Reversibility and network algorithms

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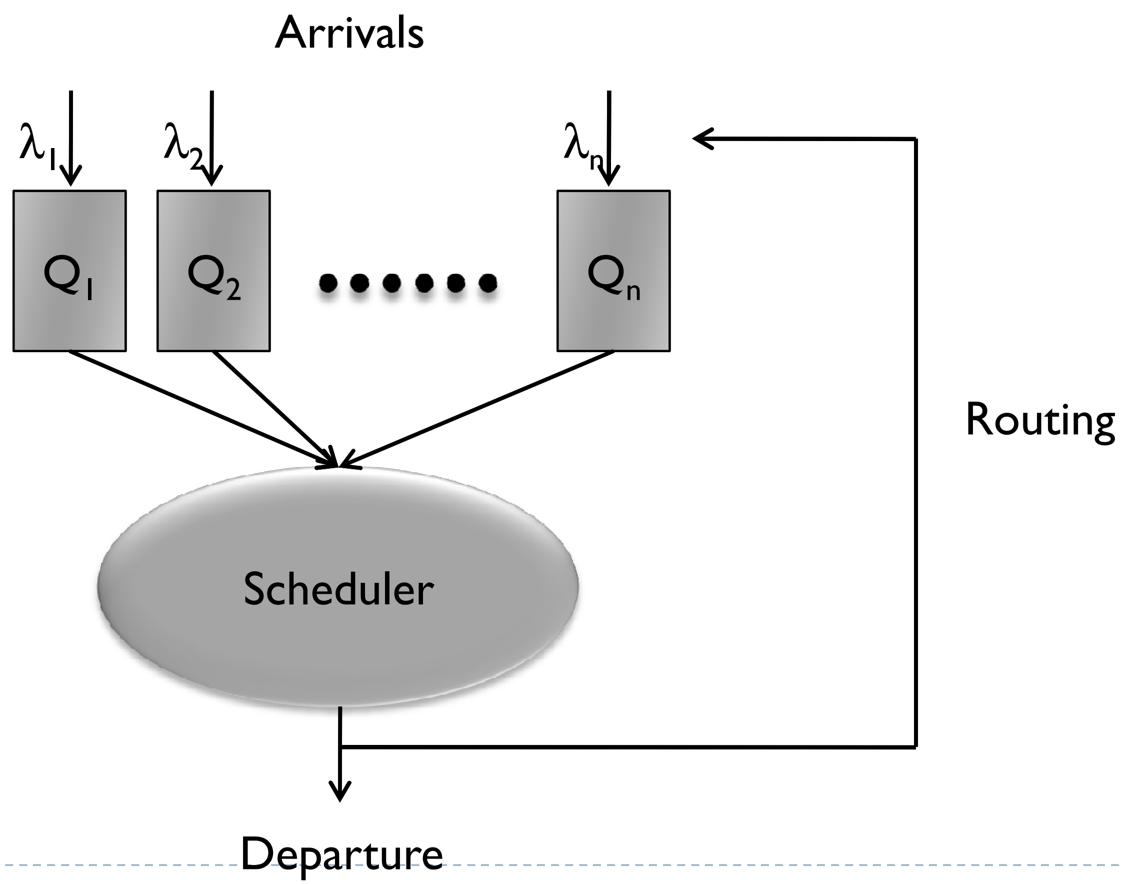
Switched network: model of interest

- Stochastic processing network of Harrison '00
  - Switched networks: *discrete-time* instances

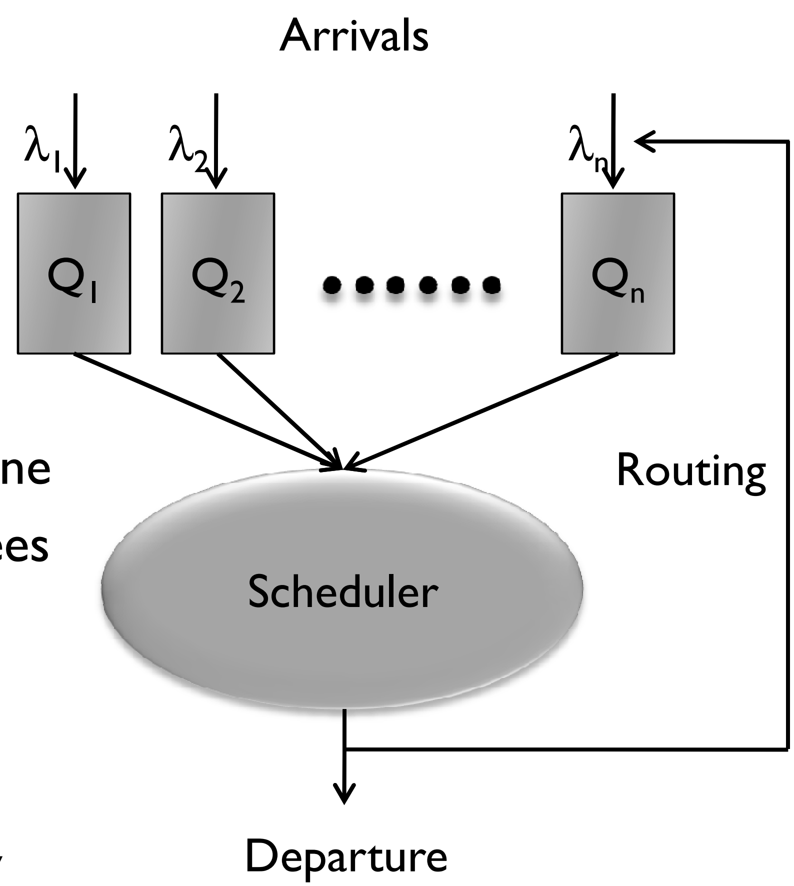


Switched network

- Example: dynamic resource sharing
  - Communication
    - Bandwidth sharing model of Internet
    - Wireless multi-hop a la mesh-network
  - Computation-Storage
    - Cloud facility or data-center
  - Human Resource (HR)
    - Project management in large industries
  - Transportation
    - Road traffic signaling

Switched network

- Basic operational task
  - Scheduling or sharing of resources
    - Among various contending entities
  - Examples
    - Which laptop transmits over WiFi
    - Disk/CPU allocation to a Virtual Machine
    - Project assignments to skilled employees
    - Signaling mechanisms on road
  - Network performance
    - Depends crucially on scheduling policy



Network performance

- Three metrics
  - Capacity
    - What is the effective resource
  - Queue-size, latency or delay
    - How long does it take to get serviced
  - Complexity
    - What sorts of implementations are feasible
- Interest is in understanding
  - Trade-offs between these metrics

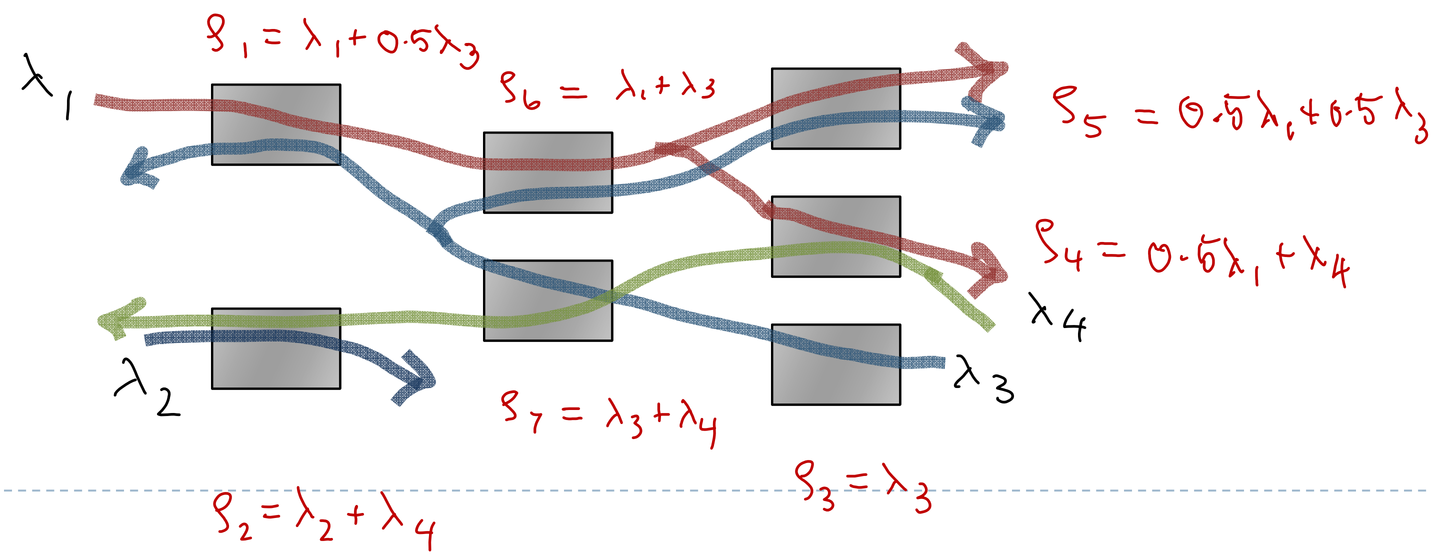
Rest of the talk

- Role of reversibility (product-form distributions) in
  - Design and analysis of scheduling algorithms

- Specifically, we shall discuss
  - Scheduling *inside* queues
    - To achieve low network-wide delay
  - Scheduling *resources* among queues
    - To achieve low network-wide delay
  - Implementing scheduling policies
    - To achieve low-complexity, distributed design

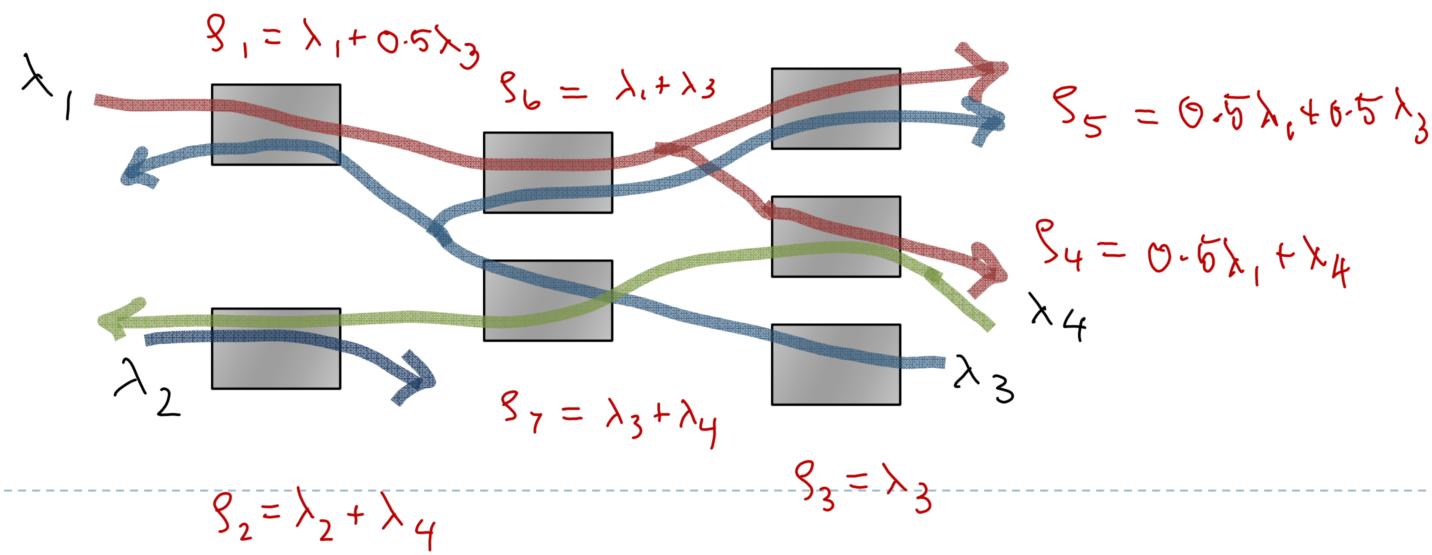
Network without constraints

- Network of n queues
  - Exogenous Poisson packet arrival process for each queue
    - Packets are of unit size (require unit amount of service)
  - Each queue can serve packets in discrete time
    - One packet per unit time (= time slot)
    - *Without any further constraint*
  - Served packets depart or join another queue



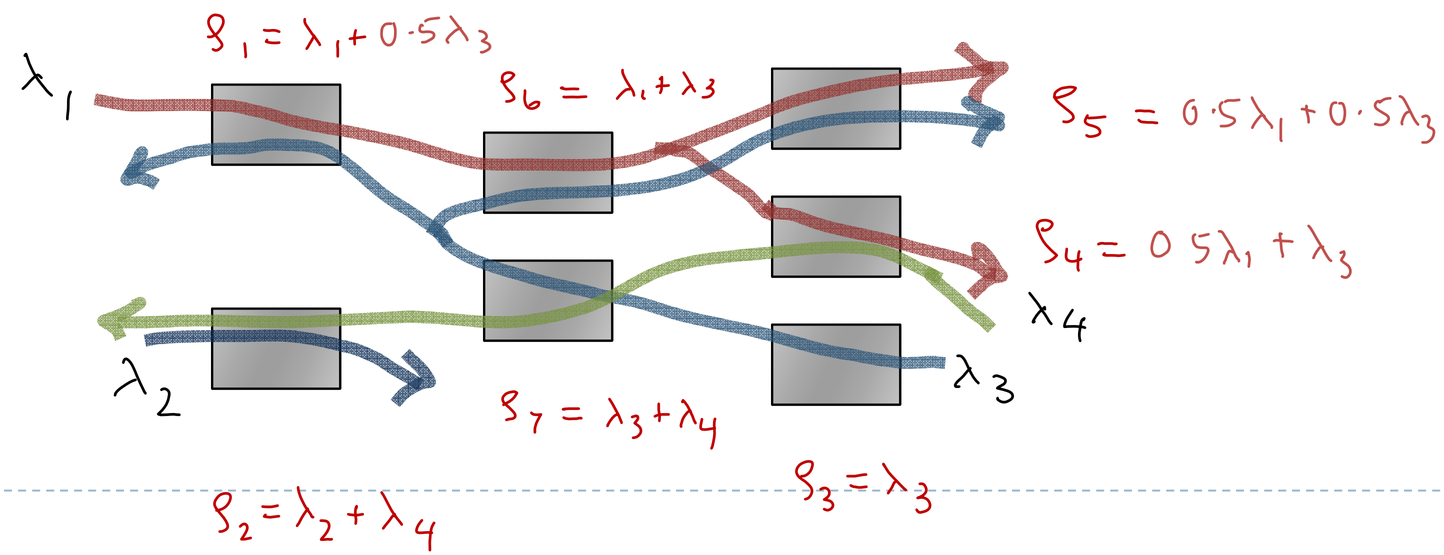
Network without constraints

- Network of n queues
  - Exogenous Poisson packet arrival process for each queue
  - Each queue can serve one packet per time slot
    - Without any further constraint
  - Scheduling required *inside* each queue
    - To decide which amongst the waiting packets to serve first



Network without constraints

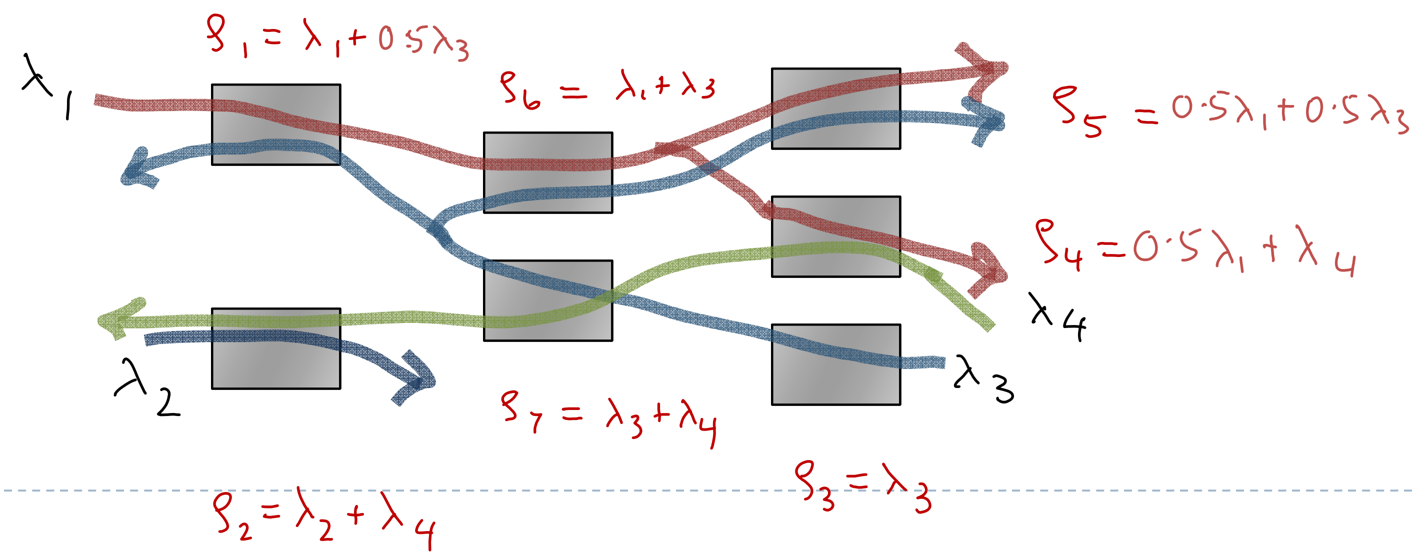
- Network of n queues in *continuous* time
  - Exogenous Poisson packet arrival process for each queue
  - Each queue has unit service capacity
  - Scheduling *inside* each queue as per
    - Pre-emptive Last In First Out (PL)
    - Which may serve a packet *in parts* unlike in discrete time



Network without constraints

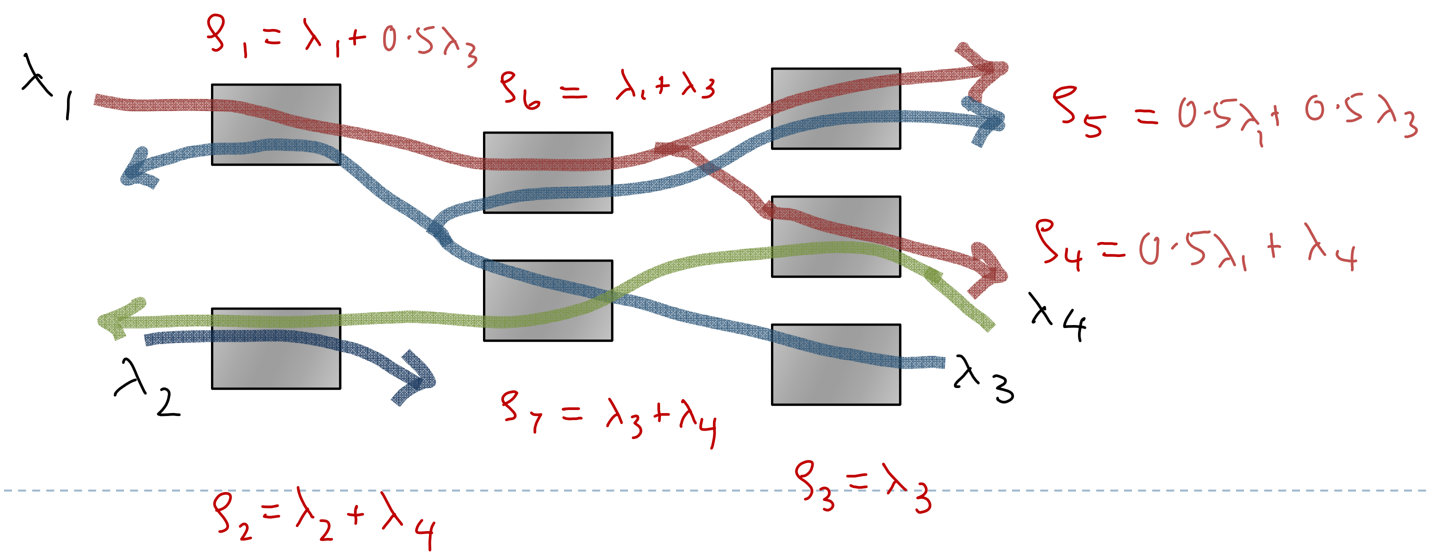
- Network of n queues in *continuous* time
  - PL Scheduling *inside* each queue
    - Quasi-reversible queues (cf. Kelly '78)
  - Stationary distribution is *product-form* (cf. BCMP '74, Kelly '78)

$$\mathbb{P}(Q_1=k_1,\dots,Q_7=k_7) \propto \prod_{j=1}^{7} \mathbb{P}(Q_j=k_j) \sim \prod_{j=1}^{7} \rho_j^{k_j}$$



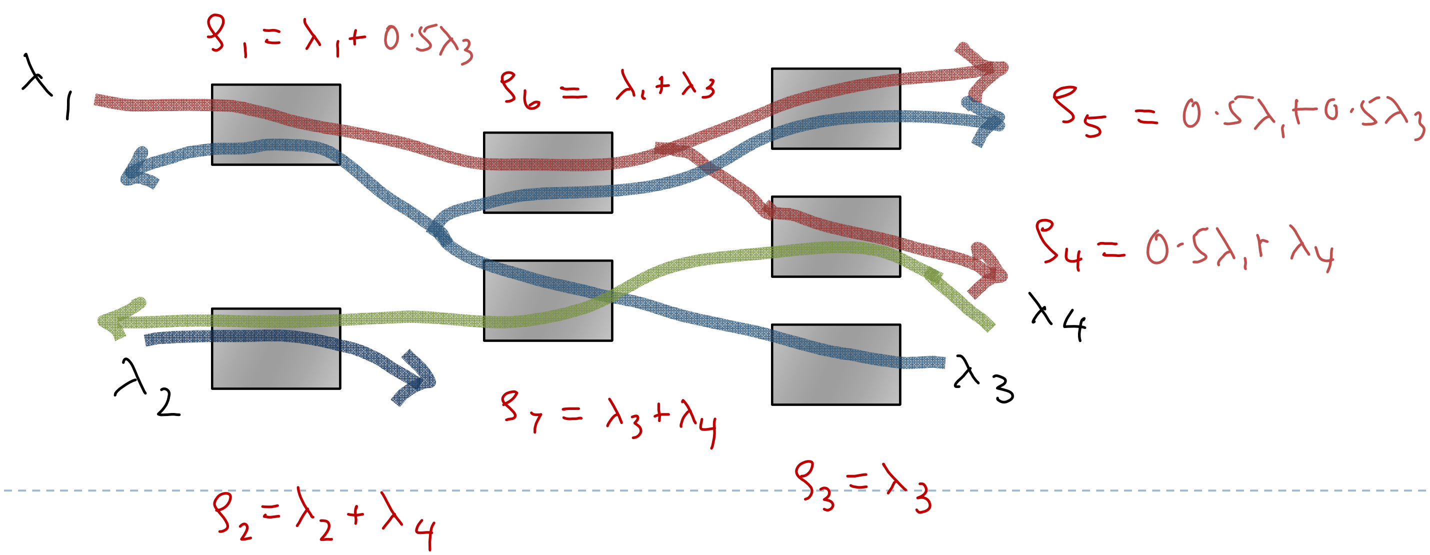
Network without constraints

- Network of n queues in *continuous* time
  - PL Scheduling *inside* each queue
  - The *product-form* distribution implies that
    - The average delay $E[D_i] = \sum_{j:\, j \in i} \frac{1}{1-\rho_j}$ for each route i
    - If all $\rho_j = \rho$, then delay of route i scales as (num of hops)/(1-ρ)



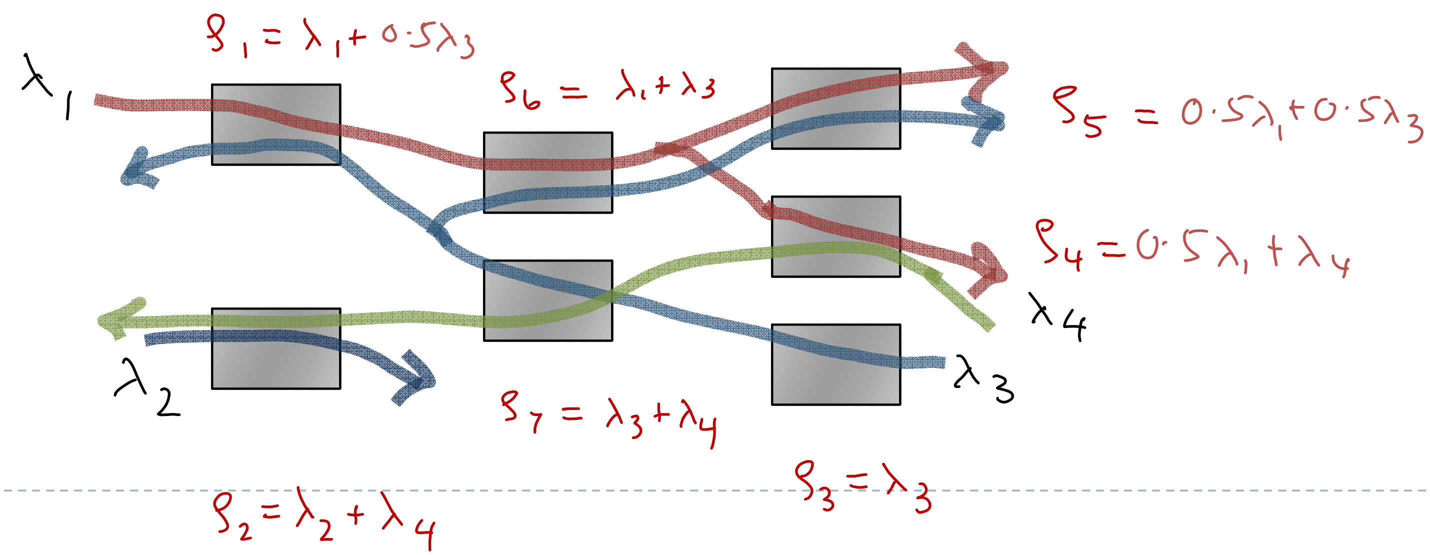
Network without constraints

- Network of n queues in *continuous* time
  - PL Scheduling *inside* each queue
  - The *product-form* distribution implies that
    - The average delay of route i scales as (num of hops)/(1-ρ)
  - Can we obtain similar performance for *discrete* time setting ?
    - That is, serving each packet in entirety



Network without constraints

- Emulation Lemma.
  - It is possible to design scheduling at each queue so that
    - The time a packet departs from *each* queue in discrete time network
    - Is at most 1 more than that in the corresponding
      - continuous time network with each node operating as per PL policy
  - This "coupling" is distribution independent



Emulation Lemma

- The scheduling algorithm in discrete time network
  - Schedule at each queue as per the Last In First Out policy
  - With respect to $\lceil A \rceil$, where A is the arrival time of a packet
    - In this queue in the continuous time network operating with PL policy
  - Ties broken as per continuous time network

- In summary
  - By simulating continuous time network (in a causal manner)
    - It is possible to achieve delay per (packet-)flow
    - That is proportional to (num of hops)/$(1-\rho)$

Network without constraints

- The achievable delay scaling
  - (num of hops)/(1-ρ)
- For M/M/1 queues in tandem
  - This is the best achievable
- For queues in tandem serving packets
  - Delay scales as (num of hops) + 1/(1-ρ)
    - The "pipe-lining" effect
- Question: which is the right scaling?
  - Single "bottleneck" link entirely avoids this

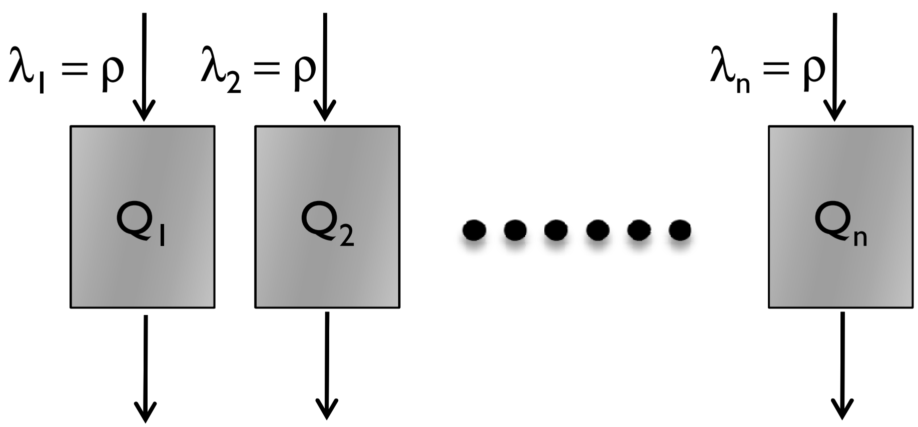
Network with constraints

- Network of n queues
  - Exogenous Poisson packet arrival process for each queue
    - Packets are of unit size (require unit amount of service)
  - Each queue can serve packets in discrete time
    - One packet per unit time (= time slot)
  - Scheduling constraints
    - Let $\sigma=[\sigma_i] \in \{0,1\}^n$ be subset of queues served
    - Then
      - σ must satisfy certain constraints : represented by $\sigma \in \mathbf{S} \subseteq \{0,1\}^n$

- Question: how does the "optimal" queue-size/delay scale
  - Depending upon **S** and gap to the capacity $(1-\rho)$

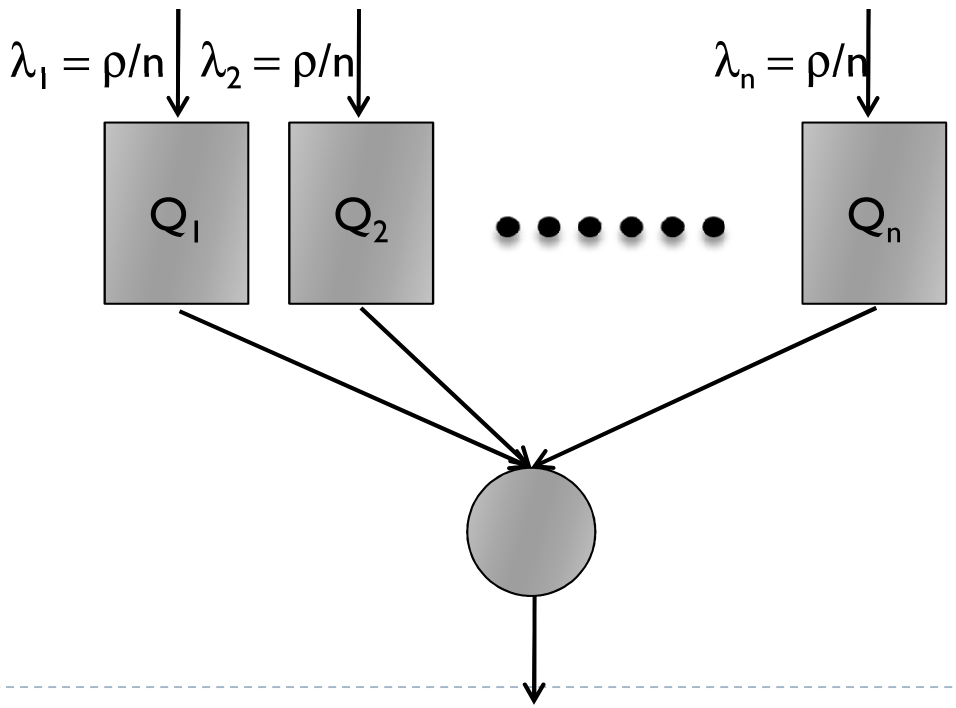
Network with constraints

- Example 1:
  - Parallel queues, n of them
  - The net average queue-size $Q_1 + \ldots + Q_n \approx n/(1-\rho)$



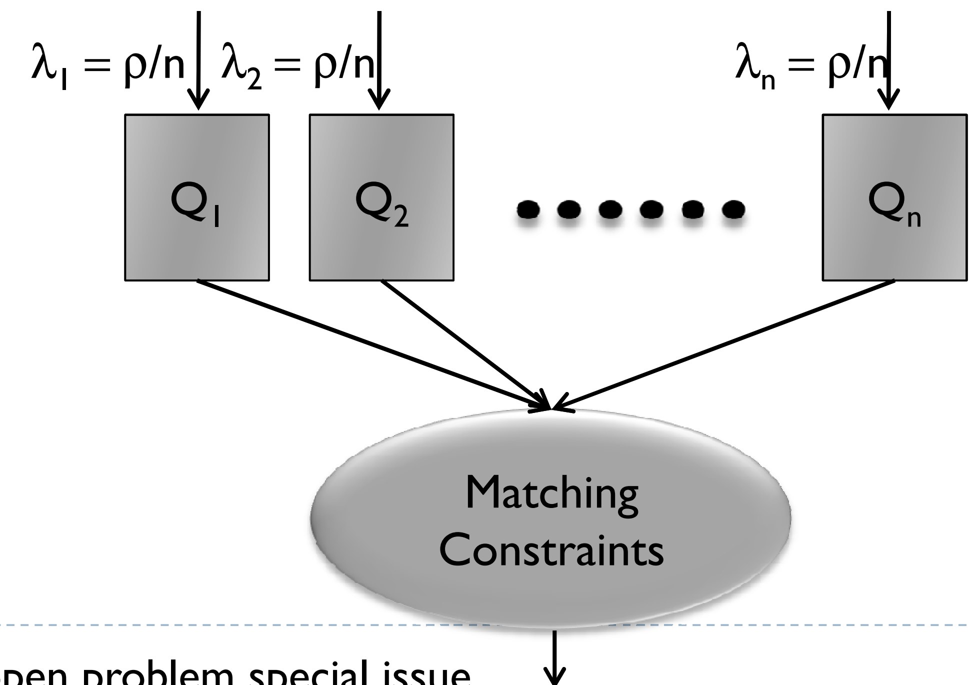
Network with constraints

- Example 2:
  - One server, n queues
  - The net average queue-size: $Q_1 + \ldots + Q_n \approx 1/(1-\rho)$



Network with constraints

- Example 3:
  - N x N switch: $n=N^2$ queues
  - Average queue-size: $Q_1+\ldots+Q_n$ *conjectured** to be $N/(1-\rho)$
    - Known upper bound: $N^2/(1-\rho)$
    - Known lower bound: $N/(1-\rho)$



- * = QUESTA open problem special issue

Network with constraints

- Network of n queues
  - With scheduling constraints represented by
    - Schedule $\sigma \in \mathbf{S} \subseteq \{0,1\}^n$
- The convex hull of **S** is the capacity region
  - Let it be represented as (polytope)
    - $\Lambda = \{x \in [0,1]^n : Ax \leq C\}$ with
      - A non-negative m x n matrix
      - C non-negative valued m-vector
- Effectively, any scheduling policy imposes constraint
  - Service rate $\sigma \in \Lambda$ (with abuse of notation)

Network with constraints

- Proportional fair policy: each time
  - Choose schedule so that induced service rate σ is such that
    - It maximizes objective $\Sigma_i \, Q_i \log \sigma_l$ over all $\sigma \in \Lambda$
  - This is achieved by a simple randomized policy
    - Find σ that solves above optimization problem
    - Decompose σ as convex combination of actions in **S**
      - $\sigma = \Sigma_k \alpha_k \pi_k$ for $\pi_k \in$ **S** with $\Sigma_k \alpha_k = 1$
    - Choose π_k with probability α_k

- This has been well analyzed by
  - Bonald-Massoulie '01, Kelly-Williams '04, Massoulie '06, Kang-Kelly-Lee-Williams '08, Ye-Yao '08

Network with constraints: prop. fair

- Kang-Kelly-Lee-Williams '08
  - Considered *heavy traffic* limit of such a network
    - With *multiple* links bottle-necked
    - Assumed
      - Matrix A full rank
      - Local traffic condition: for each j, there exists i s.t. $A_{ij} > 0, A_{ij'} = 0$ for all $j' \neq j$
  - Characterized product-form stationary distribution
    - For diffusion approximation
    - Further, it is limit of stationary distribution of the original system
    - That is, *exchange of limits* is valid

      (Shah-Tsitsiklis-Zhong '11)

Network with constraints: prop. fair

- The product-form stationary distribution implies
  - The average queue-size is

$$\mathbb{E}[Q_i] \approx \lambda_i \sum_j \frac{A_{ji}}{c_j - (A\lambda)_j}$$

$$\leq \left|\{ j : A_{ji} \neq 0 \}\right| \cdot \max_j \left(\frac{\lambda_i A_{ji}}{c_j - (A\lambda)_j} \right)$$

  - And, for any policy

$$\mathbb{E}[Q_i] \geq \max_j \frac{\lambda_i A_{ji}}{c_j - (A\lambda)_j}$$

- That is, prop. fair is optimal
  - Up to the "number of hops" (Kang-Kelly-Lee-Williams '08)

Network w constraints: prop. fair

- Back to conjecture for switch
  - Assuming the KKLW '08 holds for N x N switch
    - Using Proportional fair scheduling policy
  - The net average queue-size would turn out to be
    - $2N/(1-\rho)$: matches the conjecture !

- Recent progress (Shah-Tsitsiklis-Zhong 'xx)
  - For uniform loading with $(1-\rho) = 1/N$
    - We show that the net average queue-size is $N^{17/6}$
    - Recall (for $(1-\rho) = 1/N$)
      - What was known: N^3
      - Conjecture is: N^2

Network w constraints: implementation

- A reasonable policy
  - At each time choose schedule $\sigma \in \mathbf{S}$ such that
    - It maximizes objective $\Sigma_l F(\sigma_i)$
    - For some function F which may depends on queue-size, etc.

- Implementation:
  - How to choose this schedule each time
    - Using simple algorithm
      - Low complexity
      - Minimal data-structure
    - Preferably in a distributed manner
      - With little protocol co-ordination overhead

Network w constraints: implementation

- Product-form distribution
  - Consider a Markov chain on **S** with stationary distribution

$$\mathbb{P}(\sigma) \propto \exp\left(\sum_i F(\sigma_i) \right)$$

  - Then
    - Variational characterization of such distribution suggests

$$\mathbb{E}_{\mathbb{P}}\left[\sum_i F(\sigma_i) \right] \geq \left(\max_{\pi \in \mathbf{S}} \sum_i F(\pi_i) \right) - \log |S|$$

  - That is, effectively by *sampling* schedule at each time
    - As per stationary distribution of this Markov chain is what we want

Network w constraints: implementation

- Two issues
  - Designing Markov chain with such product-form distribution
    - Reversible construction a la Metropolis-Hasting's Rule
    - The transitions of such a Markov chain are essentially distributed
      - Separable objective is particularly useful for this property
  - Sampling from stationary distribution of Markov chain
    - The objective keeps changing every time
    - And Markov chain makes only few transitions per unit time
    - By choice of slowly varying objective F
      - It is possible to essentially sample from stationary distribution at all times (Shah-Shin '08, '10; Jiang-Walrand '08)

Discussion

- Reversible networks are useful
  - Primarily because of their product-form stationary distribution
    - Calculate average delay
      - Network without constraints
      - Network with constraints using proportional fair policy
    - Choose schedule that maximizes appropriate objective

- Reversible networks are, however, too specific
  - Therefore, *approximate* characterization can be quite useful
    - In expanding scope of these results
  - One such approximation is obtained means of
    - "Comparison" property (Shah-Shin-Tetali '11)