UNIVERSITY OF TWENTE.

Fair sharing of capacity in Jackson networks

Workshop in honour of Frank Kelly - Eurandom - April 28, 2011

Judith Timmer Werner Scheinhardt

Introduction (1)

Consider

- Some network (topology, traffic characteristics, etc.)
- Some performance measure (throughput, response time, etc.)

- Optimization (by sharing capacity, rerouting, etc.)
  can be done…

…by a single operator!

Introduction (2)

- Suppose one operator **per queue** !
- Optimization can still be done…

…if operators are willing to cooperate!

- But: individual objectives (max profit, min cost)
- Need for incentives to cooperate: sharing profit/cost

"Everybody happy ?"

→ Combine queueing theory and cooperative game theory

Literature

- Few papers on queueing systems and cooperative games
  - González, Herrero (2004)
  - García-Sanz et al. (2008)

  - Yu, Benjaafar, Gerchak (2009)

  - Anily, Haviv (2008)
  - Karsten, Van Houtum, Slikker (2011)

- All: pooling
- We: keep network as is

More literature…

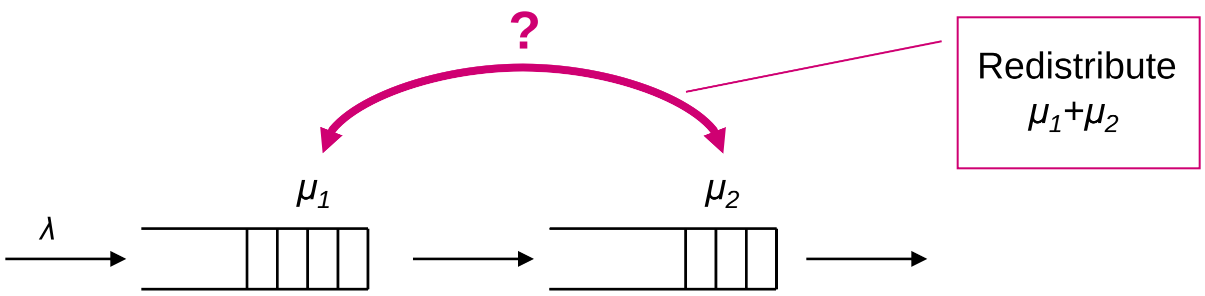
- Gibbens and Key (2008):
  Coalition Games and Resource Allocation in Ad-Hoc Networks
- Gibbens, Kelly, Cope and Whitehead (1991):
  Coalitions in the International Network
- Kelly, Massoulié, Walton (2009):
  Resource pooling in congested networks: proportional fairness and product form

Outline of remainder

- Model(s)
- Main questions:
  - How to share capacity? (known)
  - How to share the cost?
- Tandem game
- Jackson game
- Concluding remarks

Model (Tandem case)

- Network: Jackson tandem queue, n nodes
- Traffic: arrival rate λ, service rates μ_i ($> \lambda$)
- Cost: expected # jobs in system (per node: $\lambda / (\mu_i - \lambda)$)
- Cooperation: redistribute total capacity to optimise performance



Model (Jackson case)

- Network: general Jackson network, n nodes
- Traffic: local arrival rates λ_i, service rates μ_i ($> \lambda_i$)
- Cost: expected # jobs in system (per node: $\lambda_i / (\mu_i - \lambda_i)$)
- Cooperation: redistribute total capacity to optimise performance

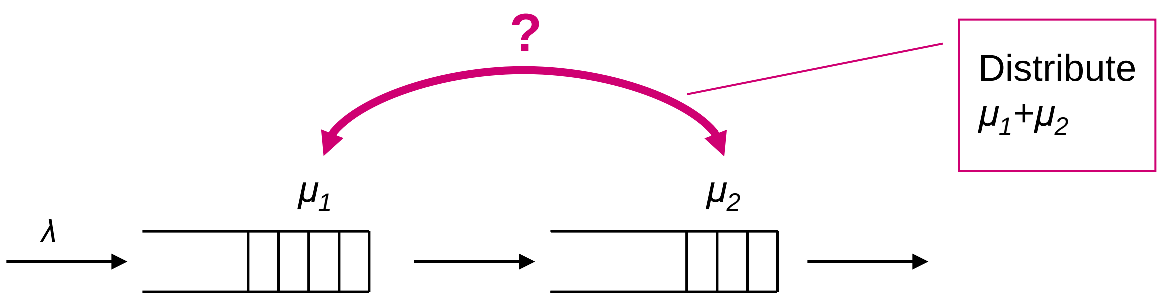
N.B. local arrival rates λ_i follow from external arrival rates λ_i^0 and routing probabs p_{ij}, regardless of capacity redistribution:

Solution of traffic equations $\lambda_j = \lambda_j^0 + \sum_{i \in N} p_{ij} \lambda_i$

Main questions:

- How to share capacity? (answer is known)

  → minimal cost



- How to share the cost?

  → "Everybody happy"

How to share capacity?

- *N:* set of players / nodes (= {1,2,…,*n*})
- *S*: any coalition of players (subset of *N*)

Redistribute (for any coalition S) the total service capacity to obtain minimal cost *c(S)*:

$$c(S) = \min_{m_i} \sum_{i \in S} \frac{\lambda}{m_i - \lambda}$$

$$s.t. \quad \sum_{i \in S} m_i = \sum_{i \in S} \mu_i$$

$$m_i > \lambda_i$$

- Solution:

$$m_i = \sum_{i \in S} \mu_i / |S| =: \bar{\mu}_S \quad \Rightarrow \quad c(S) = |S| \frac{\lambda}{\bar{\mu}_S - \lambda}$$

→ Cost function for 'Tandem game'

How to share the cost? (→ tandem game)

- Definition:

  A **tandem game** is a cost game (N,c) with the set of nodes N= {1,2,…,*n*} as player set.

  The cost c(S) of coalition S is given by $c(S) = |S| \dfrac{\lambda}{\bar{\mu}_S - \lambda}$

- Property:

  Tandem game is subadditive (incentive to cooperate):

  For disjoint *S, T:* $c(S \cup T) \leq c(S) + c(T)$

How to share the cost? (→ the core)

$$c(S) = |S| \frac{\lambda}{\bar{\mu}_S - \lambda}$$

- How to share total cost *c(N)*?
- According to any element in the core C(N,c):

$$C(N,c) = \left\{ y \in R^N : \sum_{i \in N} y_i = c(N), \sum_{i \in S} y_i \le c(S) \quad \forall S \right\}$$

- NB: core empty?

2-node tandem game

- **Theorem**
  In a 2-node tandem game, the core is never empty

  Proof: follows from subadditivity

- **Example:** $\lambda=1$, $\mu_1=2$, $\mu_2=5$

  Cost function:

S	{1}	{2}	{1,2}
$c(S)$	1	1/4	4/5

  Core is convex combination of (1,-1/5) and (11/20, 1/4)

- → Node may receive payment in core allocation (!)

3-node tandem game

- **Theorem**
  If $\mu_1 \geq \mu_2 \geq \mu_3$ then the marginal vector m^σ is in the core if $\sigma(1)=2$, i.e. node 2 goes first.

  (These may be the *only* marginal vectors in the core)

- **Example:** $\lambda=2$, $\mu_1=5$, $\mu_2=4$, $\mu_3=3$

  Cost function:

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$c(S)$	2/3	1	2	8/5	2	7/3	3

  $m^{(2,1,3)}$ = (3/5, 1, 7/5) $m^{(2,3,1)}$ = (2/3, 1, 4/3)

  If x in the core: $x_2 \leq 1, \quad x_1 + x_3 \leq 2, \quad x_1 + x_2 + x_3 = 3 \quad \Rightarrow \quad x_2 = 1 = c(2)$

  → no strict gain for node 2 (reason: $\mu_2 = \bar{\mu}_N$)

- In 4-node tandem games, the core may contain no marginal vectors…

4-node tandem game

- **Example:** λ=1, μ_1=8, μ_2=6, μ_3=4, μ_4=2

  Cost function:

S	{1}	{2}	{3}	{4}	{1,2}	{1,3}	{1,4}	{2,3}	{2,4}	{3,4}
$c(S)$	1/7	1/5	1/3	1	1/3	2/5	1/2	1/2	2/3	1
S	{1,2,3}	{1,2,4}	{1,3,4}	{2,3,4}	{1,2,3,4}					
$c(S)$	3/5	9/13	9/11	1	1					

- No marginal vectors in the core… which is non-empty!

m^{1234}	m^{1243}	m^{1324}	m^{1342}	m^{1423}	m^{1432}	m^{2134}	m^{2143}	m^{2314}	m^{2341}	m^{2413}	m^{2431}
0.143	0.143	0.143	0.143	0.143	0.143	0.133	0.133	0.100	0	0.026	0
0.190	0.190	0.400	0.182	0.192	0.182	0.200	0.200	0.200	0.200	0.200	0.200
0.267	0.308	0.257	0.257	0.308	0.318	0.267	0.308	0.300	0.300	0.308	0.333
0.400	0.359	0.200	0.418	0.357	0.357	0.400	0.359	0.400	0.500	0.467	0.467

m^{3124}	m^{3142}	m^{3214}	m^{3241}	m^{3412}	m^{3421}	m^{4123}	m^{4132}	m^{4213}	m^{4231}	m^{4312}	m^{4321}
0.067	0.067	0.100	0	-0.182	0	-0.500	-0.500	0.026	0	-0.182	0
0.200	0.182	0.167	0.167	0.182	0	0.192	0.182	-0.333	-0.333	0.182	0
0.333	0.333	0.333	0.333	0.333	0.333	0.308	0.318	0.308	0.333	0	0
0.400	0.418	0.400	0.500	0.667	0.667	1	1	1	1	1	1

n-node tandem game

- Consider tandem game *(N,c)* with cost $c(S) = |S| \frac{\lambda}{\bar{\mu}_S - \lambda}$
- Proposed cost sharing rule: node *i* pays

$$x_i = c(N) \cdot \left(\frac{1}{|N|} - \left(\frac{\mu_i - \lambda}{\sum_{j \in N} (\mu_j - \lambda)} - \frac{1}{|N|} \right) \right)$$

- **Theorem**
  The above cost sharing rule belongs to the core *C(N,c)*.

  Proof: follows

"Everybody happy !"

Jackson case: how to share capacity? (known)

- *N*: set of players / nodes (= {1,2,…,*n*})
- *S*: any coalition of players (subset of *N*)

Redistribute (for any coalition S) the total service capacity to obtain minimal cost *c(S)*:

$$c(S) = \min_{m_i} \sum_{i\in S} \frac{\lambda_i}{m_i - \lambda_i}$$
$$s.t. \quad \sum_{i\in S} m_i = \sum_{i\in S} \mu_i$$
$$m_i > \lambda_i$$

- Solution:

$$m_i = \lambda_i + \frac{\sqrt{\lambda_i}}{\sum_{k\in S}\sqrt{\lambda_k}} \sum_{k\in S}(\mu_k - \lambda_k) \quad \Rightarrow \quad c(S) = \frac{\left(\sum_{k\in S}\sqrt{\lambda_k}\right)^2}{\sum_{k\in S}(\mu_k - \lambda_k)}$$

→ Cost function for 'Tandem game'

Jackson case: how to share the cost? (→ Jackson game)

- Definition:

  A **Jackson game** is a cost game (N,c) with the set of nodes N= {1,2,…,*n*} as player set.

  The cost c(S) of coalition S is given by
  $$c(S)=\frac{\left(\sum_{k\in S}\sqrt{\lambda_k}\right)^2}{\sum_{k\in S}(\mu_k-\lambda_k)}$$

- Jackson game is subadditive (incentive to cooperate):

  For disjoint *S, T:* $c(S\cup T)\le c(S)+c(T)$

- Core empty?

$$C(N,c)=\left\{y\in R^N:\sum_{i\in N}y_i=c(N),\ \sum_{i\in S}y_i\le c(S)\quad \forall S\right\}$$

Jackson case: "relative excess capacity values"

- Role of capacities μ_i (or "excess capacities" μ_i - λ_i) from the tandem game is here played by so-called "r-values":

$$r_i = \frac{\mu_i - \lambda_i}{\sqrt{\lambda_i}}, \qquad \bar{r}_S = \frac{\sum_{i \in S} (\mu_i - \lambda_i)}{\sum_{i \in S} \sqrt{\lambda_i}} = \sum_{i \in S} \frac{\sqrt{\lambda_i}}{\sum_{k \in S} \sqrt{\lambda_k}} r_i$$

- It follows that:

$$c(S) = \frac{\left(\sum_{k \in S} \sqrt{\lambda_k} \right)^2}{\sum_{k \in S} (\mu_k - \lambda_k)} = \frac{\sum_{k \in S} \sqrt{\lambda_k}}{\bar{r}_S}$$

2-node Jackson game

- **Theorem**

  In a 2-node Jackson game, the core is never empty

  Proof: follows from subadditivity

- **Example:** $\lambda_1=1, \quad \lambda_2=4, \quad \mu_1=\mu_2=5$

  Cost function:

S	{1}	{2}	{1,2}
$c(S)$	1/4	4	9/5

  Core is convex combination of (-11/5, 4) and (1/4, 31/20)

- → Node may get paid in core allocation

3-node Jackson game

- **Theorem**
  If $r_1 \geq r_2 \geq r_3$ then the marginal vector m^σ is in the core if $\sigma(1)=2$, i.e. node 2 goes first.
  (These may be the *only* marginal vectors in the core)

- No strict gain for node 2 if $r_2 = \bar{r}_N$. In that case
  $$c(\{2\})/c(N) = \sqrt{\lambda_2}/(\sqrt{\lambda_1} + \sqrt{\lambda_2} + \sqrt{\lambda_3}),$$
  and the core is the convex hull of $m^{(2,1,3)}$ and $m^{(2,3,1)}$

Miscellaneous results for *n*-node Jackson game

- For disjoint coalitions S and T with $\bar{r}_S = \bar{r}_T$, we have

  $$c(S \cup T) = C(S) + C(T)$$

  (no incentive/need to cooperate)

- If all nodes have equal r-value *r*, the core consists of a single allocation *x* with

  $$x_i = \frac{\lambda_i}{\mu_i - \lambda_i} = \frac{\sqrt{\lambda_i}}{r}$$

Main result for *n*-node Jackson game

- Jackson game *(N,c)* with cost function
$$c(S)=\frac{\left(\sum_{k\in S}\sqrt{\lambda_k}\right)^2}{\sum_{k\in S}(\mu_k-\lambda_k)}$$
- Cost sharing rule: node *i* pays

$$x_i=c(N)\cdot\left(2\frac{\sqrt{\lambda_i}}{\sum_{j\in N}\sqrt{\lambda_j}}-\frac{\mu_i-\lambda_i}{\sum_{j\in N}(\mu_j-\lambda_j)}\right)$$

- **Theorem**
  The above cost sharing $(x_1, \ldots, x_n)$ belongs to the core *C(N,c)*.

"Everybody happy !"

n-node Jackson game (cont'd)

Theorem

The cost sharing rule $x_i = c(N)\cdot\left(2\frac{\sqrt{\lambda_i}}{\sum_{j\in N}\sqrt{\lambda_j}} - \frac{\mu_i-\lambda_i}{\sum_{j\in N}(\mu_j-\lambda_j)}\right)$ belongs to the core.

Proof:

- $$\sum_{i\in N} x_i = c(N)\cdot\sum_{i\in N}\left(2\frac{\sqrt{\lambda_i}}{\sum_{j\in N}\sqrt{\lambda_j}} - \frac{\mu_i-\lambda}{\sum_{j\in N}(\mu_j-\lambda)}\right) = c(N)$$

- $$\sum_{i\in S} x_i \le c(S)$$

$$\Leftrightarrow\ 2\frac{\sum_{i\in S}\sqrt{\lambda_i}}{\sum_{j\in N}\sqrt{\lambda_j}} - \frac{\sum_{i\in S}(\mu_i-\lambda_i)}{\sum_{j\in N}(\mu_j-\lambda_j)} \le \frac{c(S)}{c(N)}$$

$$\Leftrightarrow\ 2 \le \frac{\sum_{j\in N}\sqrt{\lambda_j}}{\sum_{i\in S}\sqrt{\lambda_i}}\left(\frac{c(S)}{c(N)} + \frac{\sum_{i\in S}(\mu_i-\lambda_i)}{\sum_{j\in N}(\mu_j-\lambda_j)}\right)$$

$$\Leftrightarrow\ 2 \le \frac{\bar{r}_N}{\bar{r}_S} + \frac{\bar{r}_S}{\bar{r}_N}$$

So each coalition S is strictly better off, unless $\bar{r}_S = \bar{r}_N$!

Concluding remarks

- *n*-node tandem/Jackson games:
  - Nonempty core: cooperation is beneficial.
  - Specific cost sharing rule found.
  - Possible payment to nodes with large capacities

Future work:

- Other networks
- Other cost functions (performance measures)
- Other ways of cooperation (e.g. change routing)

“Everybody happy ?”