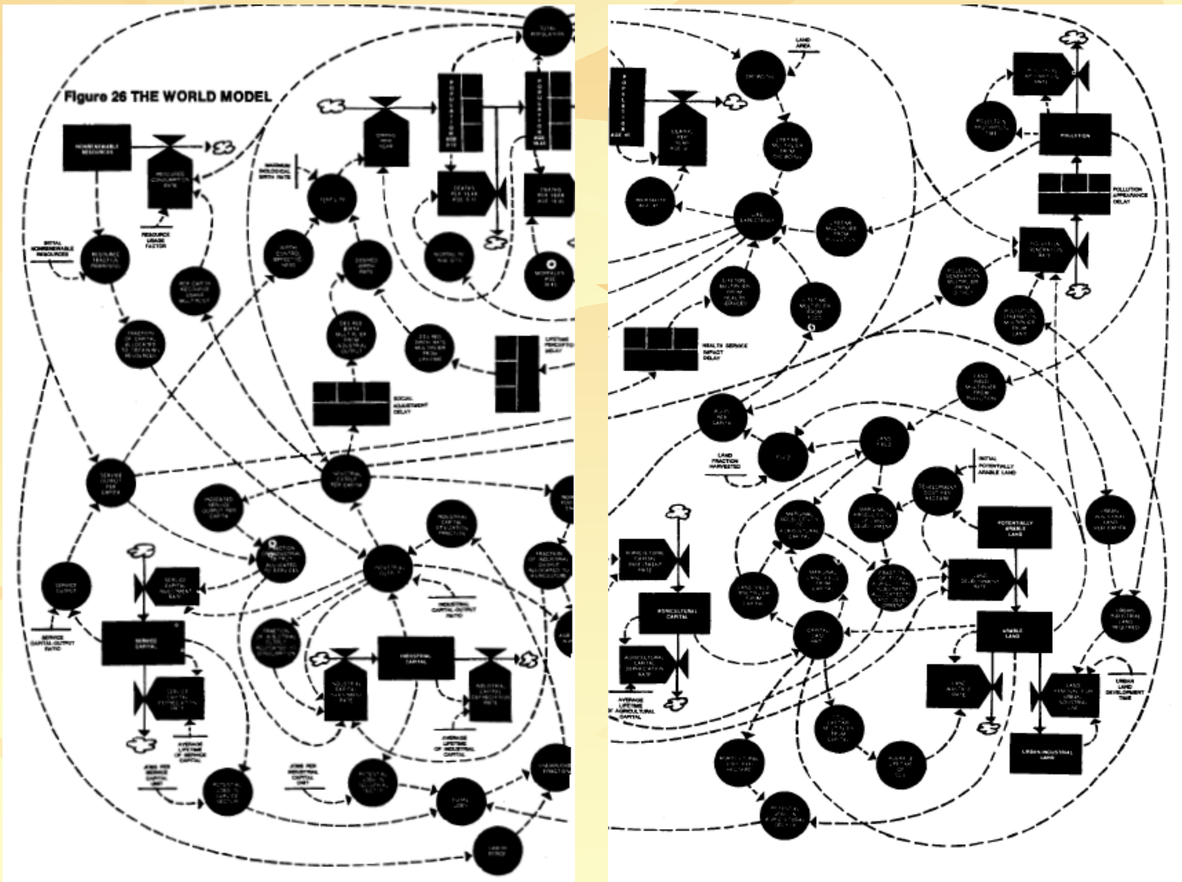
World model



Meadows et al. (1972)

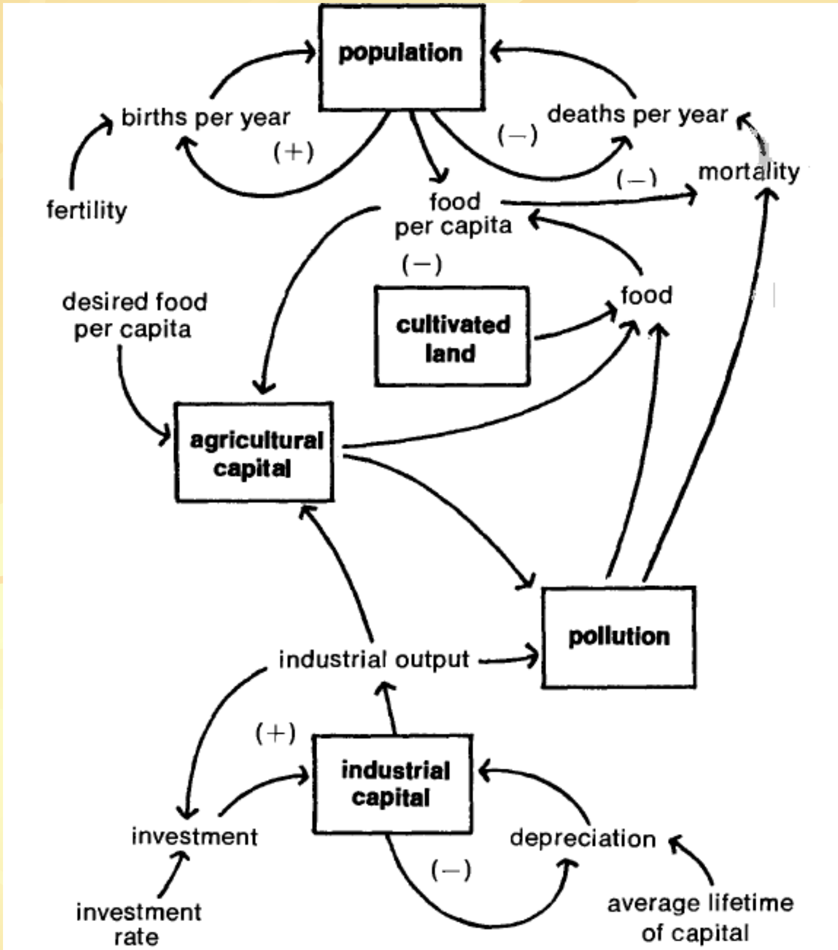
- The "standard" world model run assumes no major change in the physical, economic, or social relationships that have historically governed the development of the world system.
- All variables plotted here follow historical values from 1900 to 1970.
- Food, industrial output, and population grow exponentially until the rapidly diminishing resource base forces a slowdown in industrial growth.
- *Because of natural delays in the system, both population and pollution continue to increase for some time after the peak of industrialization.*
- Population growth is finally halted by a rise in the death rate due to decreased food and medical services

World model



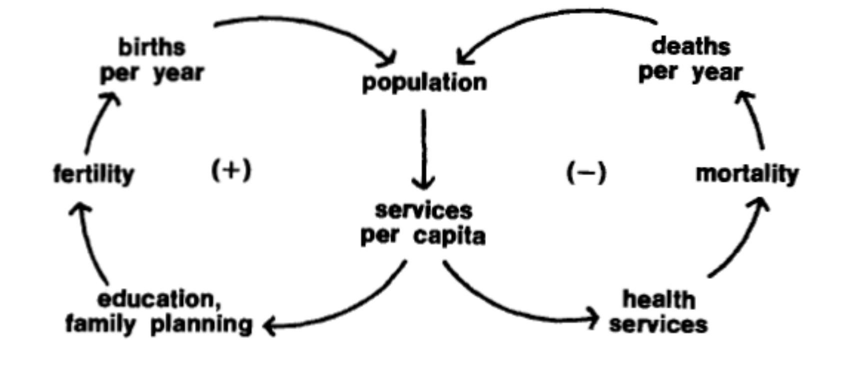
Meadows et al. (1972)

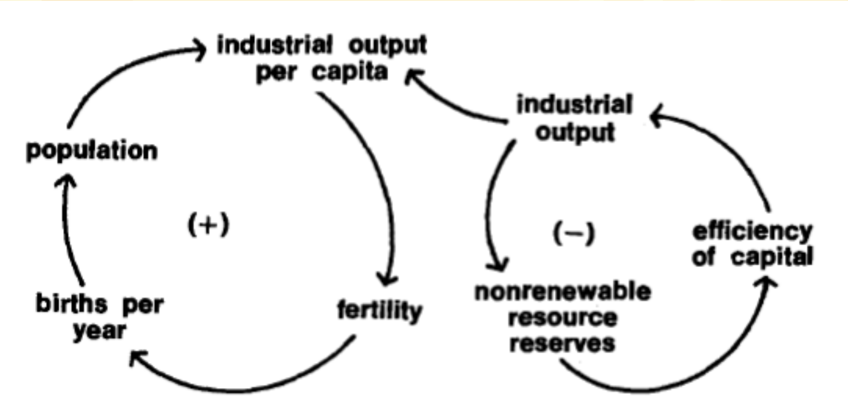
Club of Rome



Meadows et al. (1972)

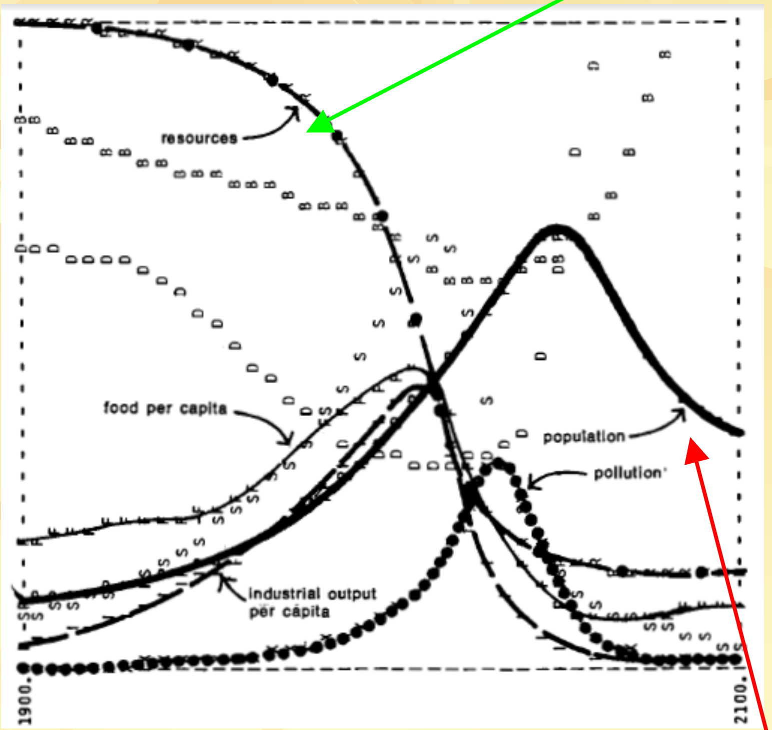
2-loops WM

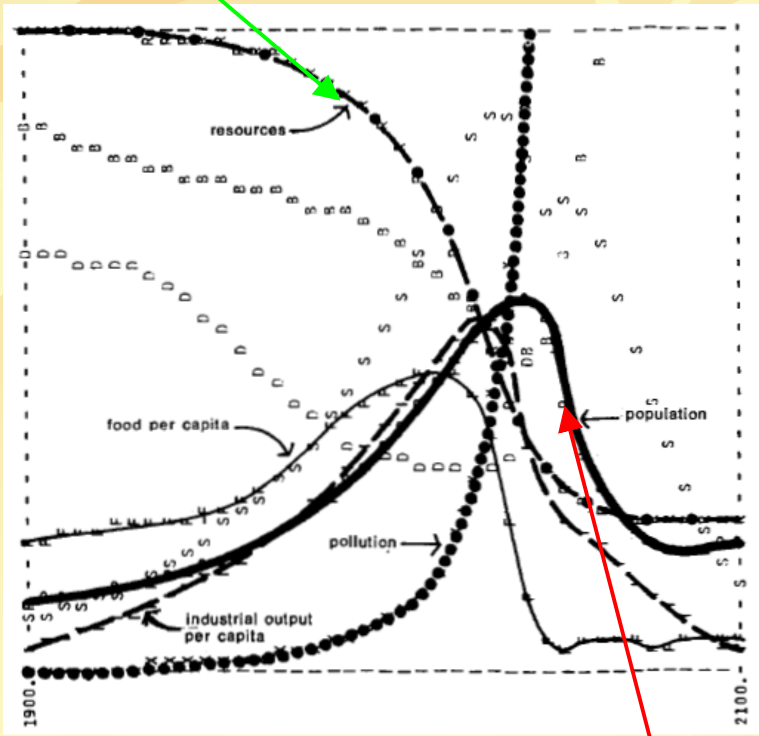




Meadows et al. (1972)

Doubling resources

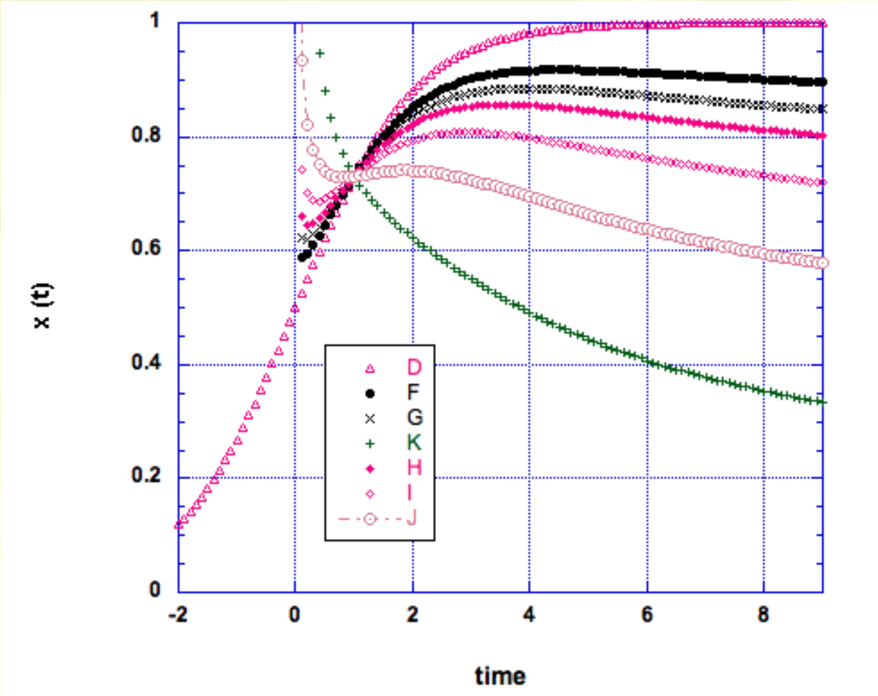


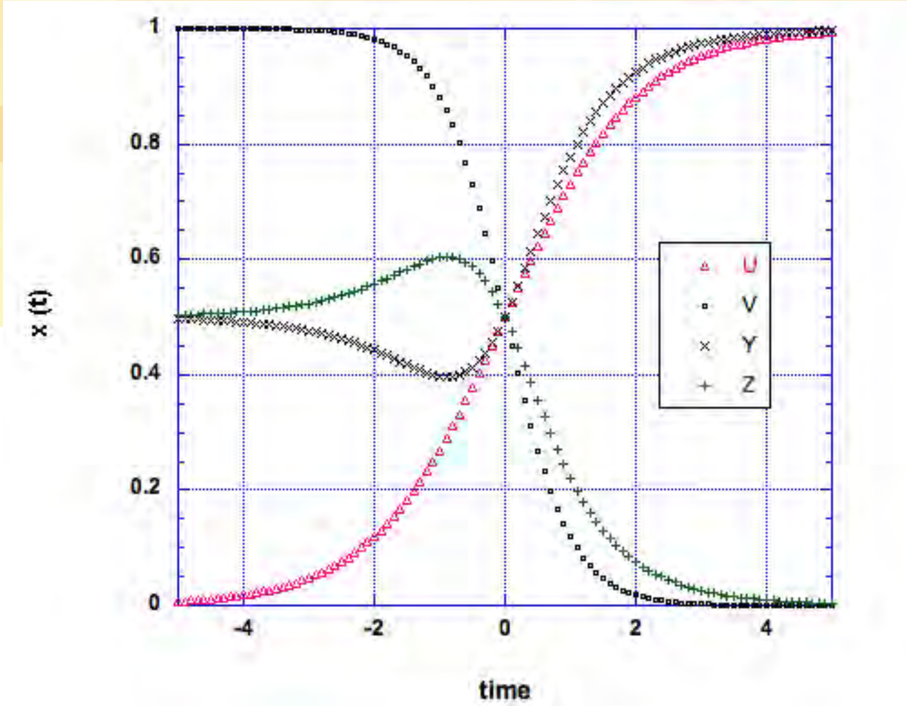


Meadows et al. (1972)

t-dependent Carrying Capacity

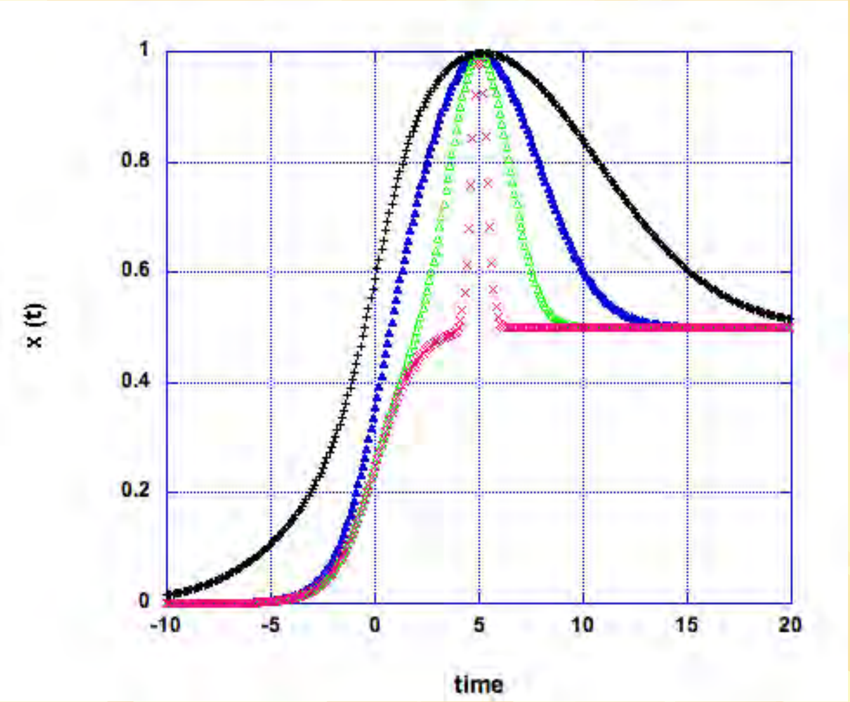
- *power* law





- *tanh* law

What I get



In summary, V to V+

$$\frac{dx}{x(1-x)} = r\,t$$

$$\frac{dy}{y\ [k-y]\ f(t)} = \frac{b}{k}\,dt$$

$$\frac{dy}{a\,y + b\,y^2 + c\,y^3 + \ \ldots.} \simeq \frac{r}{M}\,dt$$

$$\frac{dy}{a\,y + b\,y^2 + c\,y^3 + \ \ldots.} \simeq \frac{r(t)}{M(t)}\,dt$$

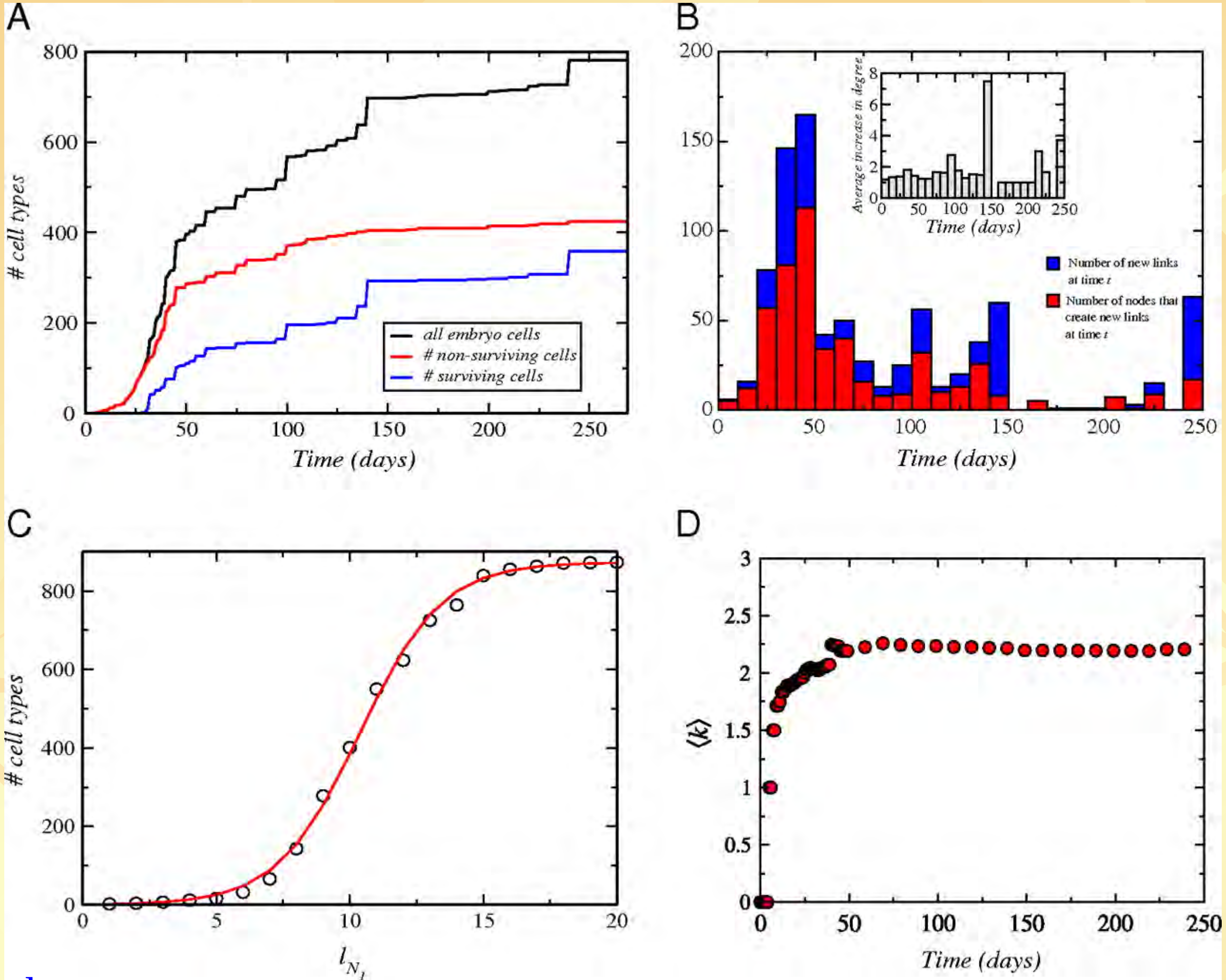
$$\frac{dy}{a(y-y_a)(y-y_b)(y-y_c)} = \frac{r}{k}\,dt$$

Conclusion

Toward one single simple dynamics equation

- Extended logistic equation
  - Growth rate time dependence
  - Carrying capacity dependence
- Exogenous and endogenous causes
  - External field interactions
  - Intra-community interactions
- Gompertz or Verhulst : asymmetric behavior
  - Initial/ final levels or conditions
- Pearl or Bass extension
- Time delay; anticipation
- More complicated but still reductive :
  - Lotka-Volterra
  - (Vitanov-Ausloos)

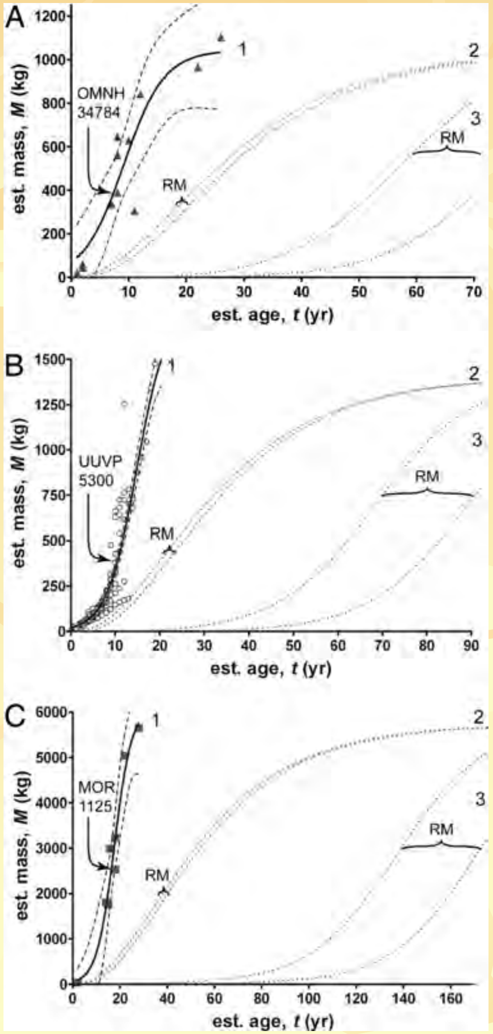
Cell number



NHCD number

Galvão et al. (2010)

Mass growth



(A) Tenontosaurus,
(B) Allosaurus,
(C) Tyrannosaurus
Lee & Werning (2008)