# The inverse cubic distributions from the point process model

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**with**

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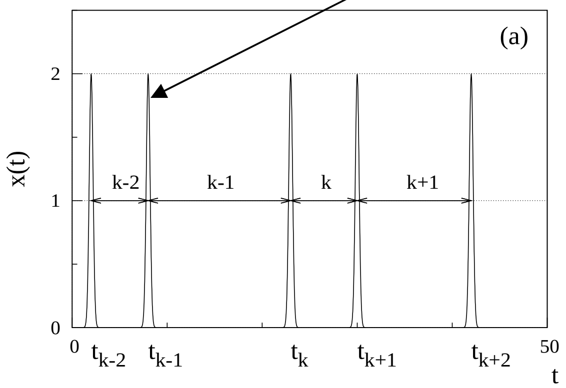
# Focus of the talk

- ✓ **Simple models,**
- **based on the point process model of $1/f^{\beta}$ noise**
- **and the nonlinear stochastic differential equations,**
- ✓ **generating the long-range processes with the inverse cubic cumulative distribution; and**
- **q-exponential and**
- **q-Gaussian distributed signals with**
- **$1/f^{\beta}$ power spectrum,**
- **exhibiting bursts and observable in**

  **long-term memory time series**
- ✓ **is proposed and analyzed**

**Starting from the autoregressive point process model**

$$\tau_{k+1} = \tau_k + \gamma \tau_k^{2\mu-1} + \sigma \tau_k^{\mu} \varepsilon_k .$$



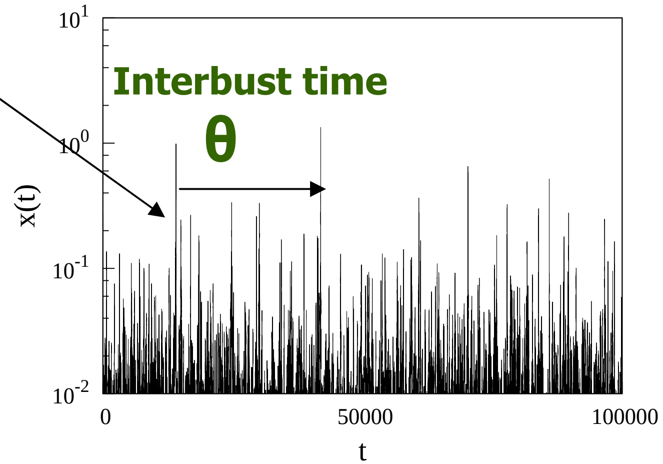

**we derive a class of the nonlinear stochastic differential equations**

$$\frac{dx}{dt_s} = \Gamma x^{2\eta-1} + x^{\eta} \xi(t_s)$$

**which generate bursting, power-law distributed, q-exp and q-Gaussian signals and $1/f^{\beta}$ noise**

$$P(x) \sim \frac{1}{x^{\lambda}}, \quad \lambda = 2(\eta - \Gamma)$$

$$S(f) \sim \frac{1}{f^{\beta}}, \quad \beta = 2 - \frac{2\Gamma + 1}{2\eta - 2}.$$




**The signal**

# THE POINT PROCESS MODEL

**The signal of the model consists of pulses or events**

$$I(t) = \sum_k A_k(t - t_k)$$

**In a low frequency region and for long-range correlations we can restrict analysis to the noise originated from the correlations between the occurrence times $t_k$.**

**Therefore, we can simplify the signal to the point process**

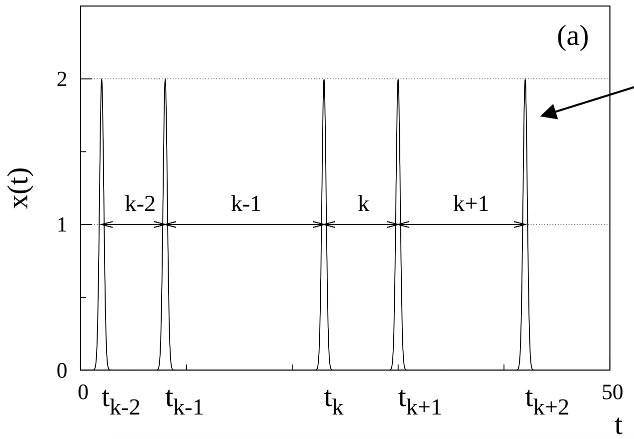
**The point process**

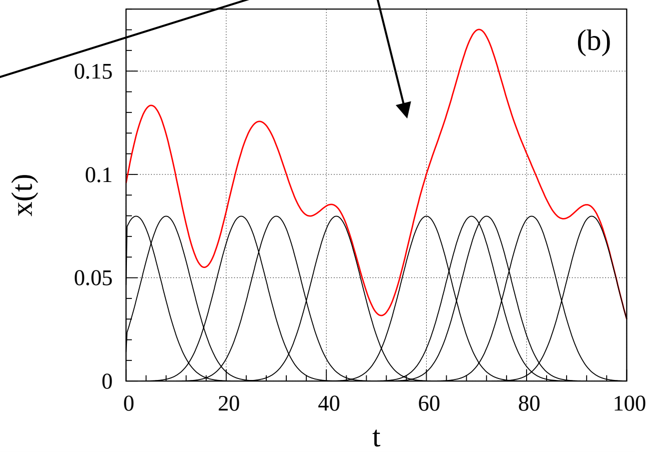
$$I(t) = \bar{a} \sum_k \delta(t - t_k)$$

**is primarily and basically defined by the occurrence times $t_1$, $t_2$, …$t_k$…**

**Or by the interevents times** $\tau_k = t_{k+1} - t_k$

# Power spectral density of the point process




$$S(f) = \lim_{T\to\infty} \left\langle \frac{2}{T} \int_{t_i}^{t_f} \int_{t_i}^{t_f} I(t')\, I\left(t''\right) e^{i\omega\left(t''-t'\right)} \mathrm{d}t'\mathrm{d}t'' \right\rangle$$

$$= \lim_{T\to\infty} \left\langle \frac{2\bar{a}^2}{T} \sum_{k=k_{\min}}^{k_{\max}} \sum_{q=k_{\min}-k}^{k_{\max}-k} e^{i\omega\Delta(k;q)} \right\rangle$$

where $T = t_f - t_i \gg \omega^{-1}$ is the observation time, $\omega = 2\pi f$, and

**may be calculated directly**

$$\Delta(k;q) \equiv t_{k+q} - t_k = \sum_{i=k}^{k+q-1} \tau_i$$

# Stochastic multiplicative point process

**Quite generally the dependence of the mean interpulse time on the occurrence number *k* may be described by the general Langevin equation with the drift coefficient** $d(\tau_k)$

**and a multiplicative noise** $b(\tau_k)\xi(k)$

$$\frac{d\tau_k}{dk} = d(\tau_k) + b(\tau_k)\xi(k).$$

**Replacing the averaging over *k* by the averaging over the distribution of the interpulse times** $\tau_k$, $P_k(\tau_k)$, **we have the power spectrum**

$$S(f) = 4\bar{I}^2\bar{\tau}\int_0^\infty d\tau_k P_k(\tau_k)\,\mathrm{Re}\int_0^\infty dq\exp\left\{i\omega\left[\tau_k q + d(\tau_k)\frac{q^2}{2}\right]\right\}$$

$$= 2\bar{I}^2\frac{\bar{\tau}}{\sqrt{\pi f}}\int_0^\infty P_k(\tau_k)\,\mathrm{Re}\left[e^{-i\left(x-\frac{\pi}{4}\right)}\,\mathrm{erfc}\sqrt{-ix}\right]\frac{\sqrt{x}}{\tau_k}d\tau_k$$

✓**B. K., et all. Phys. Rev. E 71,** 051105 **(2005)**

# Nonlinear stochastic differential equation generating 1/f noise

$$\tau_{k+1} = \tau_k + \sigma\varepsilon_k, \; S(f) \propto 1/f$$

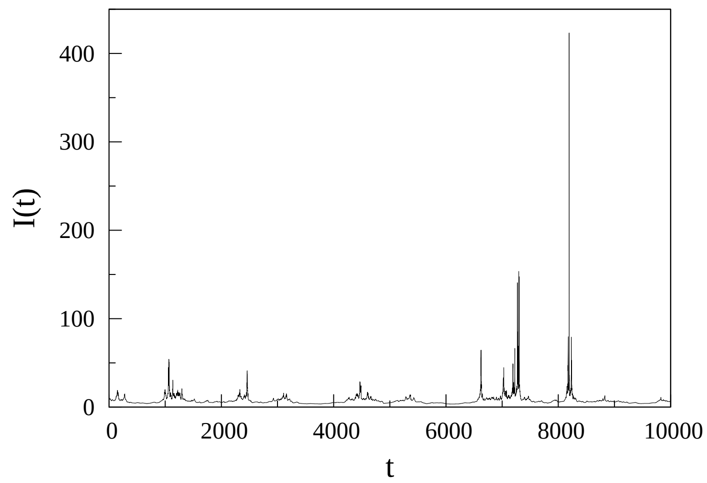
$$\frac{d\tau_k}{dk} = \sigma\xi(k) \quad \langle \xi(k)\xi(k') \rangle = \delta(k-k')$$

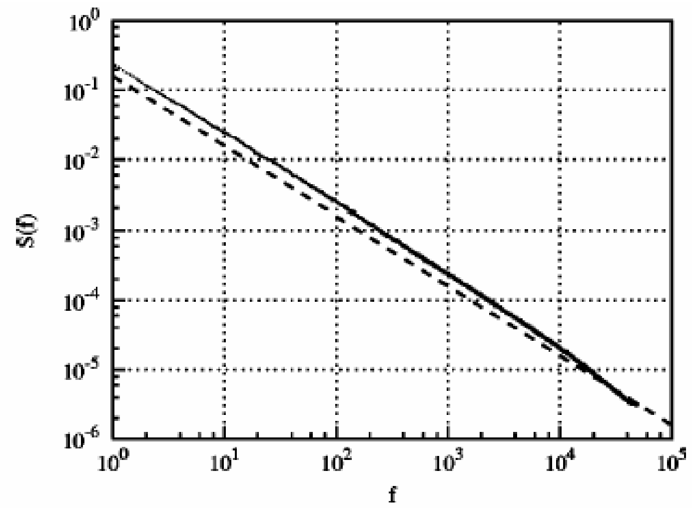
$$\mathrm{d}t = \tau_k \mathrm{d}k, \qquad x = a/\tau_k$$

$$\frac{dx}{dt} = x^4 + x^{5/2}\xi(t), \quad S(f) \propto 1/f$$

$$P(x) \sim \frac{1}{x^3}$$

**1/*f* noise and power-law distribution**




✓**B. K. and J. Ruseckas, Phys. Rev. E 70,** 020101(R) **(2004)**

**Therefore, the simplest iterative equation**

$$\tau_{k+1} = \tau_k + \sigma \varepsilon_k$$

**(with the appropriate boundary conditions)**

**generating the pure 1/f noise,**

**corresponds to the *inverse squared***

$$P_{>}(x) \sim x^{-2}$$

***cumulative* distribution.**

# Inverse cubic law

**One of stylized facts emerging from statistical analysis of *financial markets* is the *inverse cubic law* for the *cumulative* distribution of a number of events of trades and of the logarithmic price change.**


- P. Gopikrishnan, M. Meyer, L. A. N. Amaral, and H. E. Stanley, *Eur. Phys. J. B*, 3, p. 139, 1998.
- S. Solomon and P. Richmond, *Physica A*, 299, p. 188, 2001.
- X. Gabaix, P. Gopikrishnan, V. Plerou, and H. E. Stanley, *Nature*, 423, p.267, 2003.
- B. Podobnik, D. Horvatic, A. M. Petersen, and H. E. Stanley, *PNAS*,
- 106, p. 22079, 2009.
- G.-H. Mu and W.-X. Zhou, *Phys. Rev. E*, 82, 066103, 2010.

**Here, we search for the simplest stochastic differential equation, generating the long-range processes with the *inverse cubic cumulative distribution.***

**A simple model, based on the point process model of $1/f^{\beta}$ noise, is proposed and analyzed.**

✓**B. K. and M. Alaburda, to be published (2011).**

The simplest equations generating *the inverse cubic law* of the cumulative distribution, $P_{>}(x) \sim x^{-3}$, are

$$\tau_{k+1} = \tau_k + \sigma \tau_k^{-1/2} \varepsilon_k \,, \qquad (8)$$

$$d\tau(t) = \frac{\sigma}{\tau(t)} dW \,, \qquad (9)$$

and

$$d\tau(t) = \sigma_x x(t)\, dW \,, \qquad (10)$$

where $x(t) = a/\tau(t)$ and $\sigma_x = \sigma/a$.

Equation (10) reveals the particularly obvious meaning, i.e., *the intensity of fluctuations of the interevent time* $\tau(t)$ *is proportional to the intensity of the process* $x(t) \propto 1/\tau(t)$.

The cumulative distribution $P_>(x)$ of $x$ is

$$P_>(x) = \int_x^\infty P(x)\,dx$$

$$\simeq \operatorname{erf}\left(\frac{x_{\min}}{x}\right) - \frac{2x_{\min}}{\sqrt{\pi}x}\exp\left(-\frac{x_{\min}^2}{x^2}\right) \tag{14}$$

$$= \frac{x_{\min}^3}{x^3}\gamma^*\left(\frac{3}{2}, \frac{x_{\min}^2}{x^2}\right).$$

Here $\gamma^*(a, z)$ is the regularized lower incomplete gamma function. Consequently

$$P_>(x) \simeq \frac{4x_{\min}^3}{3\sqrt{\pi}x^3}, \qquad x \gg x_{\min}, \tag{15}$$

**Inverse cubic**

and we find out *the inverse cubic law*.

Further we can consider a more realistic model assuming that $\tau_k$ is a time-dependent average interevent time of the Poissonian-like process with the time-dependent rate. Within this assumption the actual interevent time $\tau_j$ is given by the conditional probability [17], [22]

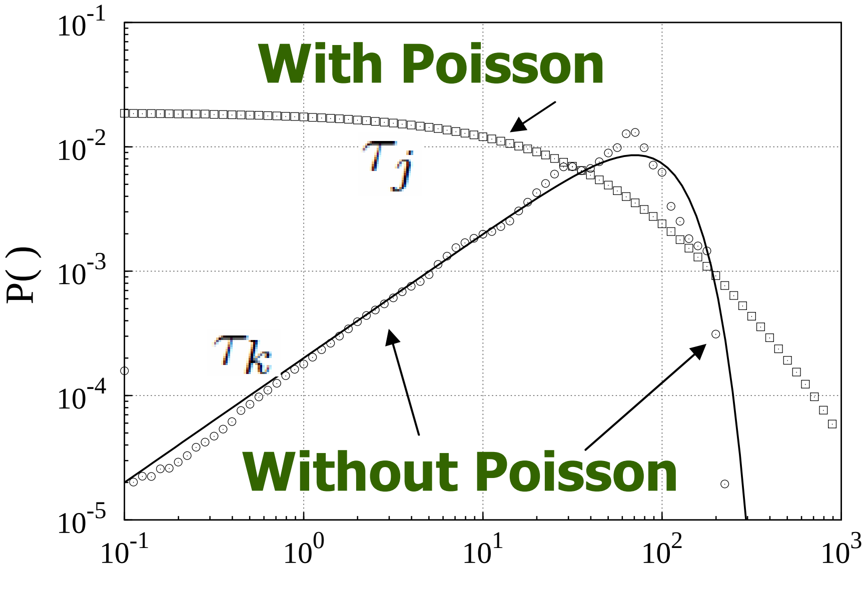
$$\varphi(\tau_j|\tau_k) = \frac{1}{\tau_k} e^{-\tau_j/\tau_k} , \qquad (16)$$

similar to the non-homogeneous Poisson process. In such a case, the distribution of the actual interevent time $\tau_j$ is expressed analogically to the superstatistical schemes [30],

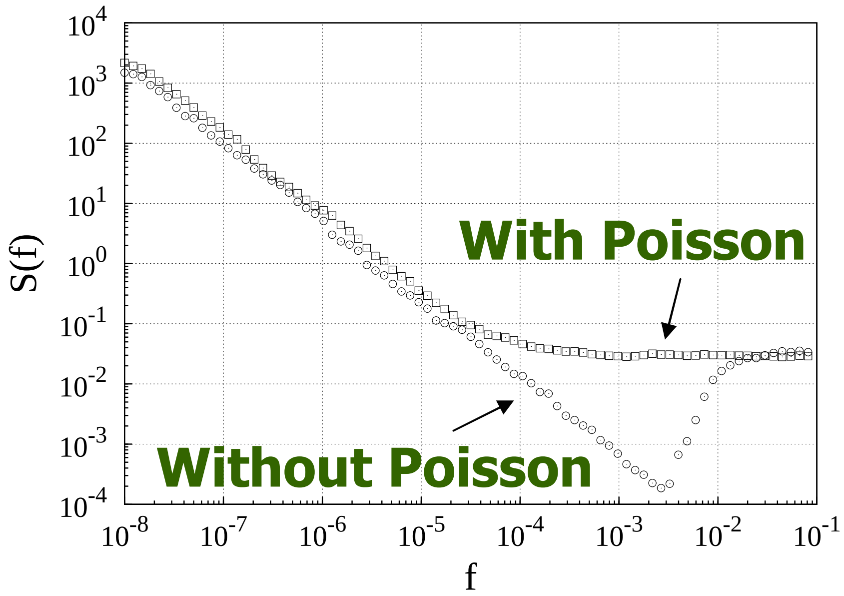
$$P_j(\tau_j) = \int \varphi(\tau_j|\tau_k) P_k(\tau_k) d\tau_k . \qquad (17)$$

The generalized model (16) and (17) represents a more realistic situation, because the concrete event occurs at random time (like in the Poisson case), however, the average interevent time is slowly (Brownian-like) modulated.

This additional stochasticity of the actual interevent time $\tau_j$ by randomization (16) of the concrete occurrence times does not influence on the low frequency power spectra of the signal.

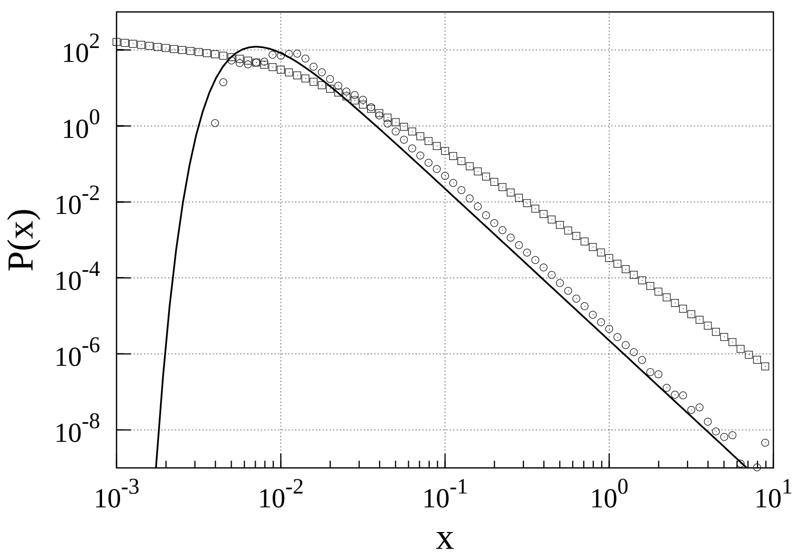


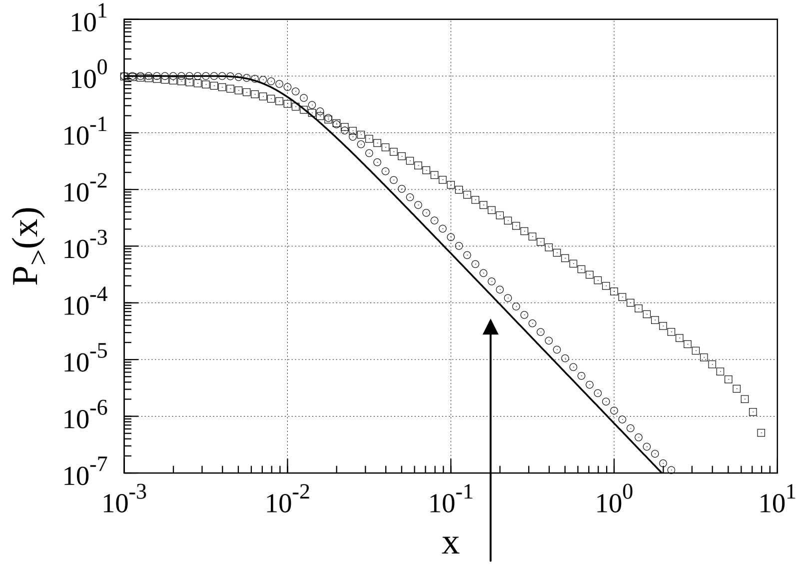

Distribution of interevent times $\tau_k$ and $\tau_j$




Spectral density

# Distribution of the signal depends, however, on the additional Poissonian-like stochasticity



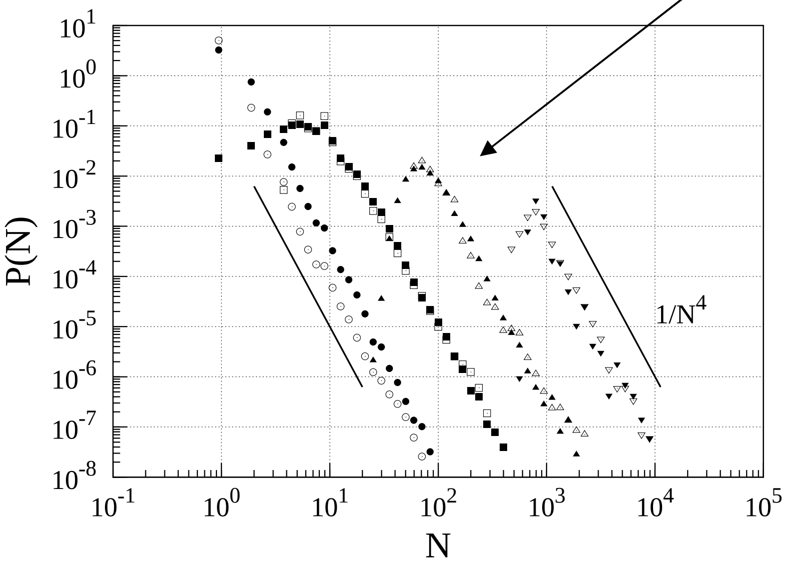

***Inverse cubic law***

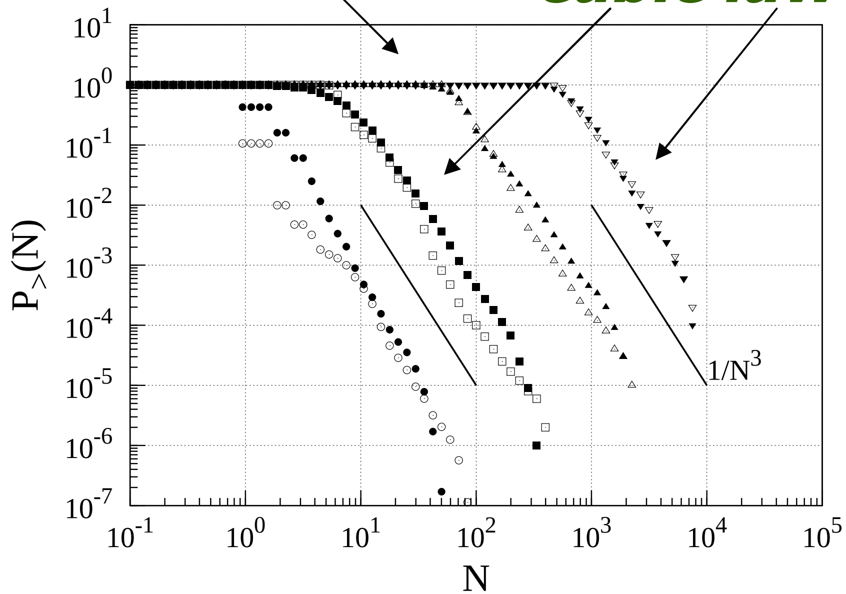
Variable $x(t) = 1/\tau(t)$ represents the formal instantaneous process and does not contain any scale of time. Actually, one measures the number of events $N$ in the definite time window $\tau_w$, e.g., the trading activity, as a number of events in some time interval, or the return at time lag $\tau_w$. These quantities are represented as the integral of the variable $x(t)$ in time interval

$$N(t) = \int_{t}^{t+\tau_w} x(t')\, dt' \quad (18)$$

**Does not depend on the additional Poissonian stochasticity**

**We obtain the *Inverse cubic law***



## GENERALIZATION OF THE MODEL

For modeling the long-range processes with $\beta < 1$ and with the power-law correlation function [20]

$$C(t) \sim \frac{1}{t^{1-\beta}} \tag{19}$$

we should modify Eqs. (8)-(10) assuming the simple additive Brownian motion of small interevent time, keeping the same dependence for large $\tau(t)$. For this purpose, instead of (9) we propose equation

$$d\tau = \sigma \frac{1}{\tau_c + \tau} dW\ , \tag{20}$$

where $\tau_c$ is a crossover parameter, separating the two kinds of the stochastic motion: (i) the simple Brownian motion for $\tau \ll \tau_c$ and (ii) the model of Section II for $\tau \gg \tau_c$.

Eq. (20) with restrictions at $\tau = \tau_{\min}$ and at $\tau = \tau_{\max}$

$$d\tau = \sigma^2 \left( \frac{\tau_{\min}^2}{\tau^2} - \frac{\tau^2}{\tau_{\max}^2} \right) \frac{dt}{\tau \left( \tau_c + \tau \right)^2} + \sigma \frac{dW}{\tau_c + \tau} . \qquad (21)$$

may be solved using a variable step of integration

$$\Delta t_i = \frac{\kappa^2}{\sigma^2} \left( \tau_c + \tau_i \right)^2 \tau_i^2 \, , \; \kappa \ll 1, \qquad (22)$$

$$\tau_{i+1} = \tau_i + \kappa^2 \left( \frac{\tau_{\min}^2}{\tau_i^2} - \frac{\tau_i^2}{\tau_{\max}^2} \right) \tau_i + \kappa \tau_i \varepsilon_i \, . \qquad (23)$$

The steady-state distribution density $P_k\left(\tau_k\right)$ in $k$ -space of interevent time $\tau_k$, instead of (12), for $\tau_{\min} \ll \tau_c \ll \tau_{\max}$ is

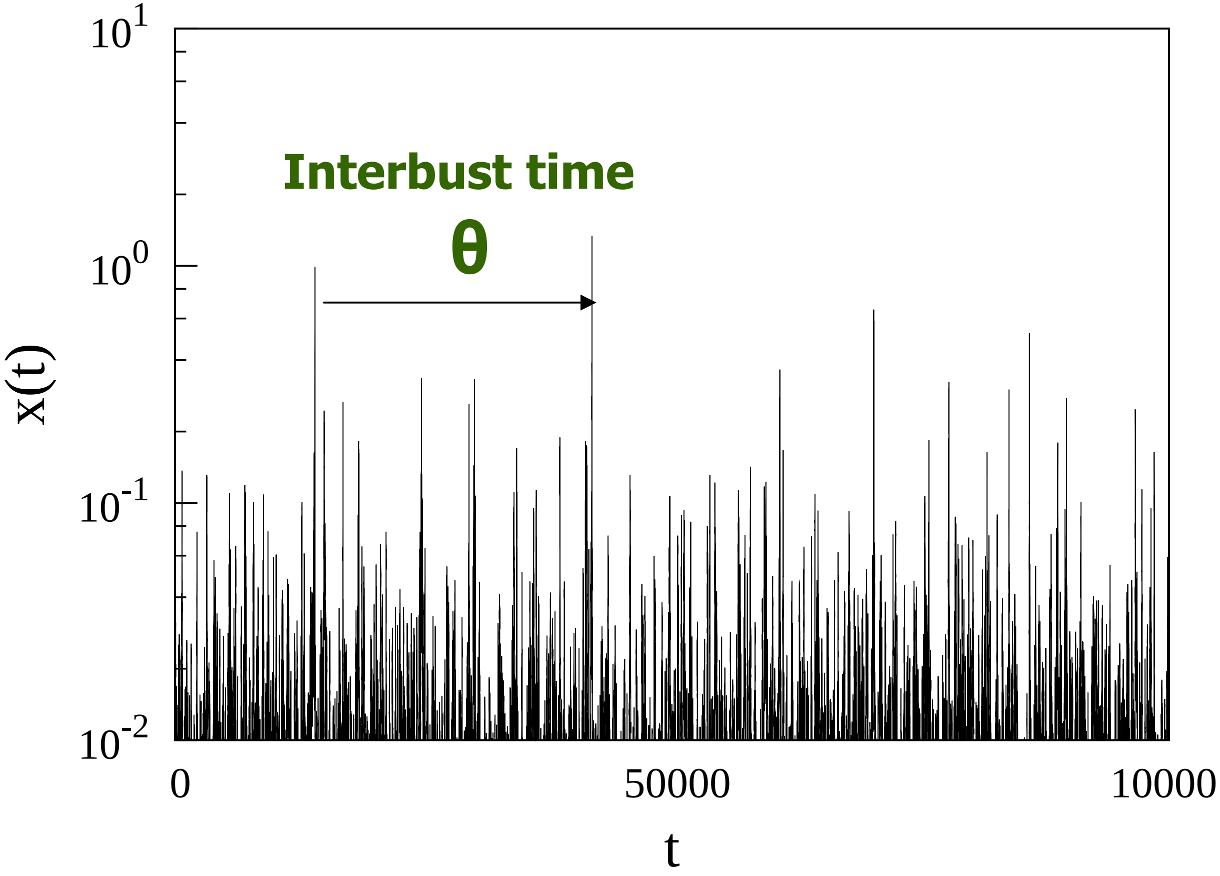
$$P_k\left(\tau_k\right) \simeq \frac{2\left(\tau_c + \tau_k\right)^2}{\tau_{\max}^2 \tau_k} \exp\left( - \frac{\tau_{\min}^2}{\tau_k^2} - \frac{\tau_k^2}{\tau_{\max}^2} \right) . \qquad (24)$$

The steady-state distribution of the intensity of the process $x(t)$, exponentially restricted at small $x_{\min} = 1/\tau_{\max}$ and large $x_{\max} = 1/\tau_{\min}$, is

$$P(x) \simeq \frac{4x_{\min}^3 (x_c + x)^2}{\sqrt{\pi} x^4} \exp\left(-\frac{x_{\min}^2}{x^2} - \frac{x^2}{x_{\max}^2}\right). \qquad (25)$$

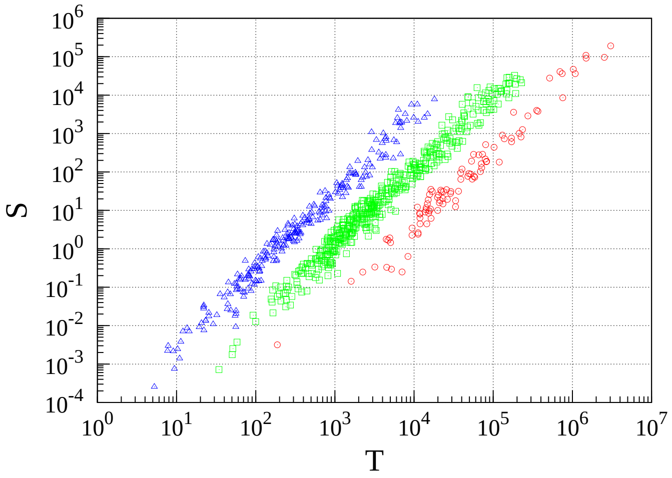
The cumulative distribution $P_>(x)$ of $x$ for $x < x_c$ is given by the same Eq. (14). The average intensity of the process $\langle x \rangle = \langle \tau_k \rangle^{-1}$, where $\langle \tau_k \rangle \simeq \frac{\sqrt{\pi}}{2} \tau_{\max}$. The counting of events may be calculated according to the same Eq. (18).

# Numerical results. Secondary structure the signal

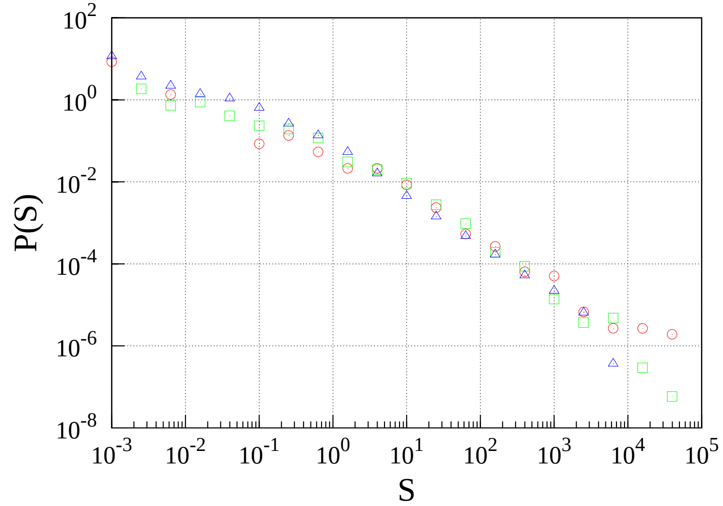



**Typical signal**

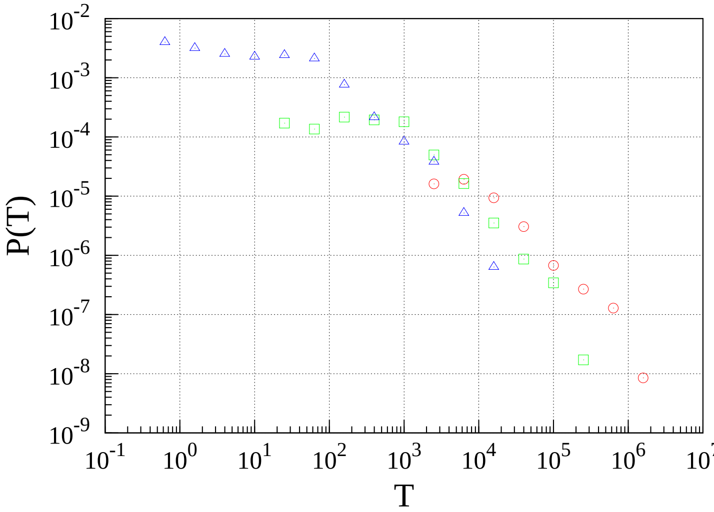
# Numerical results. Secondary structure the signals



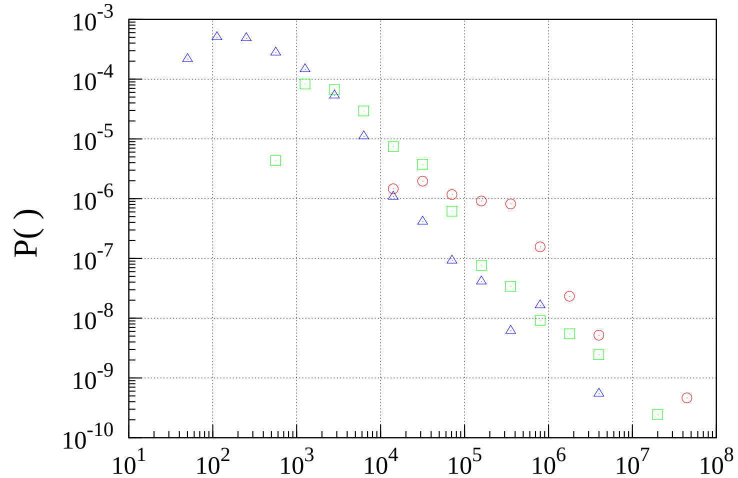

**Burst size *S* vs burst duration *T***




**Distribution of bust size *P*(*S*)**




**Distribution of bust durations *P*(*T*)**




**Distribution of interbust time *P*(*θ*)**

# q-exponential distribution and 1/f noise

$$dx = \left(\eta - \frac{1}{2}\lambda\right)(x_m + x)^{2\eta-1}\, dt + (x_m + x)^{\eta}\, dW$$

(i) is linear for small $x \ll x_m$ ,

**(ii) restrict divergence of power-law distribution of x at x=0**

$$P(x) = \frac{(\lambda-1)x_m^{\lambda-1}}{(x_m+x)^{\lambda}} = \frac{(\lambda-1)}{x_m}\exp_q\left\{-\lambda\frac{x}{x_m}\right\}, \quad x > 0$$

q-exponent

and

(iii) generate signals with $1/f^{\beta}$ spectrum:

**Analytical calculations from the related point process model**

$$S(f) \approx \frac{A}{f^{\beta}}, \quad \frac{1}{2} < \beta < 2, \quad 4 - \eta < \lambda < 1 + 2\eta,$$

$$\beta = 1 + \frac{\lambda - 3}{2(\eta - 1)}, \quad \eta > 1,$$

$$A \approx \frac{(\lambda-1)\,\Gamma(\beta - 1/2)\, x_m^{\lambda-1}}{2\sqrt{\pi}\,(\eta-1)\sin(\pi\beta/2)}\left(\frac{2+\lambda-2\eta}{2\pi}\right)^{\beta-1}$$

**B. K. and M. Alaburda, J. Stat. Mech. P02051 (2009)**

# q-Gaussian distribution

$$\mathrm{d}x = \left(\eta - \frac{1}{2}\lambda\right)\left(x_m^2 + x^2\right)^{\eta-1} x\mathrm{d}t + \left(x_m^2 + x^2\right)^{\eta/2}\mathrm{d}W, \quad \eta > 1, \quad \lambda > 1$$

$$P(x) = \frac{\Gamma\left(\frac{\lambda}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\lambda-1}{2}\right)x_m}\left(\frac{x_m^2}{x_m^2+x^2}\right)^{\lambda/2} = \frac{\Gamma\left(\frac{\lambda}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\lambda-1}{2}\right)x_m}\exp_q\left\{-\lambda\frac{x^2}{2x_m^2}\right\}$$

**Regular distribution of signal for *x* > 0, *x* = 0 and *x* < 0.**

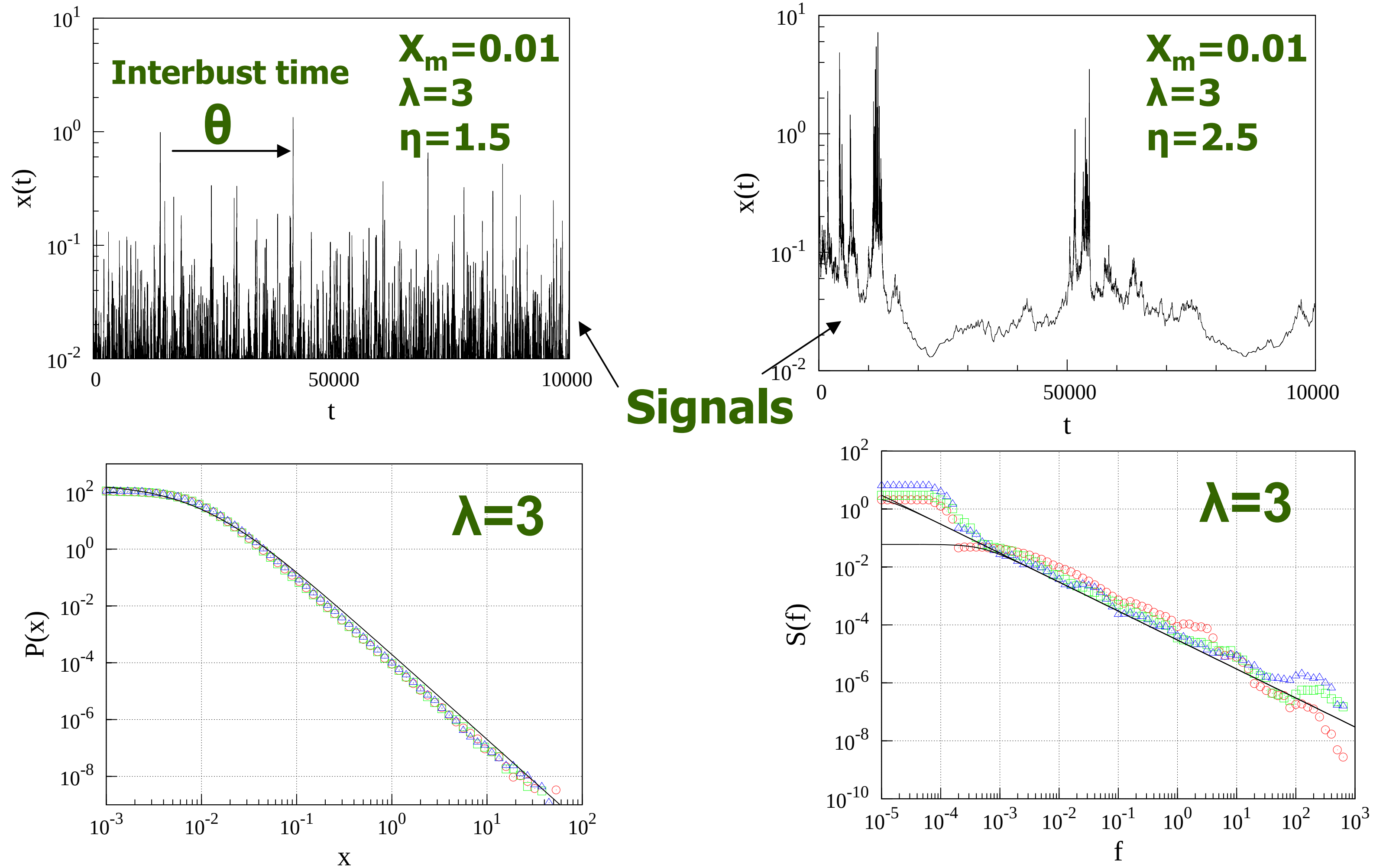
**V.Gontis, B.Kaulakys, and J.Ruseckas, AIP Conf. Proc. 1129, 563 (2009)**

$$S(f) = \frac{A}{(f_0^2 + f^2)^{\beta/}} = \exp_q\left\{-\beta\frac{f^2}{2f_0^2}\right\} \qquad \beta = 1 + \frac{\lambda - 3}{2(\eta - 1)}$$

$$C(s) = \int_0^\infty S(f)\cos(2\pi f s)\mathrm{d}f = \frac{A\sqrt{\pi}}{\Gamma(\beta/2)}\left(\frac{\pi s}{f_0}\right)^h K_h(2\pi f_0 s)$$

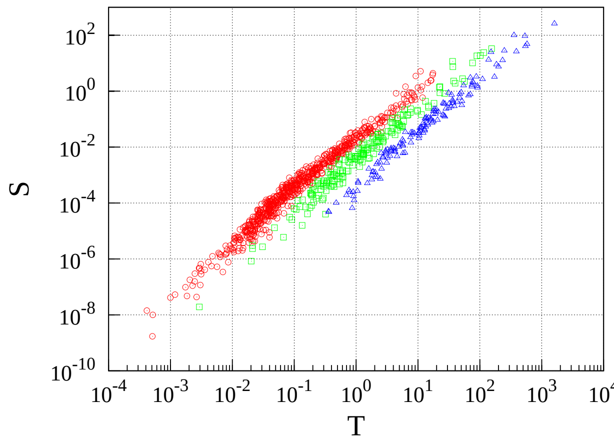
$$F(s) = F_2^2(s) = \left\langle |x(t+s) - x(t)|^2 \right\rangle = 2[C(0) - C(s)] = 4\int_0^\infty S(f)\sin^2(\pi s f)\mathrm{d}f.$$

# Numerical results. Secondary structure the signals

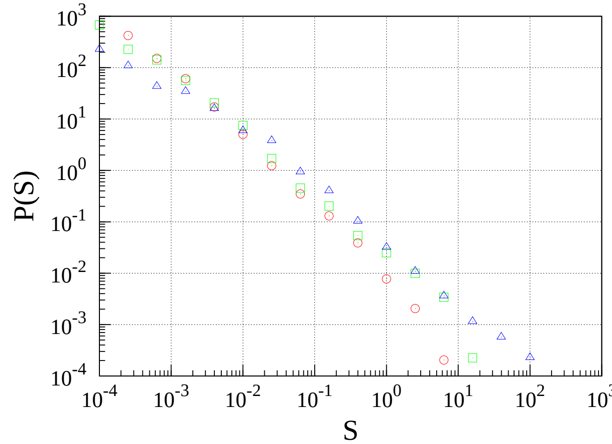



$\eta = 1.5$ (circles), $\eta = 2$ (squares) and $\eta = 2.5$ (triangles)
in comparison with the analytical results (solid lines)

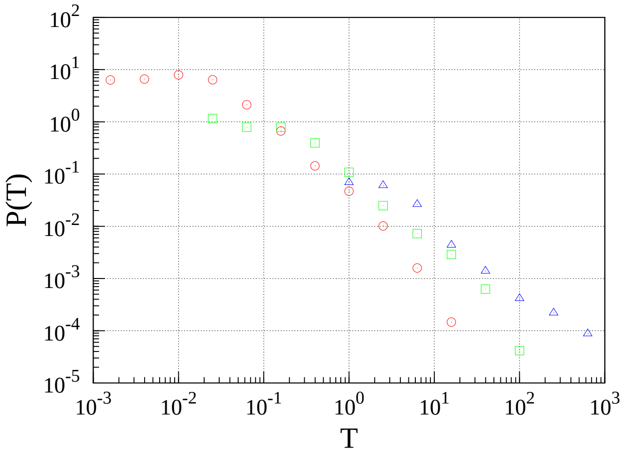
# Numerical results



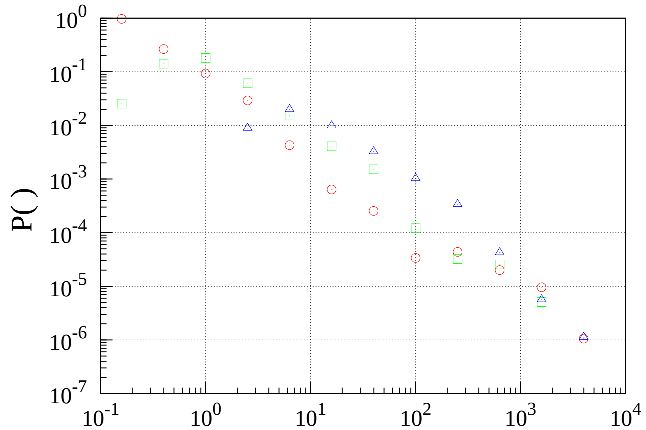

**Burst size *S* vs burst duration *T***




**Distribution of bust size *P*(*S*)**




**Distribution of bust durations *P*(*T*)**




**Distribution of interbust time *P*($\theta$)**

# Some conclusions

- A simple stochastic differential equation
- may generate the inverse cubic
- q-exponential and
- q-Gaussian distributed signals with
- $1/f^{\beta}$ power spectrum,
- exhibiting bursts, similar to the crackling processes and
- observable in long-term memory time series.

# References

B. K. and M. Alaburda, J. Stat. Mech. P02051 (2009).

V. Gontis, B. K. and J. Ruseckas, AIP Conf. Proc. 1129, 563 (2009).

B. K. and J. Ruseckas, Phys. Rev. E 81 031105 (2010).

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