Minimal agent based models as a microscopic reasoning of nonlinear stochastic models

Vygintas Gontis, Aleksejus Kononovicius and Bronislovas Kaulakys

Institute of Theoretical Physics and Astronomy, Vilnius University, Lithuania,
vygintas@gontis.eu, www.gontis.eu

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2. The nonlinear stochastic differential equations as a background of financial fluctuations
3. Kirman's herding model as a statistical background of microscopic description
4. Introduction of macroscopic feedback on microscopic behavior
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6. Kirman's herding as a background of financial fluctuations
7. Herding model with three groups of agents
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The class of non-linear SDE with power law statistics

$$\mathrm{d}x = (\eta - \tfrac{\lambda}{2})x^{2\eta-1} + x^{\eta}\mathrm{d}W_s$$

$$P(x) \sim x^{-\lambda}, \quad S(f) \sim \frac{1}{f^{\beta}}, \quad \beta = 1 - \frac{3-\lambda}{2\eta - 2}$$

$$P(x) \sim x^{-\lambda} \exp\left\{ - \left(\frac{x_{min}}{x}\right)^{m} - \left(\frac{x}{x_{max}}\right)^{m} \right\}$$

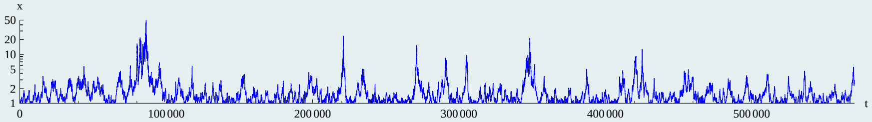
$$\mathrm{d}x = \left(\eta - \tfrac{\lambda}{2} + \tfrac{m}{2}\left\{ \left(\frac{x_{min}}{x}\right)^{m} - \left(\frac{x}{x_{max}}\right)^{m} \right\}\right) x^{2\eta-1} + x^{\eta}\mathrm{d}W_s$$

Publications

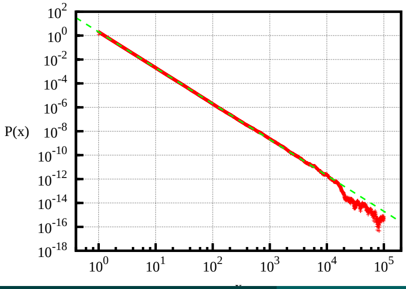
- Kaulakys, B.; Gontis, V. & Alaburda, M. (2005), *Physical Review E*, 71, 051105.
- Kaulakys, B.; Ruseckas, J.; Gontis, V. & Alaburda, M. (2006), *Physica A*, 365, p. 217-221.
- Ruseckas, J. & Kaulakys B. (2010), *Physical Review E*, 81, 031105.

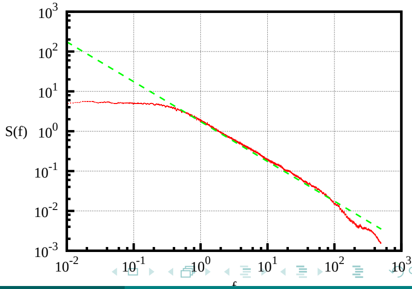
Power-law statistics arising from the nonlinear stochastic differential equations

A simple case of nonlinear SDE



$$\mathrm{d}x = x^{3/2}\mathrm{d}W$$



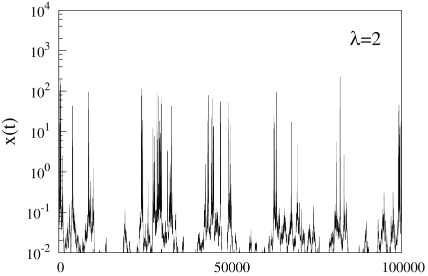


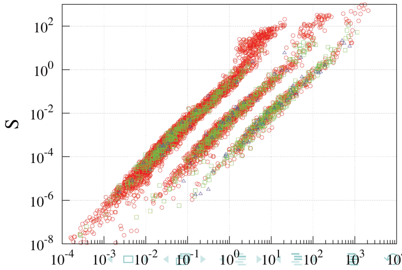
Power-law statistics arising from the nonlinear stochastic differential equations (continued)

More power-law statistics

$$\mathrm{d}x = (\eta - \frac{\lambda}{2})x^{2\eta-1}\mathrm{d}t + x^{\eta}\mathrm{d}W\,, \qquad P(x) \sim x^{-\lambda}\,, \qquad S(f) \sim 1/f^{\beta}$$

$$\beta = 1 - \frac{\lambda-3}{2\eta-2},\ S \sim T^2,\ P(S) \sim S^{-1.3},\ P(T) \sim T^{-1.5},\ P(\theta) \sim \theta^{-1.5}$$



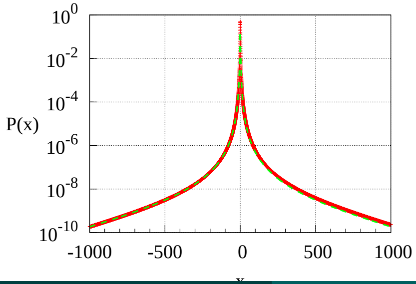


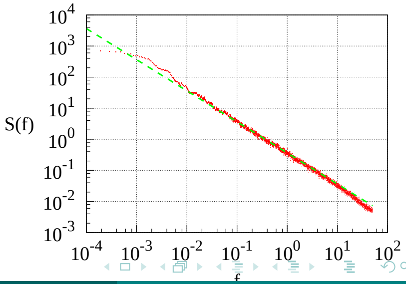
The stochastic model with a q-Gaussian PDF and power spectrum $S(f) \sim 1/f^{\beta}$

SDE with q-Gaussian PDF and power spectrum $S(f) \sim 1/f^{\beta}$

$$\mathrm{d}x = \left(\eta - \frac{\lambda}{2}\right)(1+x^2)^{\eta-1}x\mathrm{d}t + (1+x^2)^{\frac{\eta}{2}}\mathrm{d}W$$

$$x \to \frac{x}{x_0},\ P(x) \sim \left(\frac{1}{1+x^2}\right)^{\frac{\lambda}{2}},\ \beta = 1 - \frac{\lambda-3}{2\eta-2},\ \eta = \frac{3}{2},\ \lambda = 3.$$





The stochastic model of return in financial markets reproducing q-Gaussian PDF and power spectrum

The background nonlinear SDE for financial markets

$$\mathrm{d}x = \left(\eta - \frac{\lambda}{2} - (\frac{x}{x_{max}})^2\right) \frac{(1+x^2)^{\eta-1}}{((1+x^2)^{\frac{1}{2}}\epsilon + 1)^2} x \mathrm{d}t_s + \frac{(1+x^2)^{\frac{\eta}{2}}}{(1+x^2)^{\frac{1}{2}}\epsilon + 1} \mathrm{d}W_s$$

We solve SDE introducing the variable step of integration

$$h_k = \kappa^2 \frac{(\epsilon\sqrt{x_k^2+1}+1)^2}{(x_k^2+1)^{\eta-1}},$$

the differential equation transforms to the difference equation

$$x_{k+1} = x_k + \kappa^2 \left(\eta - \frac{\lambda}{2} - \left(\frac{x}{x_max}\right)^2\right) x_k + \kappa\sqrt{x_k^2+1}\varepsilon_k$$

$$t_{k+1} = t_k + \kappa^2 \frac{(\epsilon\sqrt{x_k^2+1}+1)^2}{\dots}$$

One step probabilities

$$p(X \to X+1) = (N-X)\,(\sigma_1 + hX)\,\Delta t = \pi^{+} N^2 \Delta t,$$
$$p(X \to X-1) = X\,(\sigma_2 + h(N-X))\,\Delta t = \pi^{-} N^2 \Delta t\ ,$$

can be rewritten for continuous $x = X/N$ as

$$\pi^{+}(x) = (1-x)\left(\frac{\sigma_1}{N} + hx\right)\ ,$$
$$\pi^{-}(x) = x\left(\frac{\sigma_2}{N} + h(1-x)\right)\ ,$$

where X is a number of agents exploiting chosen trading strategy, N is a total number of agents in the system. Here the large number of agents N is assumed to ensure the continuity of variable x.

Master equation for the probability density function of continuous variable x

$$\partial_t \omega(x,t) = N^2 \left\{ (\mathbf{E}-1)[\pi^-(x)\omega(x,t)] + (\mathbf{E}^{-\mathbf{1}}-1)[\pi^+(x)\omega(x,t)] \right\}.$$

With the Taylor expansion of operators $\mathbf{E}$ and $\mathbf{E}^{-\mathbf{1}}$ (up to the second term) we arrive at the approximation of the Master equation

$$\partial_t \omega(x,t) = -N\partial_x[\{\pi^+(x)-\pi^-(x)\}\omega(x,t)] + \frac{1}{2}\partial_x^2[\{\pi^+(x)+\pi^-(x)\}\omega(x,t)].$$

Introducing custom functions
$A(x) = N\{\pi^+(x) - \pi^-(x)\} = \sigma_1(1-x) - \sigma_2 x$,
$D(x) = \pi^+(x) + \pi^-(x) = 2hx(1-x) + \frac{\sigma_1}{N}(1-x) + \frac{\sigma_2}{N}x$
we get Fokker-Planck equation

$$\partial_t \omega(x,t) = -\partial_x[A(x)\omega(x,t)] + \frac{1}{2}\partial_x^2[D(x)\omega(x,t)].$$

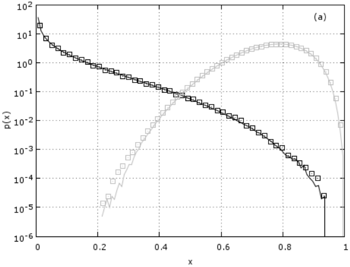
Introduction of macroscopic feedback on microscopic behavior - variable time scale

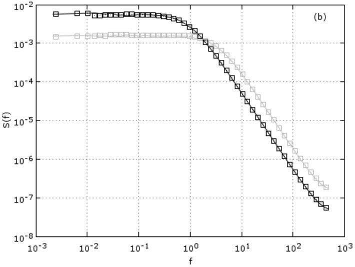
We introduce interevent time $\tau(x)$ into the transition probabilities:

$$\pi^{+}(x) = (1-x)\left[\frac{\sigma_1}{N} + \frac{hx}{\tau(x)}\right], \quad \pi^{-}(x) = x\frac{\frac{\sigma_2}{N} + h(1-x)}{\tau(x)}.$$

The same derivation produces the SDE for x

$$\mathrm{d}x = \left[\sigma_1(1-x) - \frac{\sigma_2 x}{\tau(x)}\right]\mathrm{d}t + \sqrt{\frac{2hx(1-x)}{\tau(x)}}\mathrm{d}W.$$





Bass product diffusion as a one direction Kirman process

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = [N - X(t)][p + \frac{q}{N}X(t)], \quad X(0) = 0,$$

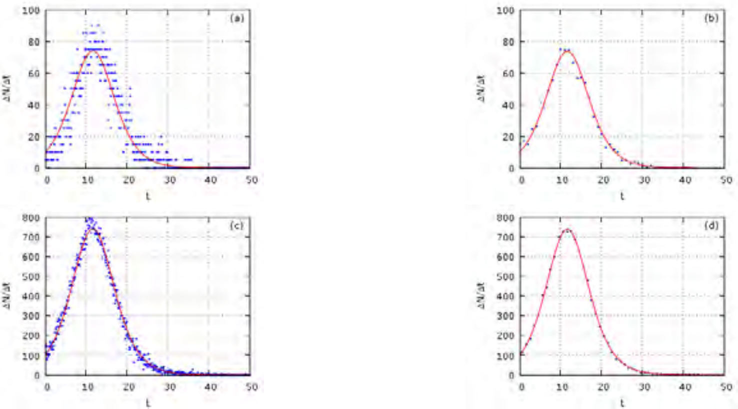
where $X(t)$ - the number of product users at time t; N is a number of potential users, p is the coefficient of innovation, q is the coefficient of imitation. It is a case when a new user adopts the product with probability

$$p(X \to X+1) = (N - X)\left(p + \tfrac{q}{N}X\right), \quad p(X \to X-1) = 0.$$

The functions defining the macroscopic system description in the limit $N \to \infty$ are as follows

$$A(x) = N\pi^{+}(x) = (1-x)(p+qx), \quad D(x) = \pi^{+}(x) = \frac{(1-x)}{N}(p+qx).$$

Comparison of macroscopic and microscopic Bass diffusion descriptions



(a) $N = 1000$, $\Delta t = 0.1$; (b) $N = 1000$, $\Delta t = 1$; (c) $N = 10000$, $\Delta t = 0.1$; (d) $N = 10000$, $\Delta t = 1$.

Definition of price and returns

We follow S. Alfarano, T. Lux, F. Wagner., Computational Economics 26 1, 2005, p. 19-49, introducing main financial definitions. Walrassian scenario based on excess demand D formed by fundamentalists N_f and chartists N_c.

$$D_f(t) = N_f(t) \ln \frac{P_f(t)}{P(t)}, \quad D_c(t) = -r_0 N_c(t)\xi(t),$$

here $N_c(t)\xi(t)$ is a difference between chartist sellers and chartist buyers, r_0 is scaling term.

$$P(t) = P_f(t) \exp\left[r_0 \frac{N_c(t)}{N_f(t)}\xi(t)\right],$$

$$r_\tau(t) = r_0 \left[\frac{x(t)}{1-x(t)}\xi(t) - \frac{x(t-\tau)}{1-x(t-\tau)}\xi(t-\tau)\right],$$

$$r(t) = r_0 y(t)\left[\xi(t) - \xi(t-\tau)\right] = r_0 \frac{x(t)}{1-x(t)}\zeta(t).$$

Stochastic differential equation for the modulating return - volatility

Continuous variable $y(t) = \frac{x(t)}{1-x(t)}$ stands for the modulating absolute return or volatility in the model. One can by Ito transform of variable arrive at the SDE for y

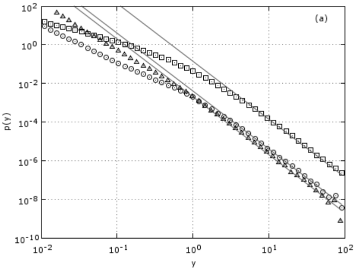
$$\mathrm{d}y = \left(\sigma_1 + y\frac{2h-\sigma_2}{\tau(y)}\right)(1+y)\mathrm{d}t_s + \sqrt{\frac{2hy}{\tau(y)}}(1+y)\mathrm{d}W_s.$$

With the assumption $\tau(y) = y^{-\alpha}$ in the limit $y >> 1$ we can consider only terms of the highest power in the SDE. This produces a simplified nonlinear SDE for y of general class

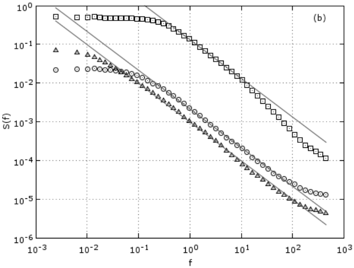
$$\mathrm{d}x = (\eta - \frac{\lambda}{2})x^{2\eta-1} + x^{\eta}\mathrm{d}W_s$$

with $\eta = \frac{3-\alpha}{2}$; $\lambda = \alpha + 1 + \frac{\sigma_2}{h}$ and $\beta = 1 + \frac{\sigma_2/h+\alpha-2}{1+\alpha}$.

Comparison of the numerical SDE solution with analytical PDF and power spectral density, $\alpha = 0$ (boxes), $\alpha = 1$ (circles) and $\alpha = 2$ (triangles), $\sigma_2/h = 2 - \alpha$.



PDF



PSD

Herding model with three groups of agents

N_f - fundamentalists, N_{opt} - chartists optimists, N_{pes} - chartists pessimists

$$\pi_{op}(N_{opt} \to N_{opt} + 1) = N_{pes}(a + bN_{opt})$$

,

$$\pi_{op}(N_{opt} \to N_{opt} - 1) = N_{opt}(a + bN_{pes})$$

.

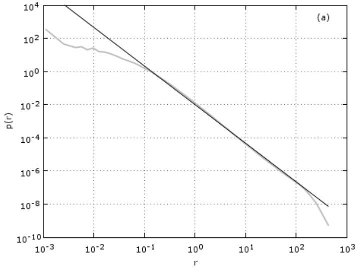
$$\xi(t) = x_{opt}(t) - x_{pes}(t) = [1 - x_{pes}(t)] - x_{pes}(t) = 1 - 2x_{pes}(t),$$

$$\mathrm{d}\xi = -2a\xi\mathrm{d}t + \sqrt{2b(1-\xi^2)}\mathrm{d}W.$$

$$r(t) = r_0[y(t)\xi(t) - y(t-\tau)\xi(t-\tau)] =$$

$$r_0\left[\frac{N_{opt}(t) - N_{pes}(t)}{N_{fund}(t)} - \frac{N_{opt}(t-\tau) - N_{pes}(t-\tau)}{N_{fund}(t-\tau)}\right].$$

Two time scales of financial fluctuations - numerical evidence



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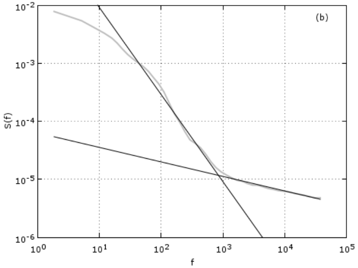

PSD

Summary

- We generalized macroscopic treatment of microscopic herding model proposed by A. Kirman:
- This reveals evident relation between Bass new product diffusion model and one directional Kirman's herding,
- Gives a microscopic background for the stochastic modeling of financial variables by the class of nonlinear stochastic differential equations.
- Further we consider financial market with three groups of heterogenous agents and two time scales of their interaction.

Thank you!