
Modeling Jump Dependence using Lévy copulas

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Outline

- Introduction
 - Lévy process
 - Lévy copula
- M-Estimator
 - Definition. Asymptotic properties
 - Simulation study: Clayton copula
- Example: MSCI equity index data (Europe)

Lévy process

□ bivariate Lévy process: a stochastic process $\mathbb{Y} = (Y_1, Y_2)$ such that

- $\mathbb{Y}_0 = (0, 0)$;
- independent and stationary increments;
- stochastically continuous.

□ by Lévy -Khintchine representation: $\mathbb{E} [e^{i\langle z, \mathbb{Y} \rangle}] = e^{\phi(z)}$, where

$$\phi(z) = i \langle \gamma, z \rangle - \frac{1}{2} \langle z, \mathbb{A}z \rangle + \int_{\mathbb{R}^2 \setminus (0,0)} \left(e^{i\langle z, y \rangle} - 1 - i \langle z, y \rangle \mathbf{1}\{|y| \leq 1\} \right) \nu(dy),$$

any Lévy process: $(\gamma, \mathbb{A}, \nu)$, with $\gamma \in \mathbb{R}^2$, $\mathbb{A} \in \mathbb{R}^{2 \times 2}$ and a positive sigma-finite measure ν on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Lévy copula

- denote $\overline{\mathbb{R}} := (-\infty, \infty]$
- a function $F: S \rightarrow \overline{\mathbb{R}}$, $S \subseteq \overline{\mathbb{R}}^2$ is **2-increasing** if

$$F(a_1, a_2) + F(b_1, b_2) - F(a_1, b_2) - F(b_1, a_2) \geq 0,$$

for all $(a_1, a_2), (b_1, b_2) \in S$ with $a_1 \leq b_1, a_2 \leq b_2$

- a function $F: \overline{\mathbb{R}}^2 \rightarrow \overline{\mathbb{R}}$ is a **Lévy copula** if
 - $F(u_1, u_2) \neq \infty$, if $(u_1, u_2) \neq (\infty, \infty)$,
 - $F(u_1, u_2) = 0$, if $u_i = 0$, for at least one $i = 1, 2$,
 - F is 2-increasing
 - $F^i(u) = u$, $i = 1, 2$, $u \in \mathbb{R}$

Lévy copula

- a function $U: \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}$, $S \subseteq \overline{\mathbb{R}}^2$ is the of a Lévy process with the Lévy measure ν if

$U(x_1, x_2) = \text{sgn}(x_1)\text{sgn}(x_2)\nu(\mathcal{I}(x_1) \times \mathcal{I}(x_2))$, where

$$\mathcal{I}(x_1) = \begin{cases} (x_1, \infty), & \text{if } x_1 \geq 0, \\ (-\infty, x_1], & \text{if } x_1 < 0. \end{cases}$$

- Lévy copula vs. copula

Lévy copula

□ **Theorem** (Kallsen and Tankov)

Let F be a bivariate Lévy copula and U_1 and U_2 the tail integrals of its marginal processes. Then there exists a bivariate Lévy process \mathbb{Y} whose components have tail integrals U_1 and U_2 , and such that

$$U^I((x_i)_{i \in I}) = F^I((U_i(x_i))_{i \in I})$$

for any non-empty $I \subseteq \{1, 2\}$. **The Lévy measure ν is uniquely determined by U_1 , U_2 and F .**

Lévy copula. Limit relation

□ **Theorem** (Kallsen and Tankov)

Let $\mathbb{X} = (X_1, X_2)$ be a bivariate Lévy process with Lévy copula F and tail integrals U_1 and U_2 . Denote by $C_t^{(\alpha_1, \alpha_2)} : [0, 1]^2 \rightarrow [0, 1]$ a copula of $(-\alpha_1 X_1, -\alpha_2 X_2)$, for $t \in (0, \infty)^2$ and $\alpha_1, \alpha_2 \in \{-1, 1\}$. Then

$$F(u_1, u_2) = \lim_{t \rightarrow 0} C_t^{(\operatorname{sgn} u_1, \operatorname{sgn} u_2)}(t|u_1|, t|u_2|) \operatorname{sgn} u_1 \operatorname{sgn} u_2,$$

for any $(u_1, u_2) \in \overline{\operatorname{Ran}(U_1) \times \operatorname{Ran}(U_2)}$

M-Estimator: Basic Assumptions

- bivariate Lévy process \mathbb{Y} observed at n distinct times separated by Δ_n :
 $\mathbb{Y}_{\Delta_n}, \mathbb{Y}_{2\Delta_n}, \dots, \mathbb{Y}_{n\Delta_n}$
- n increments: $\mathbb{Y}_{\Delta_n}, \mathbb{Y}_{2\Delta_n} - \mathbb{Y}_{\Delta_n}, \dots, \mathbb{Y}_{n\Delta_n} - \mathbb{Y}_{(n-1)\Delta_n}$
- sample $\mathbb{X}_1, \dots, \mathbb{X}_n$, where $\mathbb{X}_i = \mathbb{Y}_{i\Delta_n} - \mathbb{Y}_{(i-1)\Delta_n}$, with distribution function H and marginals H_1 and H_2
- assume that its Lévy copula F belongs to a parametric family
 $F \in \{F_\theta : \theta \in \Theta \subseteq \mathbb{R}^p\}$,

M-Estimator: Nonparametric Estimator

- nonparametric estimator of F (Laeven)

$$\hat{F}(u_1, u_2) := \text{sgn}(u_1)\text{sgn}(u_2) \frac{1}{k} \sum_{i=1}^n 1 \left\{ \begin{cases} R_i^1 > n + 1 - ku_1, & \text{if } u_1 \geq 0 \\ R_i^1 \leq k|u_1|, & \text{if } u_1 < 0 \end{cases} \right\},$$
$$\left\{ \begin{cases} R_i^2 > n + 1 - ku_2, & \text{if } u_2 \geq 0 \\ R_i^2 \leq k|u_2|, & \text{if } u_2 < 0 \end{cases} \right\}$$

- R_i^j is the rank of X_{ij} among X_{1j}, \dots, X_{nj} , $j = 1, 2$
- $k \in \{1, 2, \dots, n\}$

- $k = k_n \rightarrow \infty$ and $k/n \rightarrow 0$ when $n \rightarrow \infty$
- $\Delta_n \rightarrow 0$ and $n\Delta_n \rightarrow \infty$ when $n \rightarrow \infty$

M-Estimator: Definition

- let $T > 0$; denote $S_T := [-T, T]^2 \cap \overline{\text{Ran}(U_1) \times \text{Ran}(U_2)}$
- let $g \equiv (g_1, \dots, g_q)^T : T \rightarrow \mathbb{R}^q$, $q \geq p$, be an integrable function such that $\varphi : \Theta \rightarrow \mathbb{R}^q$ defined by

$$\varphi(\theta) := \int_T g(u) F(u; \theta) \, du$$

is a homeomorphism between Θ and its image $\varphi(\Theta)$

- M-estimator $\hat{\theta}_n$ of θ_0 is defined as the **minimizer of the criterion function**

$$Q_{k,n}(\theta) = \sum_{m=1}^q \left(\int_T g_m(x) (\hat{F}_n(x) - F(x; \theta)) \, dx \right)^2$$

M-Estimator: Asymptotic Properties

□ **existence, uniqueness and consistency**

□ **asymptotic normality**

□ conditions:

- there exists $\alpha \geq 1$ such that $k/n - \Delta_n = o(k/n)^\alpha$;
- F has continuous first derivatives F^1 and F^2 ,
- there exist $\alpha > 0$ and $c > 0$ such that, as $t \rightarrow 0$,

$$t^{-1} C_t^{(\operatorname{sgn}(u_1), \operatorname{sgn}(u_2))}(t|u_1|, t|u_2|) \operatorname{sgn}(u_1) \operatorname{sgn}(u_2) - F(u_1, u_2) = O(t^\alpha)$$

uniformly on $\{(u_1, u_2) \in \overline{\operatorname{Ran}(U_1) \times \operatorname{Ran}(U_2)} : u_1^2 + u_2^2 = c\}$,

- for the α from (AN2), $k/n - \Delta_n = O((k/n)^{1+\alpha})$ and
 $k = o(n^{2\alpha/(1+2\alpha)})$;

M-Estimator: Consistency

Theorem 1 (Consistency) *If the following conditions are satisfied,*

(C1) Θ is open,

(C2) φ is a homeomorphism from Θ to $\varphi(\Theta)$,

(C3) φ is a twice continuously differentiable,

(C4) $\dot{\varphi}(\theta_0)$ is of full rank;

(C5) $k = k_n$ is an intermediate sequence: $k \rightarrow \infty$ and $k/n \rightarrow 0$, as $n \rightarrow \infty$,

(C6) Δ_n is such that $\Delta_n \rightarrow 0$ and $n\Delta_n \rightarrow \infty$,

(C7) there exists $\alpha \geq 1$ such that $k/n - \Delta_n = o(k/n)^\alpha$;

then with probability tending to one the criterion function $Q_{k,n}$ has a unique minimizer $\hat{\theta}_n$, and $\hat{\theta}_n \xrightarrow{\mathbb{P}} \theta_0$, as $n \rightarrow \infty$.

M-Estimator: Asymptotic Normality

Theorem 2 (Asymptotic normality) *If in addition to the assumptions (C1)-(C6) of Theorem 1 the following assumptions hold,*

(AN1) *F has continuous first derivatives $F^{(1)}$ and $F^{(2)}$,*

(AN2) *there exist $\alpha > 0$ and $c > 0$ such that, as $t \rightarrow 0$,*

$$t^{-1} C_t^{(\operatorname{sgn}(u_1), \operatorname{sgn}(u_2))}(t|u_1|, t|u_2|) \operatorname{sgn}(u_1) \operatorname{sgn}(u_2) - F(u_1, u_2) = O(t^\alpha)$$

uniformly on $\{(u_1, u_2) \in \overline{\operatorname{Ran}(U_1) \times \operatorname{Ran}(U_2)} : u_1^2 + u_2^2 = c\}$,

(AN3) *for the α from (AN2), $k/n - \Delta_n = O((k/n)^{1+\alpha})$ and $k = o(n^{2\alpha/(1+2\alpha)})$;*

then as $n \rightarrow \infty$,

$$\sqrt{k}(\hat{\theta}_n - \theta_0) \xrightarrow{d} N((0, 0)^T, M(\theta_0)). \quad (2)$$

Simulation Study: Clayton Lévy Copula

- Clayton Lévy copula is given by

$$F(u_1, u_2) = \left(|u_1|^{-\zeta} + |u_2|^{-\zeta} \right)^{-1/\zeta} (\eta \mathbf{1}\{u_1 u_2 \geq 0\} - (1 - \eta) \mathbf{1}\{u_1 u_2 < 0\}),$$

with parameters $\zeta > 0$ and $\eta \in [0, 1]$

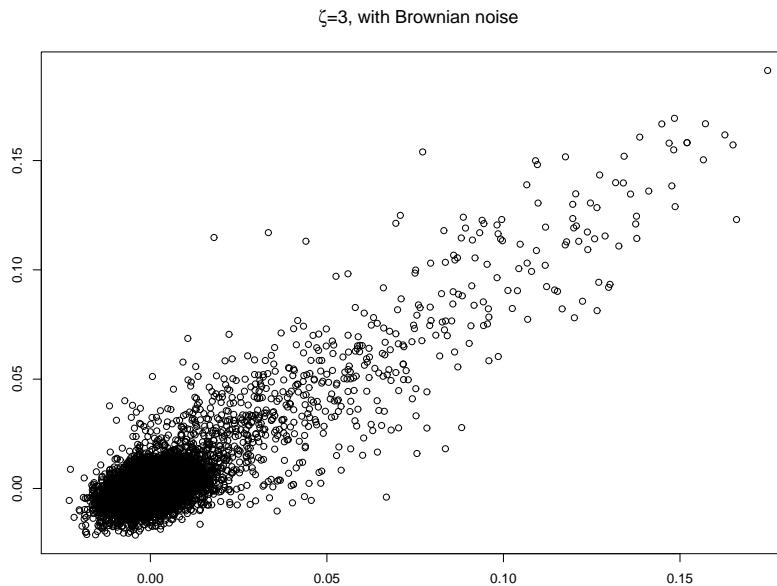
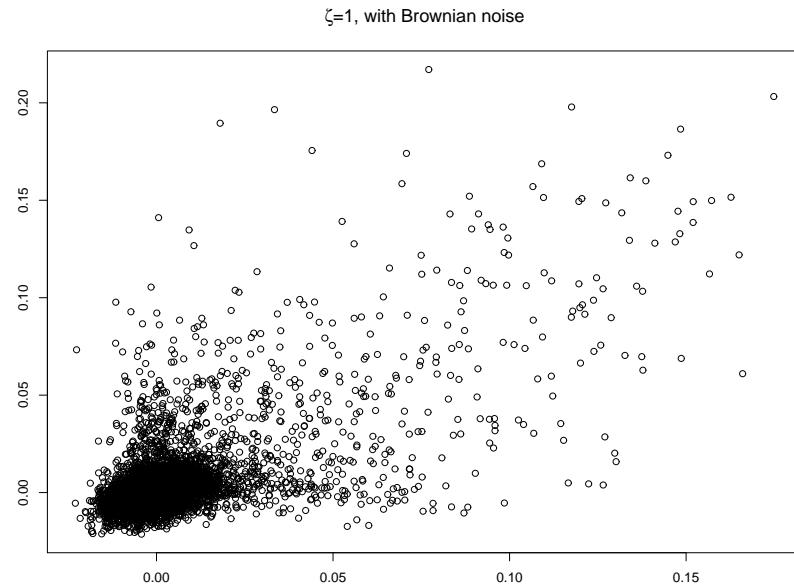
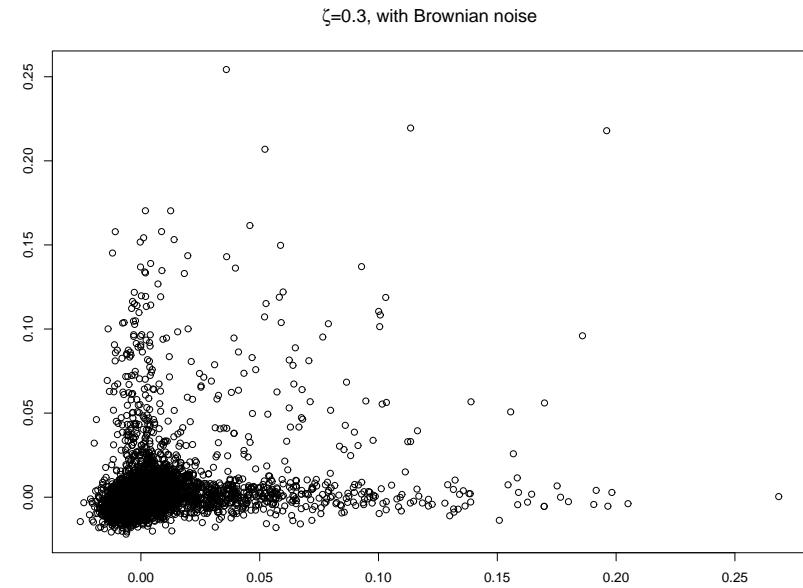
- $\zeta > 0$ unknown, $\eta = 1$ fixed
- Lévy process: bivariate Poisson jump diffusion

$$dX_t = \mu dt + \sigma dW_t + J dN_t$$

Simulation Study: Clayton Lévy Copula

- $\nu(dx) = \lambda_1 f_1(x_1; \theta_1) dx_1 + \lambda_2 f_2(x_2; \theta_2) dx_2 + \lambda_{12} f_{12}(x_1, x_2; \theta_{12}) dx_1 dx_2,$
- model parameters:
 $\mu = (0, 0)$, $\lambda_1^+ + \lambda_{12}^+ = \lambda_2^+ + \lambda_{12}^+ = 6$, $\theta_1 = \theta_2 = 1/30$,
 $\sigma = (0.1, 0.1)$, $\rho = 0.4$,
three different values for the Lévy copula parameter $\zeta \in \{0.3, 1, 3\}$
- simulations: 50 samples of size $n = 7500$, corresponding to $T_n = 30$ and $\Delta_n = 1/250$ (e.g. daily returns over 30 years)

Simulation Study: Clayton Lévy Copula



Simulation Study: Clayton Lévy Copula

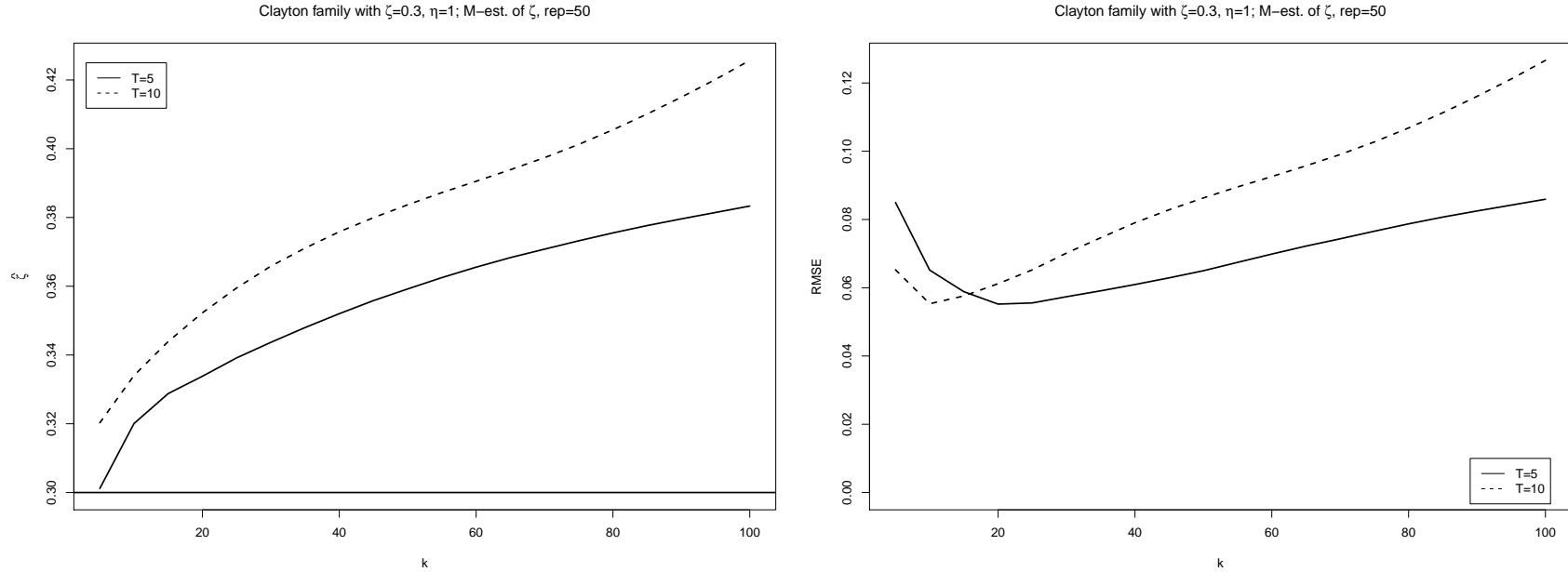


Figure 1: Estimation of $\zeta = 0.3$, for different values of k and T .

Simulation Study: Clayton Lévy Copula

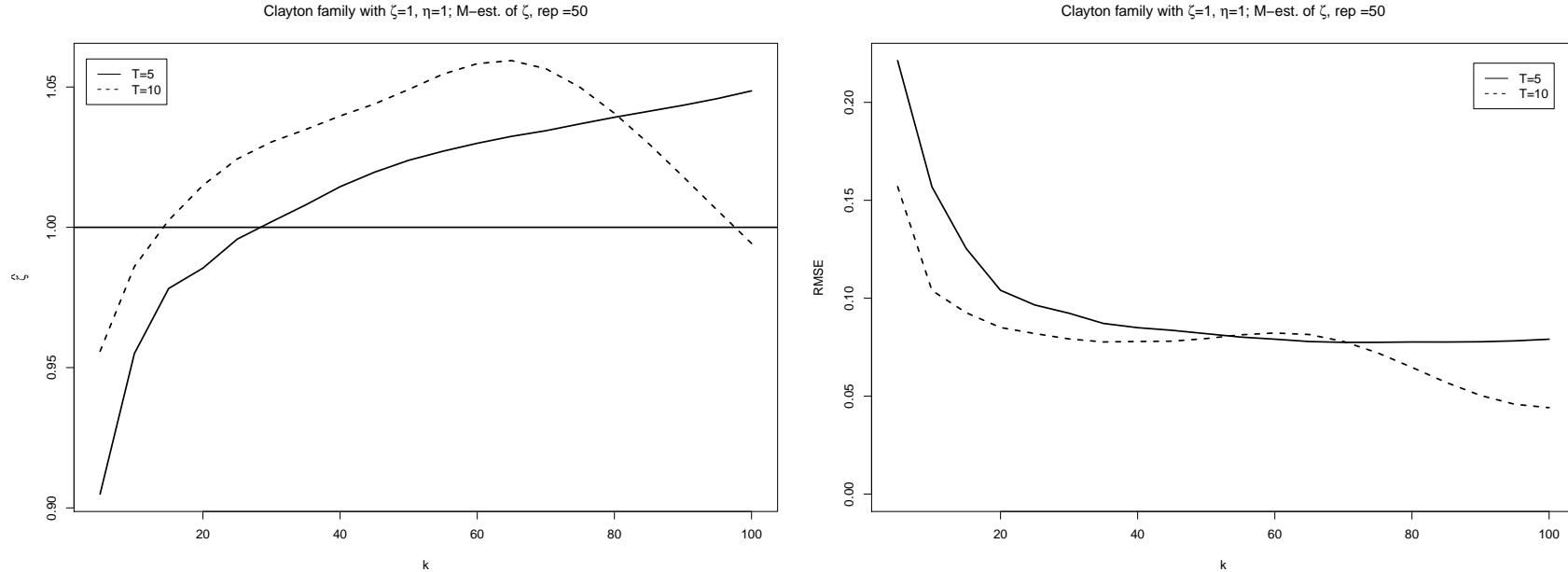


Figure 2: Estimation of $\zeta = 1$, for different values of k and T .

Simulation Study: Clayton Lévy Copula

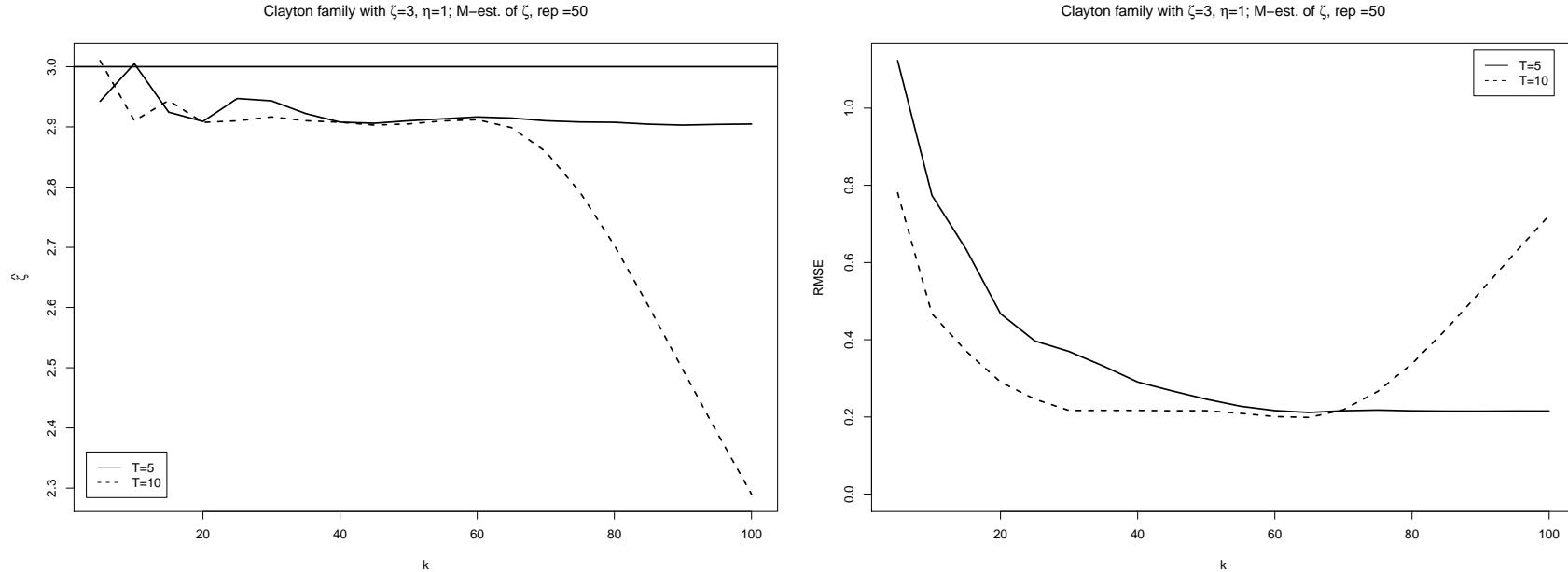


Figure 3: Estimation of $\zeta = 3$, for different values of k and T .

Simulation Study: Clayton Lévy Copula

- jump dependence coefficient: $F(1, 1) = 2^{-1/0.3} \approx 0.099$

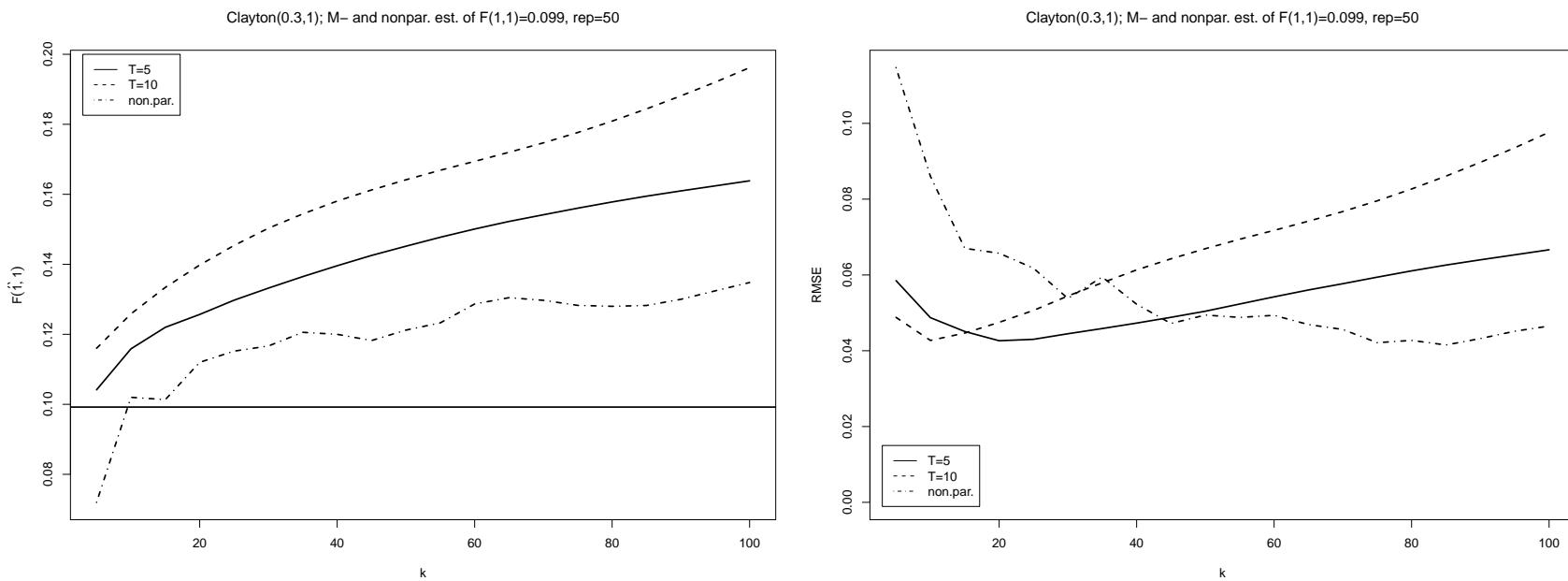


Figure 4: Estimation of $F(1, 1) = 2^{-1/0.3} \approx 0.099$, for different values of k and T .

Simulation Study: Clayton Lévy Copula

- jump dependence coefficient: $F(1, 1) = 2^{-1/1} = 0.5$

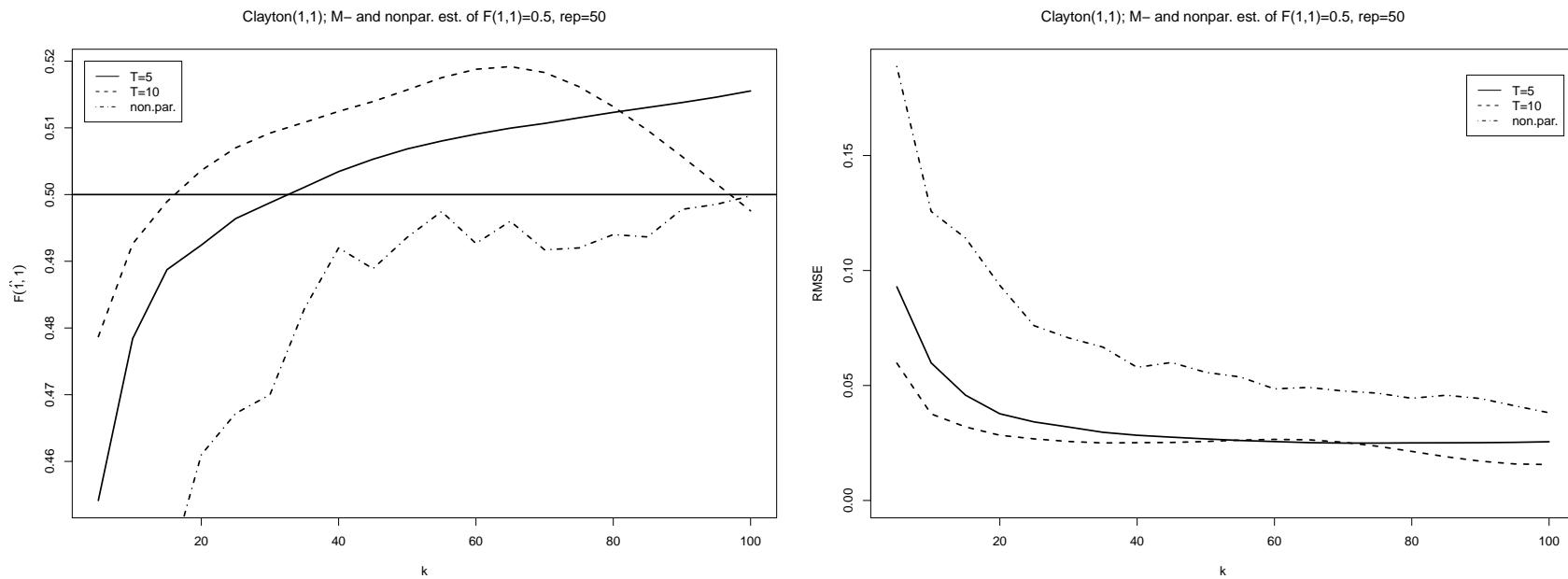


Figure 5: Estimation of $F(1, 1) = 0.5$, for different values of k and T .

Simulation Study: Clayton Lévy Copula

- jump dependence coefficient: $F(1, 1) = 2^{-1/3} \approx 0.794$

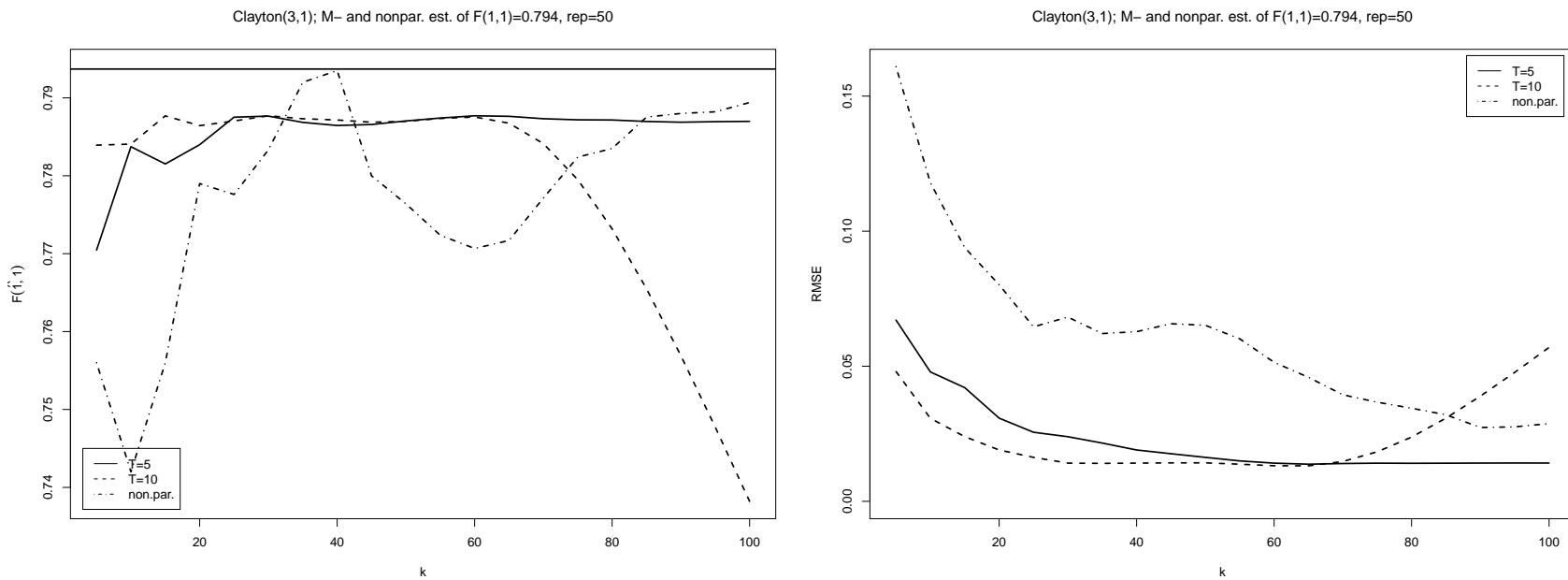


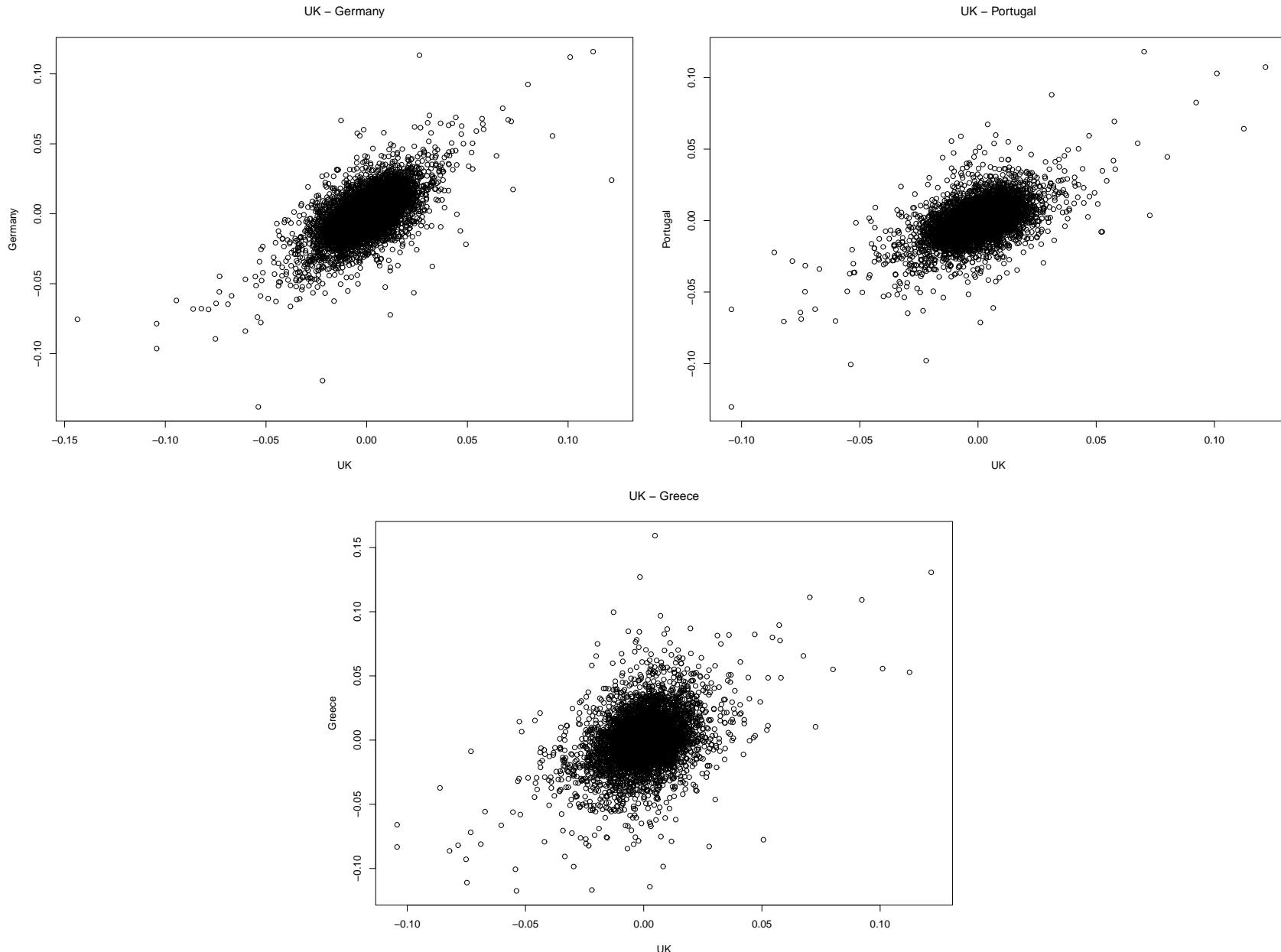
Figure 6: Estimation of $F(1, 1) = 2^{-1/3} \approx 0.794$, for different values of k and T .

Example: MSCI equity data; Europe

- we consider log-returns from the price index for the following pairs of countries:
 - UK-Germany, time period Jan 2, 1980 - Jun 30, 2011, $n = 8127$
 - UK-Portugal, time period Jan 4, 1988 - June 30, 2011, $n = 6129$
 - UK-Greece, time period Jan 4, 1988 - June 30, 2011, $n = 6129$
 - summary statistics: mean ≈ 0 , std.dev $0.012 - 0.02$

	$\hat{\zeta}, k = 30$	$F(1, 1) = 2^{-1/\hat{\zeta}}$	$\hat{F}(1, 1)$
UK-Germany	0.885	0.457	0.5
UK-Portugal	0.545	0.280	0.33
UK-Greece	0.439	0.206	0.23

Example: MSCI equity data; Europe



Example: MSCI equity data; Europe

