COLLECTIVE BEHAVIOUR IN COMPLEX NETWORKS: SCALING AND BEYOND

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Plan

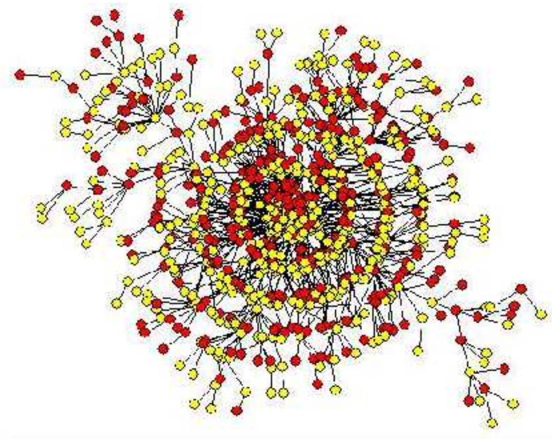
1. Motivation.

2. Introduction. Phase transitions and critical phenomena: scaling and universality.

3. How are the concepts of scaling and universality modified when a system resides on a network?

Motivations

- Academic interest. Unusual features of the Ising (Leone et al.'02, Dorogovtsev et al.'02, Tadić et al.'05), XY (Kwak et al.'07), Potts (Iglói, Turban'02) models



- Social networks as models for opinion formation (Galam'99 - '09, Sznajd'00, Sznajd-Weron'05, Staufer, Solomon'07, Kulakowski'08)

- Integrated nanosystems with nontrivial architecture (Moriarty'01, Wang et al.'03, Archer et al.'07)

Critical points in condensed matter, first experiments

Cagniard de la Tour
1777 - 1859

Thomas Andrews
1813 - 1885

Pierre Curie
1859 - 1906

C. de la Tour, 1823:

... cet état particulier exige toujous une température très-élevée, presque indépendante de la capacité du tube ...

(Ann. Chim. Phys. **22***, 410)*

T. Andrews. Bakerian lecture, 1869:

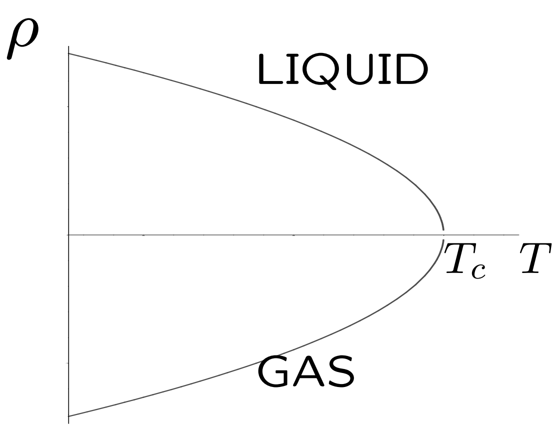
... From carbonic acid as a perfect gas to carbonic acid as a perfect liquid, the transition we have seen may be accomplished by a continuous process ... (Phil. Trans. **159***, 575)*

P. Curie, 1895:

Thèse sur les propriétés magnétiques des corps
à diverses températures, pressions et perfect liquid,
intensités de champ magnétique (Sorbonne, 1895)

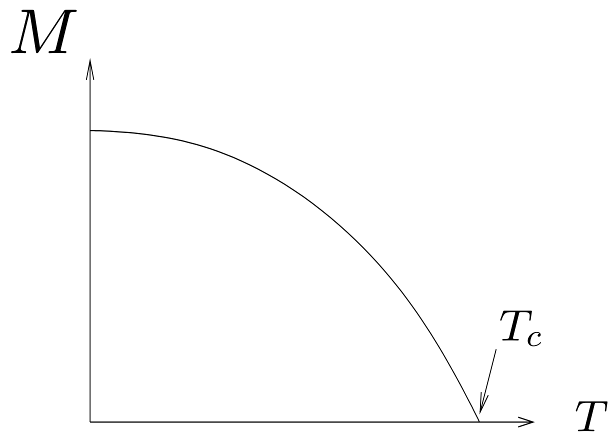
Universality

Critical point (fluids)



$$\rho,\ M \sim (T_c - T)^{\beta}$$

Curie point (magnets)

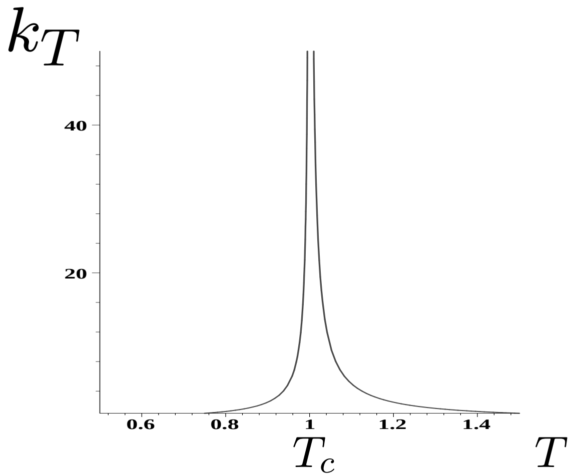


P. Curie: formal analogy between a priori unrelated physical systems (early concept of universality)

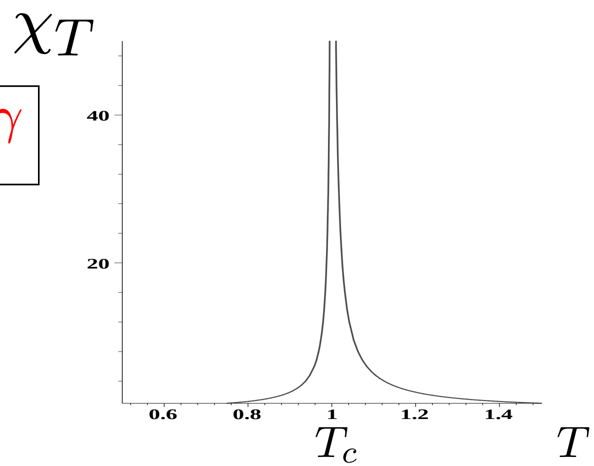
Universality

Critical point (fluids)

Curie point (magnets)



$$k_T,\, \chi_T \sim |T_c - T|^{-\gamma}$$



P. Curie: formal analogy between a priori unrelated physical systems (early concept of universality)

Power laws and critical exponents

J. D. van der Waals (1893): the notion of a critical exponent

J. D. van der Waals
1837 - 1923

For a magnet in magnetic field h in vicinity of T_C, ($\tau = |T - T_C|$):

$\underline{h = 0:}$

$$M \sim \tau^{\beta},$$
$$\chi_T \sim \tau^{-\gamma},$$
$$C_h \sim \tau^{-\alpha},$$
$$M_T = -T\left(\frac{\partial M}{\partial T}\right)_h \sim \tau^{-\omega}.$$

$\underline{\tau = 0:}$

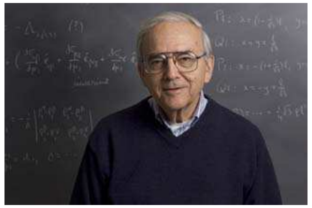
$$M \sim h^{1/\delta},$$
$$\chi_T \sim h^{-\gamma_c},$$
$$C_h \sim h^{-\alpha_c},$$
$$M_T \sim h^{-\omega_c}.$$

'Classical' theories: $\beta = 1/2$, $\gamma = 1$, $\alpha = 0$, $\delta = 3$...

J.-E. Verschaffelt
1870 - 1955

J.-E. Verschaffelt (1900): non-classical $\beta = 0.3434$, $\delta = 4.259$.

Scaling and universality

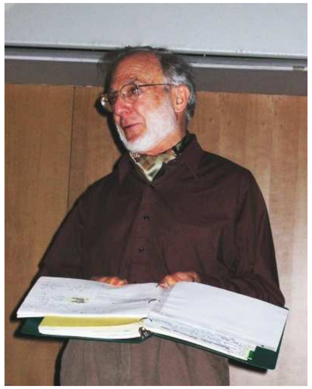


B. Widom

V. Pokrovskii

- *Scaling laws: expression of physical principles in the mathematical language of homogeneous functions.*

- *Universality: certain properties of a wide class of systems do not depend on specific details of certain system*



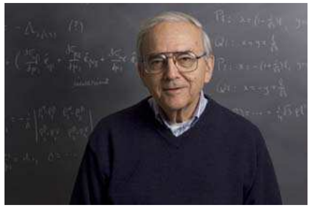
M. Fisher

L. Kadanoff

H.E. Stanley

A. Patashinskii

Scaling and universality



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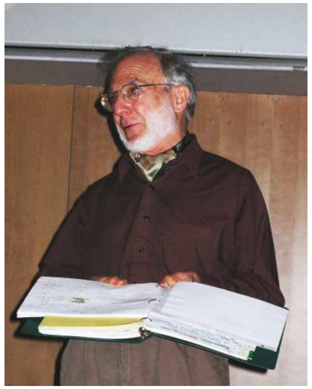
- *Scaling laws: expression of physical principles in the mathematical language of homogeneous functions.*

$$F(\tau, M) = \tau^{2-\alpha} f_{\pm}(M/\tau^{\beta})$$

- *Universality: certain properties of a wide class of systems do not depend on specific details of certain system*

'global' factors: d, n, symmetry



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How are the concepts of scaling and universality modified when a system resides on a network?

$$P(k) \sim 1/k^{\lambda}$$

- *power laws,*
- *scaling functions,*
- *logarithmic corrections.*

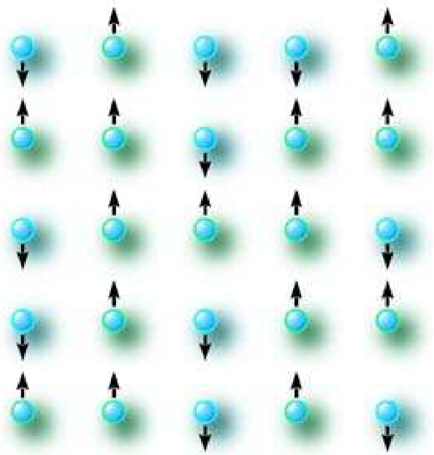
Ising model (Cologne/Hamburg/Mersch/Peoria)

Ernst Ising
1900-1998

E. Ising. Beitrag zur Theorie des Ferro- und Paramagnetism (Hamburg, 1924); Z. Phys. **31** *(1925) 253*

$$H = -\sum_{<i,j>} S_i S_j - h \sum_i S_i,$$

$$S_i = \uparrow, \quad \downarrow .$$



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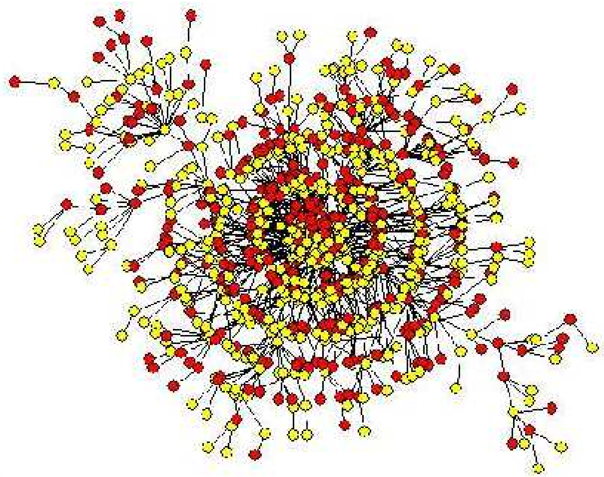


Results for the network: critical exponents

$\lambda \leq 3$, system is disordered only for $T \to \infty$:

$M \sim e^{-\eta T}$, $\lambda = 3$ and $M \sim T^{-\frac{1}{3-\lambda}}$, $2 < \lambda < 3$.

$\lambda > 3$, 2nd order phase transition with the exponents:

	α	β	γ	δ	ω	α_c	γ_c	ω_c
$\lambda \geq 5$	0	1/2	1	3	1/2	0	2/3	1/3
$3 < \lambda < 5$	$\frac{\lambda-5}{\lambda-3}$	$\frac{1}{\lambda-3}$	1	$\lambda - 2$	$\frac{\lambda-4}{\lambda-3}$	$\frac{\lambda-5}{\lambda-2}$	$\frac{\lambda-3}{\lambda-2}$	$\frac{\lambda-4}{\lambda-2}$

λ becomes a *global* parameter!

Results for networks: scaling hypothesis

Helmholts potential is a generalized homogeneous function:

$$F(\tau, M) = \tau^{2-\alpha} f_{\pm}(M/\tau^{\beta})$$

Holds only for $3 < \lambda < 5$ and for $\lambda > 5$.

Scaling functions become λ-dependent!

Magnetic eq. of state:

$$H = \frac{\partial F}{\partial M}|_T,$$

$$h(\tau, M) = \tau^{\beta\delta} H_{\pm}(x),$$

with $x = m/\tau^{\beta}$.

Entropic eq. of state:

$$S = -\frac{\partial F}{\partial T}|_M,$$

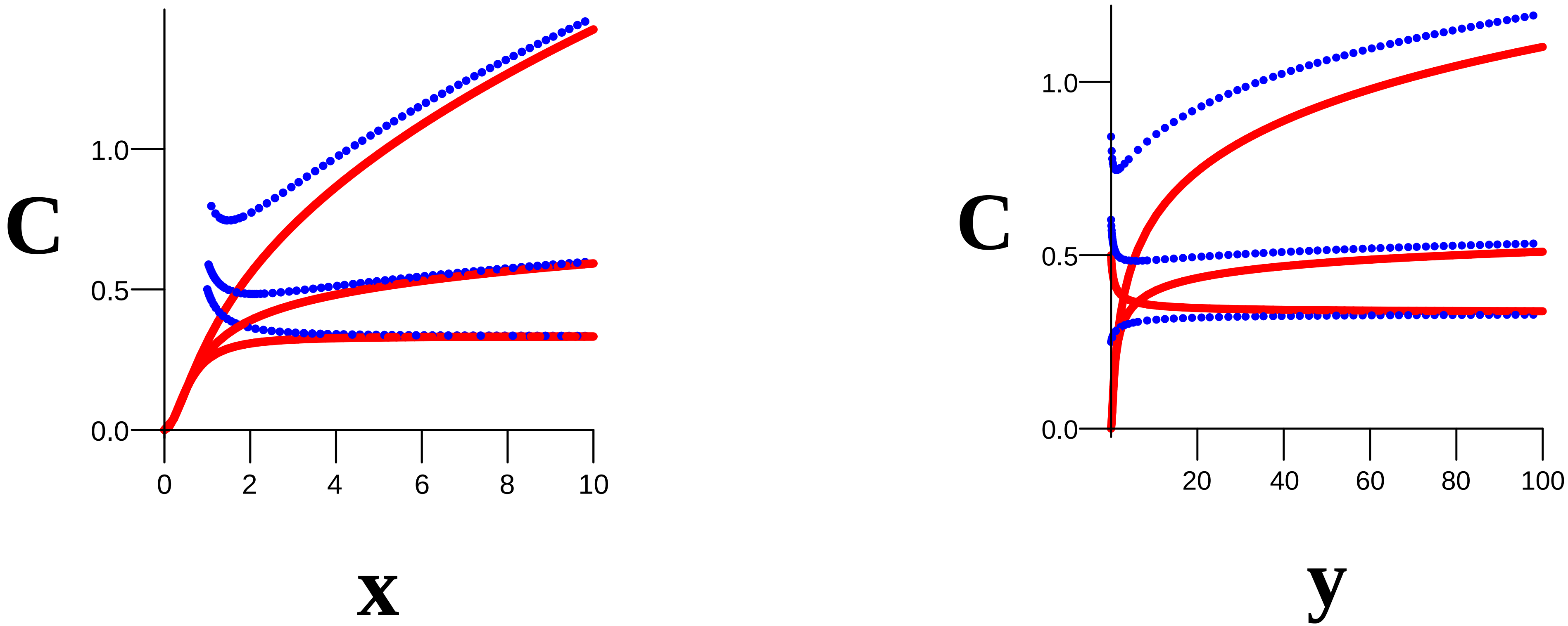
$$S(\tau, M) = \tau \mathcal{S}(x),$$

Results for networks: scaling functions and amplitude ratios

	$3<\lambda<5$	$\lambda>5$
$f_\pm$	$\pm x^2/2 + x^{\lambda-1}/4$	$\pm x^2/2 + x^4/4$
$H_\pm$	$\frac{\lambda-1}{4}x^{\lambda-2} \pm x$	$x^3 \pm x$
$\chi_\pm$	$\frac{1}{(\lambda-1)(\lambda-2)x^{\lambda-3}/4\pm 1}$	$\frac{1}{3x^2\pm 1}$
$\mathcal{S}$	$-x^2/2$	$-x^2/2$
$\mathcal{C}_\pm$	$\frac{x^2}{(\lambda-1)(\lambda-2)x^{\lambda-3}/4\pm 1}$	$\frac{x^2}{3x^2\pm 1}$
$\mathcal{M}_\pm$	$\frac{x}{(\lambda-1)(\lambda-2)x^{\lambda-3}/4\pm 1}$	$\frac{x}{3x^2\pm 1}$
A^+/A^-	0	0
Γ^+/Γ^-	$\lambda-3$	2
R_χ	1	1
R_C	0	0
R_A	$\frac{1}{\lambda-2}[\frac{4}{\lambda-1}]^{\frac{\lambda-5}{(\lambda-2)(\lambda-3)}}$	$1/3$

Results for networks: scaling functions, an example

Heat capacity scaling functions C_- (blue) and C_+ (red) as functions of the scaling variables $x = m/\tau^{\beta}$ and $y = h/\tau^{\beta\delta}$ for $\lambda > 5$, $\lambda = 4.8$ and $\lambda = 4.5$ (lower, middle and upper pairs of curves, respectively):



Results for networks: logarithmic corrections

Logarithmic corrections to scaling are found at the 'upper critical' value $\lambda = 5$:

$h = 0$:

$$M \sim \tau^{\beta} |\ln \tau|^{\widehat{\beta}},$$
$$\chi \sim \tau^{-\gamma} |\ln \tau|^{\widehat{\gamma}},$$
$$C_h \sim \tau^{-\alpha} |\ln \tau|^{\widehat{\alpha}}.$$

$\tau = 0$:

$$M \sim h^{1/\delta} |\ln h|^{\widehat{\delta}},$$
$$\chi \sim h^{-\gamma_c} |\ln h|^{\widehat{\gamma}_c},$$
$$C_h \sim h^{-\alpha_c} |\ln h|^{\widehat{\alpha}_c}.$$

For their numerical values we get:

$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\gamma}$	$\widehat{\delta}$	$\widehat{\alpha}_c$	$\widehat{\gamma}_c$
-1	$-1/2$	0	$-1/3$	-1	$-1/3$

From the Lee-Yang analysis we get the following relations for the logarithmic correction-to-scaling exponents:

$$\hat{\beta}(\delta - 1) = \delta\hat{\delta} - \hat{\gamma}, \quad \hat{\alpha} = 2\hat{\beta} - \hat{\gamma},$$

$$\hat{\gamma}_c = \hat{\delta}, \quad \hat{\alpha}_c = \frac{(\gamma + 2)(\hat{\beta} - \hat{\gamma})}{\beta + \gamma} + \hat{\gamma}.$$

Recall:

$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\delta}$	$\hat{\alpha}_c$	$\hat{\gamma}_c$
-1	$-\frac{1}{2}$	0	$-\frac{1}{3}$	-1	$-\frac{1}{3}$

Compare

with:

$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\delta}$	
-1	$-\frac{1}{8}$	$\frac{3}{4}$	$-\frac{1}{15}$	$q = 4$ Potts, $d = 2$
$\frac{4-N}{N+8}$	$\frac{3}{N+8}$	$\frac{N+2}{N+8}$	$\frac{1}{3}$	$O(N)$, $d = 4$

Conclusions

- Scaling and universality of critical phenomena are modified when system resides on a scale-free network
- Role of the high-degree vertices (hubs):
  - Rapid decay ($\lambda > 5$): usual Landau theory;
  - "Upper critical dimension" $\lambda = 5$: logarithmic corrections (scaling relations hold);
  - $3 < \lambda < 5$: non-trivial λ-dependence of the exponents and amplitude ratios;
  - $2 < \lambda \leq 3$: $T_c = \infty$.
- A comprehensive list of observables that describe the scaling and characterize the criticality in scale-free networks.
- Refs.: Phys. Rev. E **82** (2010) 011145; Phys. Rev. E **83** (2011), to appear, arXiv:1101.3680.