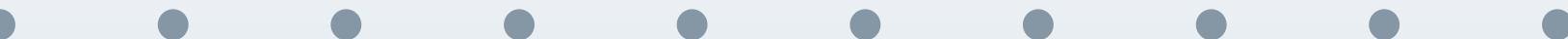


Recent results on volatility change point estimation for stochastic differential equations

Stefano M. Iacus (University of Milan &  R Core Team)

Workshop on Parameter Estimation for Dynamical Systems, Eurandom, 4-6 June 2012



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Discover from the data a structural change in the generating model.

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Discover from the data a structural change in the generating model.
Next example shows historical change points for the Dow-Jones index.

Change point problem

The change point
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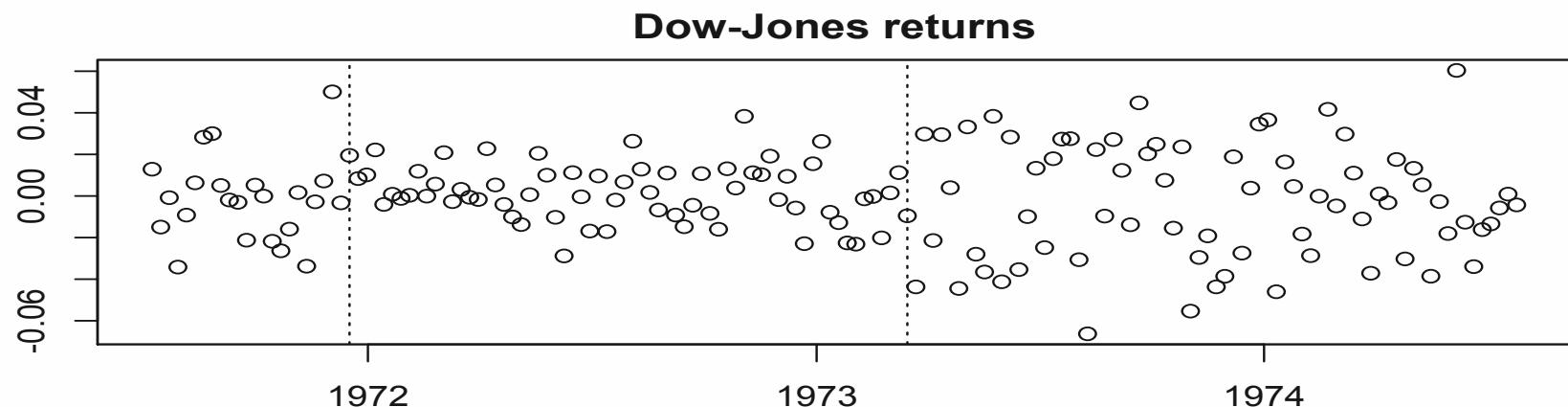
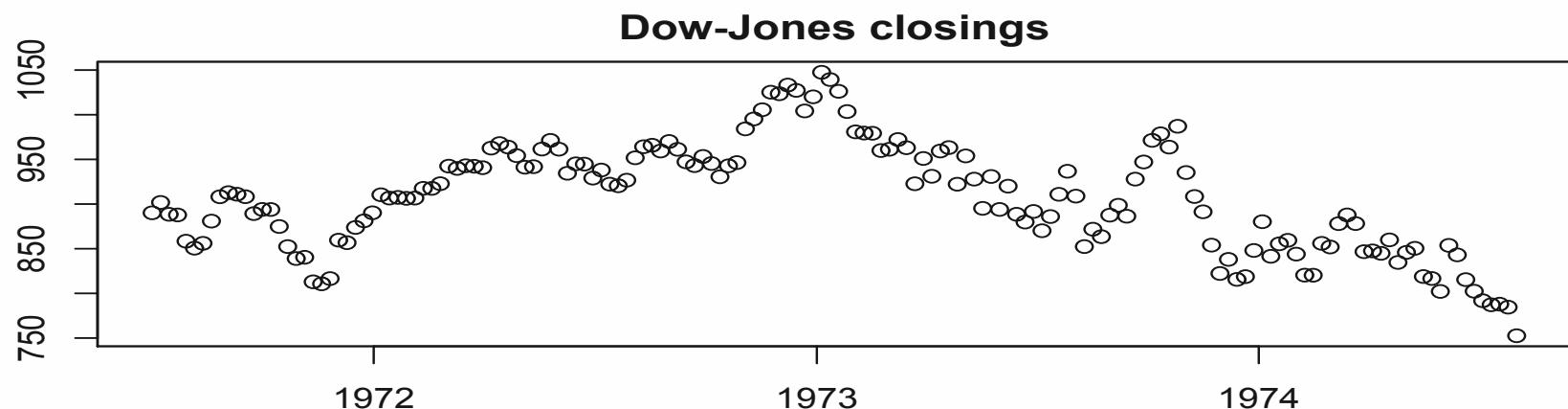
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Discover from the data a structural change in the generating model.
Next example shows historical change points for the Dow-Jones index.



break of gold-US\$ linkage (left)

Watergate scandal (right)

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Originally, the problem was considered for i.i.d samples; see Hinkley (1971), Csörgő and Horváth (1997), Inclan and Tiao (1994)

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It moved quickly to time series analysis: Bai (1994, 1997), Kim *et al.* (2000), Lee *et al.* (2003), Chen *et al.* (2005) and the papers cited therein.

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The list is vastly incomplete.

Main approaches: least squares, MLE, Bayesian and CUSUM

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For continuous time observations, Kutoyants (1994, 2004) considered change point (*in space*) problem for the drift of an ergodic diffusion process solution to

$$dX_t = S(\theta, X_t)dt + \sigma(X_t)dW_t, \quad X_0, \quad t \geq 0 \quad (1)$$

with the trend function discontinuous along the two points of the state space of X_t , say $x_*^{(1)}(\theta)$ and $x_*^{(2)}(\theta)$, $\theta \in [\alpha, \beta] \subset \mathbb{R}$ and the interest is in the estimation of θ .

Lee, Nishiyama and Yoshida (2003) considered CUSUM tests statistic for the *time* change point for the model in (1) where θ takes different values before and after a time instant τ^* .

Habibi R. (2012) considered a recursive-MLE approach for the estimation of the change point in the parameter of the drift from 1-d continuous time observations in the very special case of gBm with stochastic volatility:
 $dX_t = \theta X_t dt + \sigma_t X_t dW_t$.

Estimation of volatility under discrete time sampling

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The problem of parametric estimation of the volatility (and drift) for discretely observed diffusion processes under different sampling schemes, dates back to the works of Genon-Catalot and Jacod (1993, 1994), Yoshida (1992) and Kessler (1997).

Change point analysis for the volatility appears in the few works reviewed in the following.

The main peculiarity is the non regularity of the statistical model with respect to the structural change, i.e. the (quasi-)likelihood is not smooth with respect to the parameter τ , the change point instant. This fact add some technicalities in the proofs and different limiting distributions are obtained and rates of convergence are faster than usual \sqrt{n} .

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De Gregorio and I. (2008) considered the change point problem for the ergodic model

$$dX_t = b(X_t)dt + \sqrt{\theta}\sigma(X_t)dW_t, \quad 0 \leq t \leq T, X_0 = x_0,$$

observed at discrete time instants $t_i = i\Delta_n, i = 0, \dots, n, \Delta_n = t_{i+1} - t_i$ under the sampling scheme $\Delta_n \rightarrow 0, n \rightarrow \infty, n\Delta_n = T, T$ fixed. The coefficients b and σ are supposed to be known. The change point problem is formulated as follows

$$X_t = \begin{cases} X_0 + \int_0^t b(X_s)ds + \sqrt{\theta_1} \int_0^t \sigma(X_s)dW_s, & 0 \leq t \leq \tau^* \\ X_{\tau^*} + \int_{\tau^*}^t b(X_s)ds + \sqrt{\theta_2} \int_{\tau^*}^t \sigma(X_s)dW_s, & \tau^* < t \leq T \end{cases}$$

where $\tau^* \in (0, T)$ is the **change point** and θ_1, θ_2 two parameters to be estimated. The approach follows the lines of Bai (1994, 1997).

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Let $Z_i = \frac{X_{i+1} - X_i - b(X_i)\Delta_n}{\sqrt{\Delta_n}\sigma(X_i)}$ be the standardized residuals. Let $k_0 = [n \cdot \tau^*]$, then the LS estimator is obtained via

$$\begin{aligned}\hat{k}_0 &= \arg \min_k \left(\min_{\theta_1, \theta_2} \left\{ \sum_{i=1}^k (Z_i^2 - \theta_1)^2 + \sum_{i=k+1}^n (Z_i^2 - \theta_2)^2 \right\} \right) \\ &= \arg \min_k \left\{ \sum_{i=1}^k (Z_i^2 - \bar{\theta}_1)^2 + \sum_{i=k+1}^n (Z_i^2 - \bar{\theta}_2)^2 \right\},\end{aligned}\quad (2)$$

where

$$\bar{\theta}_1 = \frac{1}{k} \sum_{i=1}^k Z_i^2 =: \frac{S_k}{k} \quad \text{and} \quad \bar{\theta}_2 = \frac{1}{n-k} \sum_{i=k+1}^n Z_i^2 =: \frac{S_{n-k}^*}{n-k}.$$

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Solving (2) is equivalent to solve the following

$$D_k = \frac{k}{n} - \frac{S_k}{S_n}.$$

Therefore, least squares change point estimator \hat{k}_0 ($\hat{\tau}_n = \hat{k}_0/n$) is given by

$$\hat{k}_0 = \arg \max_k |D_k|$$

Theorem: Under $H_0 : \theta_1 = \theta_2$, i.e. “no change point”, we have that

$$\sqrt{\frac{n}{2}} |D_k| \xrightarrow{d} |B_0(\tau)|, \quad B_0(\tau) \text{ Brownian bridge on } [0, 1]$$

The estimator $\hat{\tau}_n$ is consistent.

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Let $\vartheta_n = |\theta_2 - \theta_1|$. Under the conditions

$$\vartheta_n \rightarrow 0, \quad \frac{\sqrt{n}\vartheta_n}{\sqrt{\log n}} \rightarrow \infty,$$

$$\frac{n\vartheta_n^2(\hat{\tau}_n - \tau^*)}{2\hat{\theta}^2} \xrightarrow{d} \arg \max_v \left\{ \mathcal{W}(v) - \frac{|v|}{2} \right\}$$

with $\hat{\theta}$ some consistent estimator of the common limit θ_0 of θ_1 and θ_2 ; \mathcal{W} is a two sided Brownian motion, i.e.

$$\mathcal{W}(u) = \begin{cases} W_1(-u), & u < 0 \\ W_2(u), & u \geq 0 \end{cases} \quad (3)$$

where W_1, W_2 are two independent Brownian motions.

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The estimators $\hat{\theta}_1, \hat{\theta}_2$ are also well behaved, i.e.

$$\sqrt{n} \begin{pmatrix} \hat{\theta}_1 - \theta_1 \\ \hat{\theta}_2 - \theta_2 \end{pmatrix} \xrightarrow{d} N(0, \Sigma), \quad \text{where} \quad \Sigma = \begin{pmatrix} 2\frac{\theta_0^2}{\tau^*} & 0 \\ 0 & 2\frac{\theta_0^2}{1-\tau^*} \end{pmatrix}.$$

The above results hold in the high frequency case $\Delta_n \rightarrow 0$ with $n \rightarrow \infty$ and $n\Delta = T$ fixed, but also for the rapidly increasing design case $n\Delta_n = T \rightarrow \infty$.

Case of unknown drift

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We assume now to observe a diffusion process that is a solution to the reduced stochastic differential equation

$$dX_t = b(X_t)dt + \sqrt{\theta}dW_t,$$

where $b(\cdot)$ is unknown and estimated using nonparametric methods. Let K be a suitable kernel, and consider the estimator

$$\hat{b}(x) = \frac{\sum_{i=1}^n K\left(\frac{X_i-x}{h_n}\right) \frac{X_{i+1}-X_i}{\Delta_n}}{\sum_{i=1}^n K\left(\frac{X_i-x}{h_n}\right)}$$

the above is a particular case of the nonparametric estimators of the drift considered in Bandi and Phillips (2003), but other approaches exist.

For fixed T , the drift $b(\cdot)$ cannot be estimated consistently, hence it is required that $T \rightarrow \infty$.

Bandwidth h_n and local time

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Let

$$\bar{L}_X(t, x) = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{\{x, x+\epsilon\}}(X_s) ds$$

be the chronological local time of X . Bandi and Phillips (2003) shown that under the following additional assumption

$$\frac{\bar{L}_X(T, x)}{h_n} \sqrt{\Delta_n \log\left(\frac{1}{\Delta_n}\right)} = o_{a.s.}(1)$$

and $h_n \bar{L}_X(T, x) \xrightarrow{a.s.} \infty$, $\hat{b}(\cdot)$ is a consistent estimator of $b(\cdot)$. In the stationary case, the above condition may be replaced by

$$\frac{T}{h_n} \sqrt{\Delta_n \log\left(\frac{1}{\Delta_n}\right)} = o_{a.s.}(1).$$

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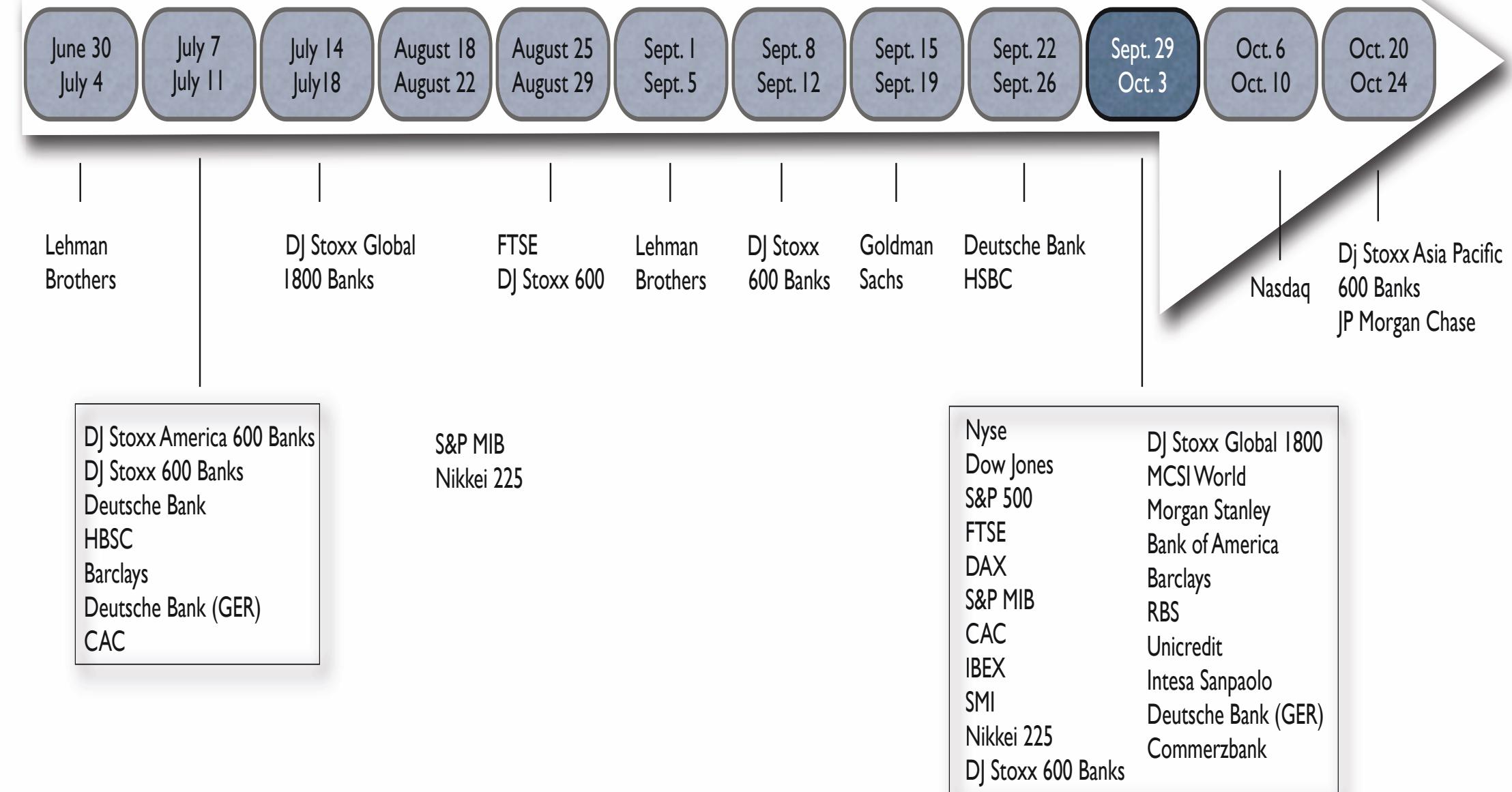
Pluggin-in the standardized residuals

$$\hat{Z}_i = \frac{X_{i+1} - X_i - \hat{b}(X_i)\Delta_n}{\sqrt{\Delta_n}}$$

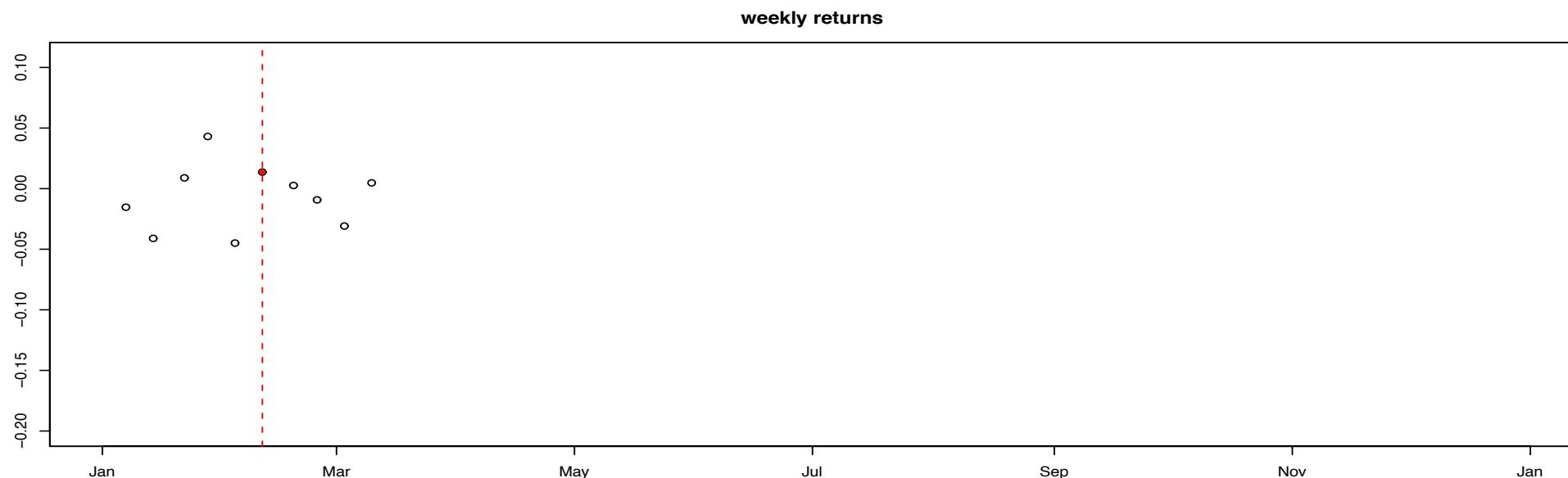
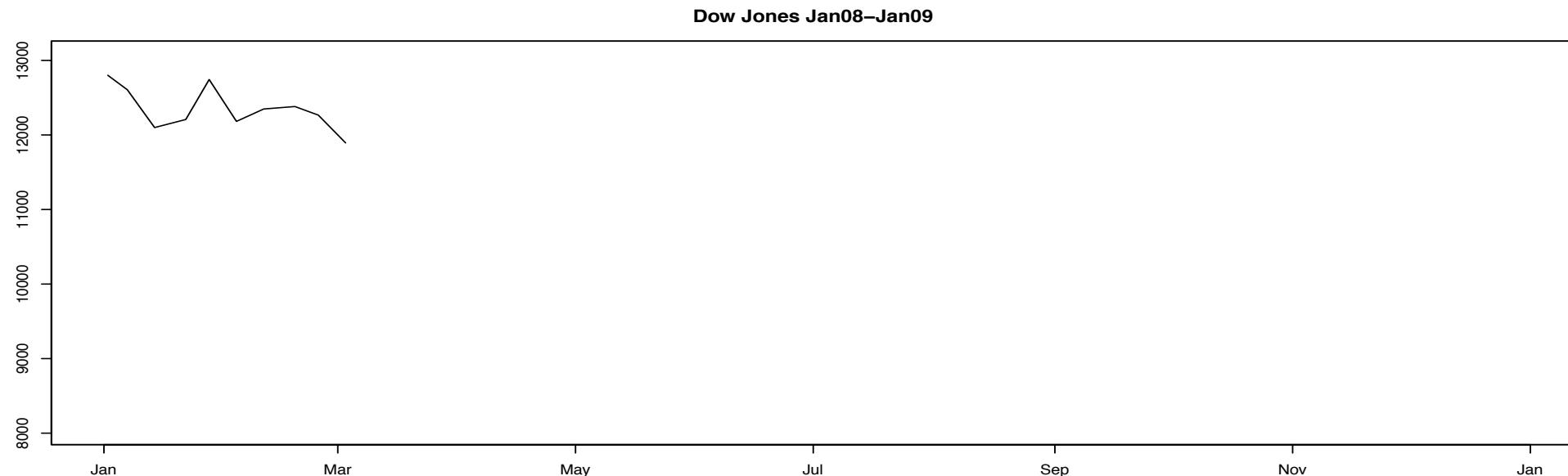
into the statistic D_k , all previous results hold.

This approach has been used in Smaldone (2009) to re-analyze the recent global financial crisis using data from different markets. It emerged that the analysis of some of the assets could have predicted by the last week of June 2008 the global structural change of late September 2008.

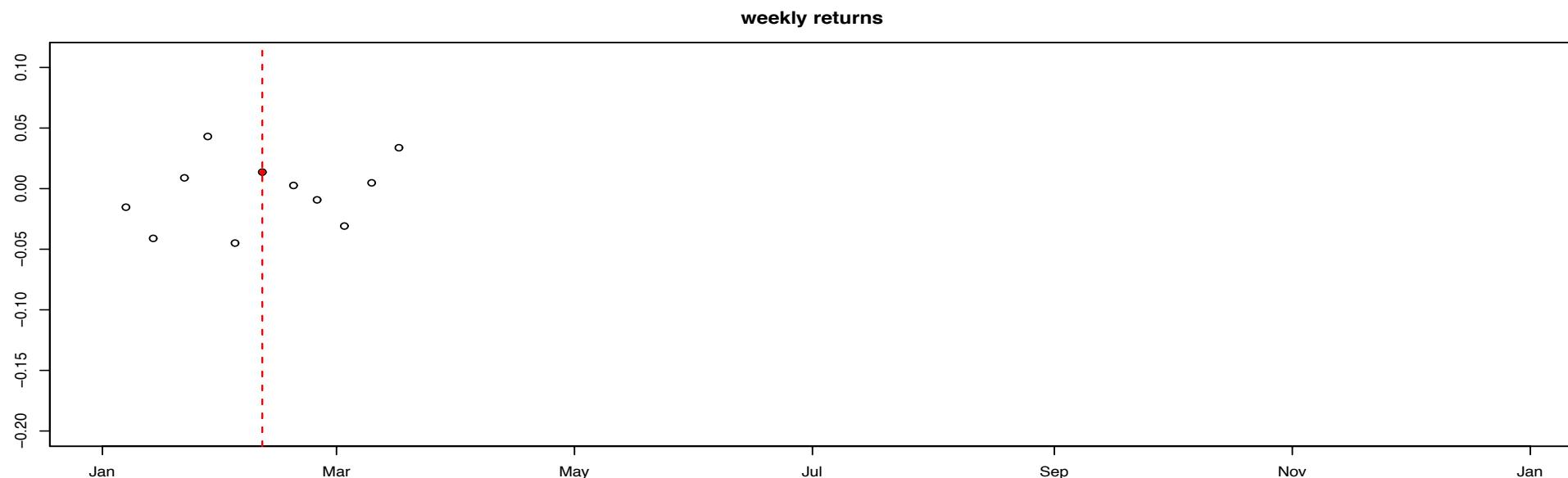
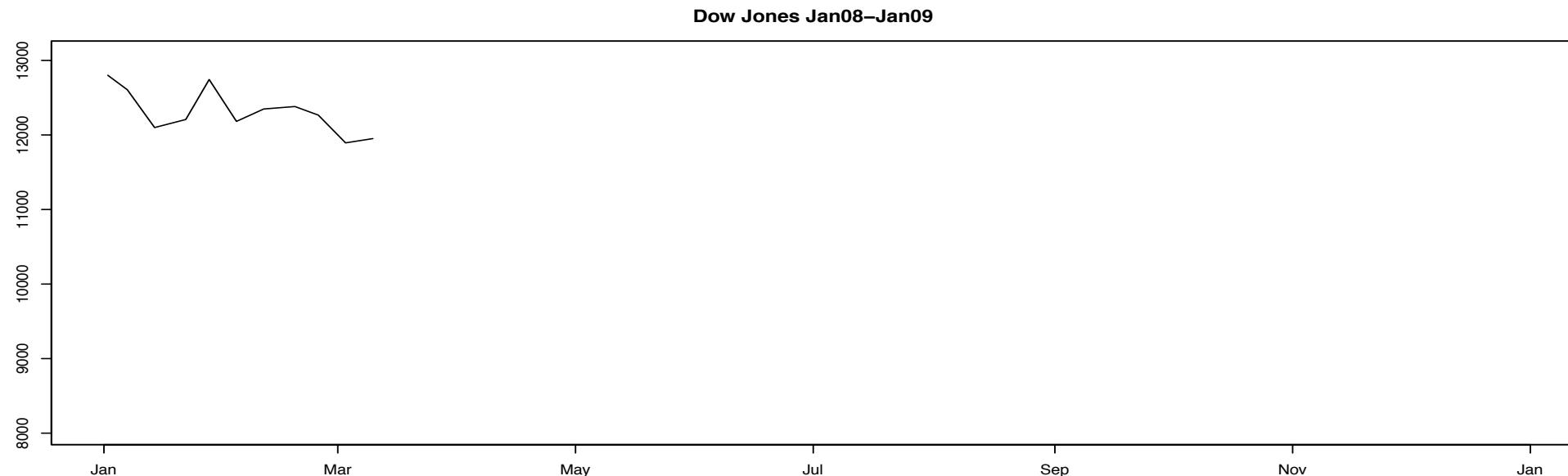
Time



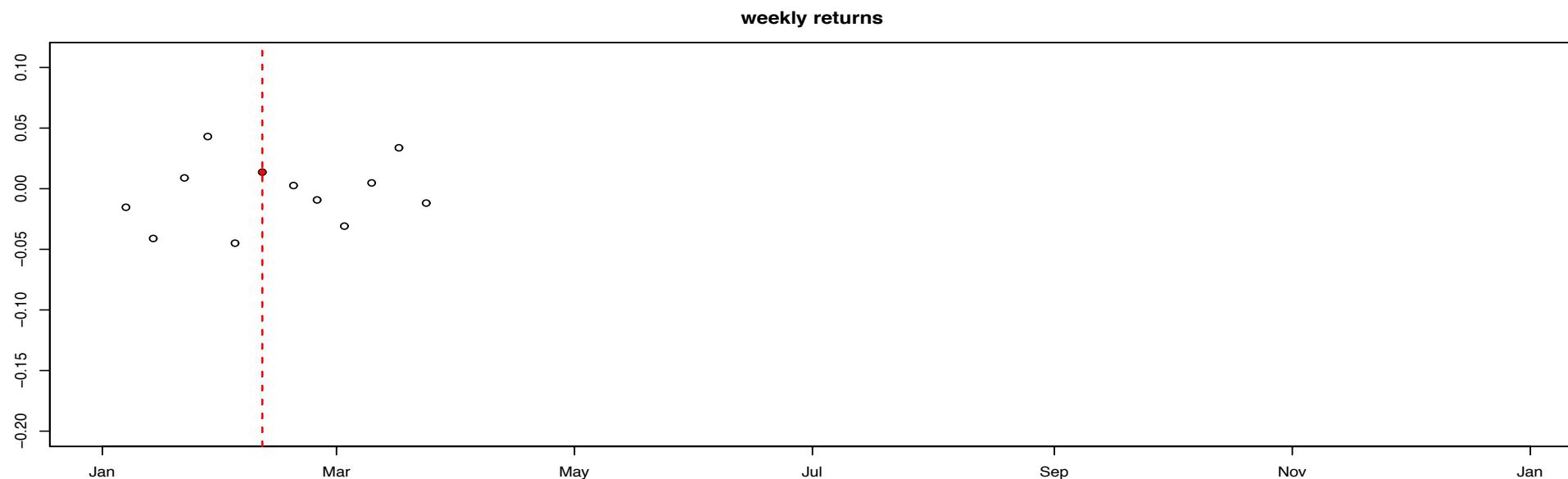
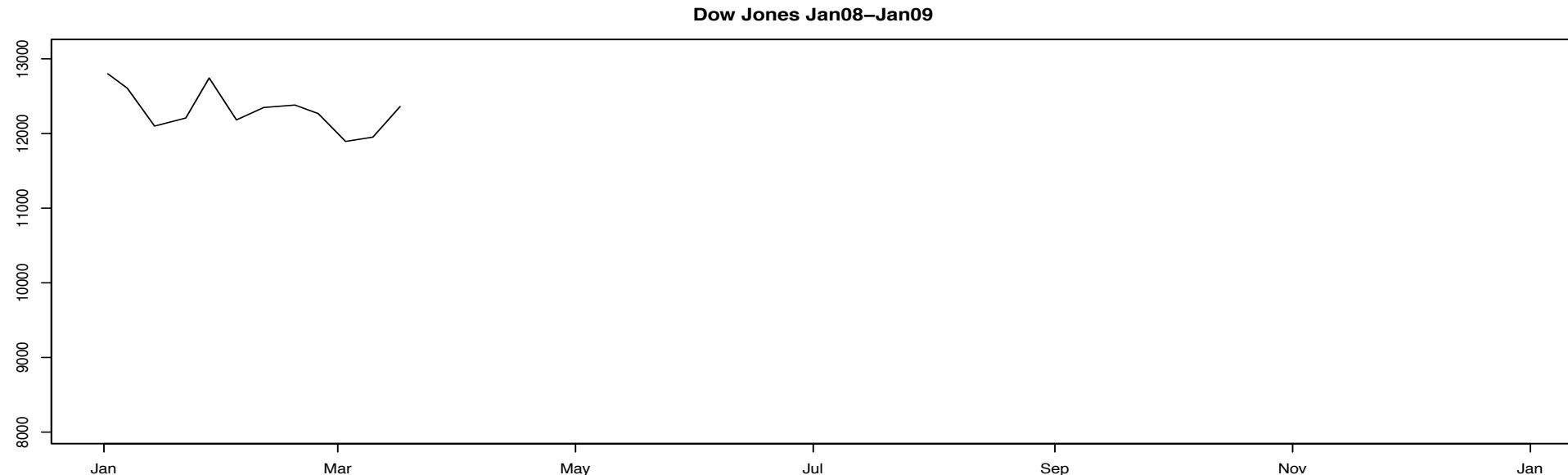
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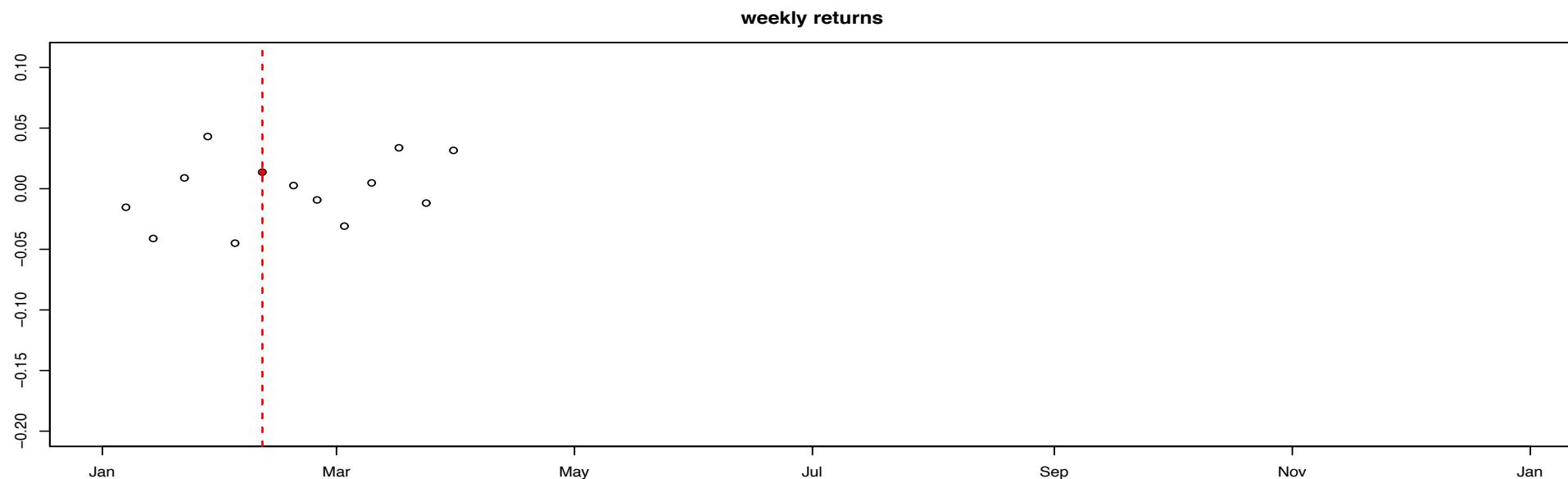
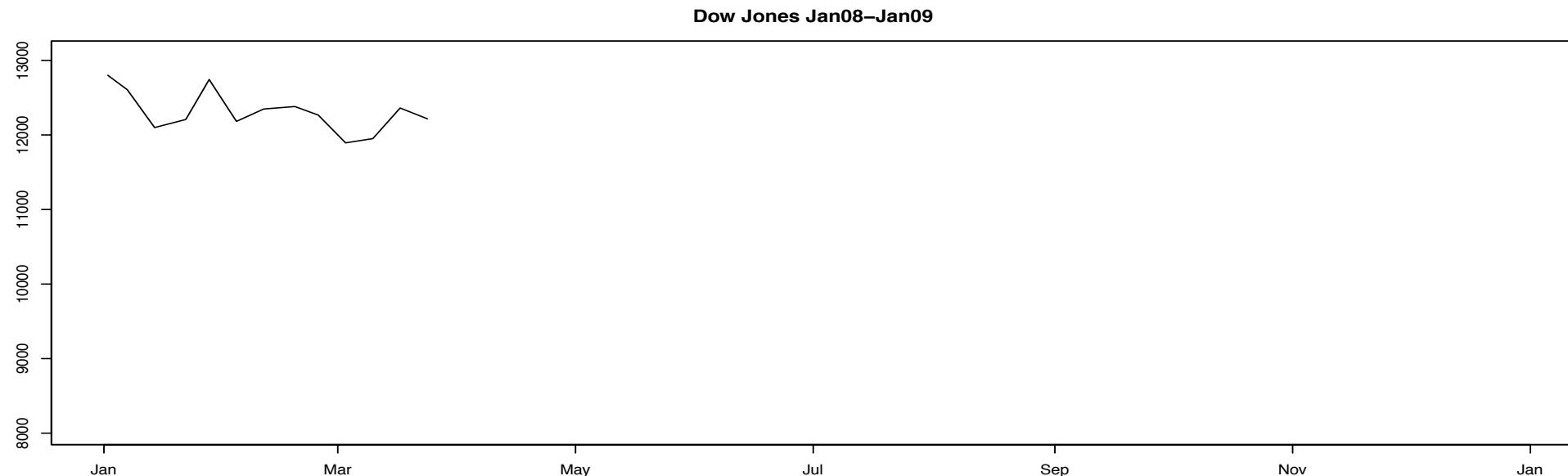
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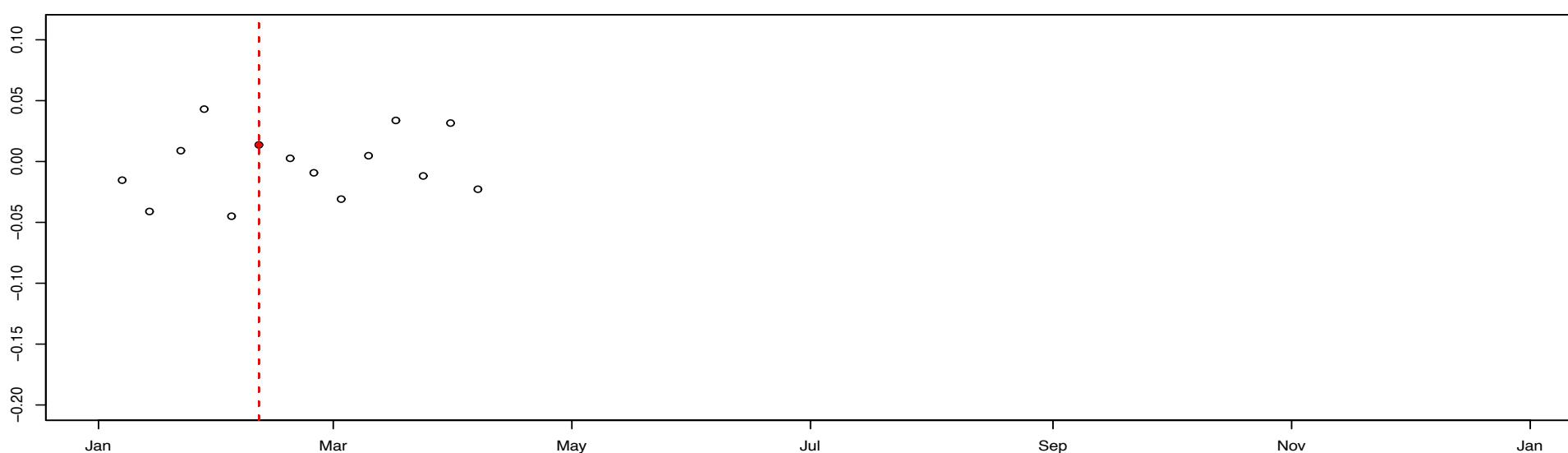
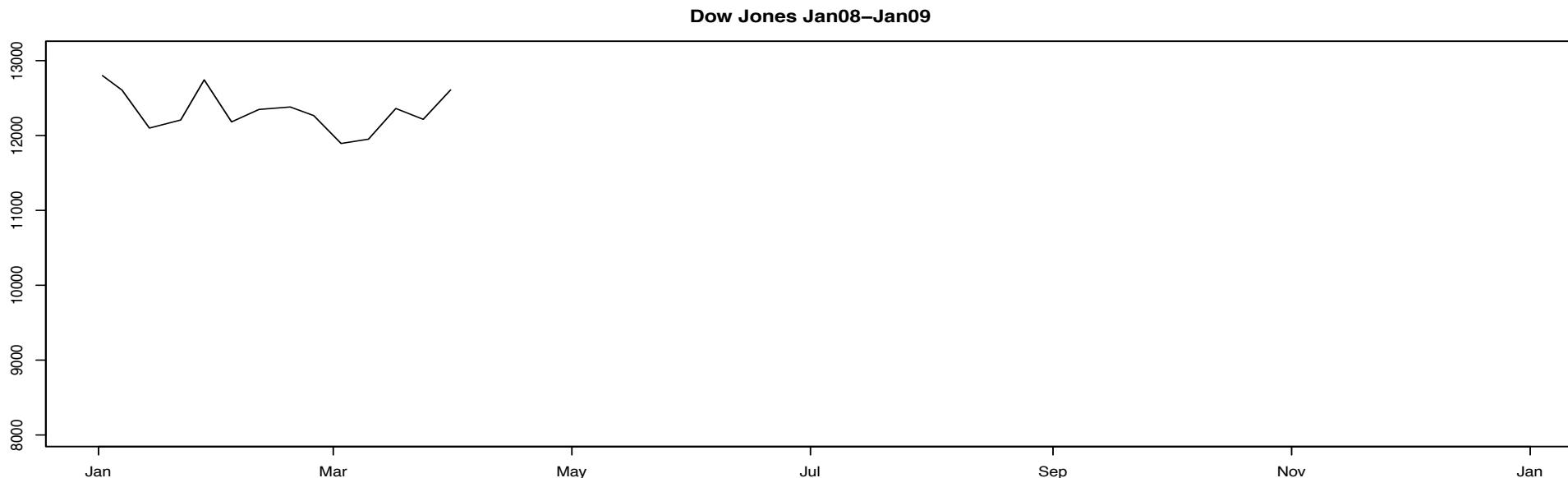
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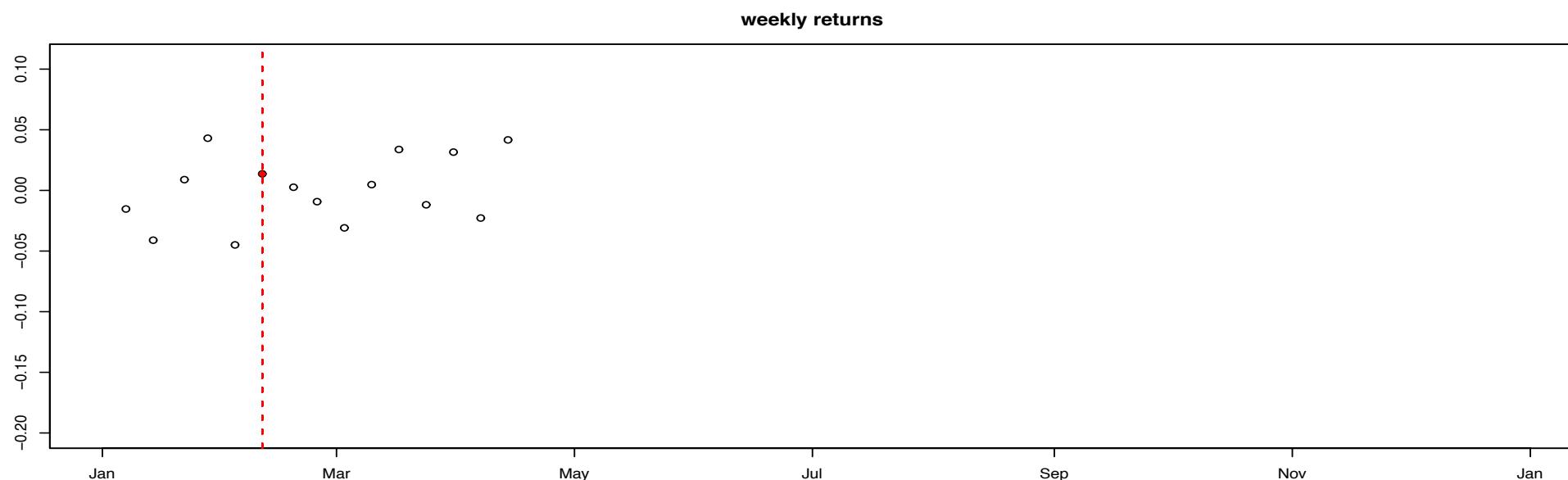
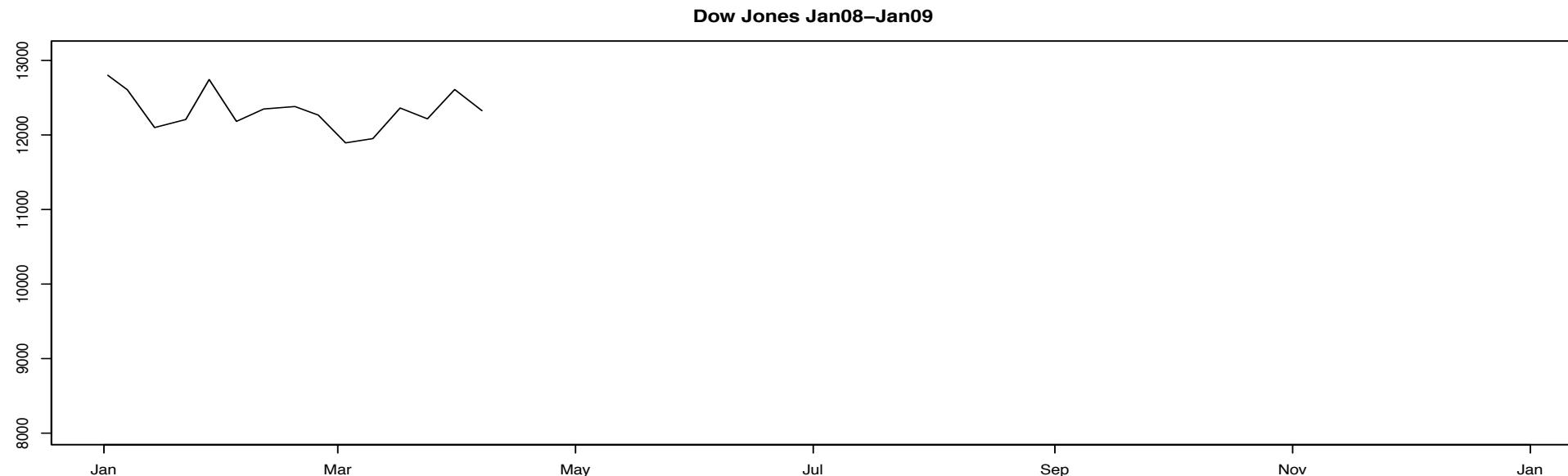
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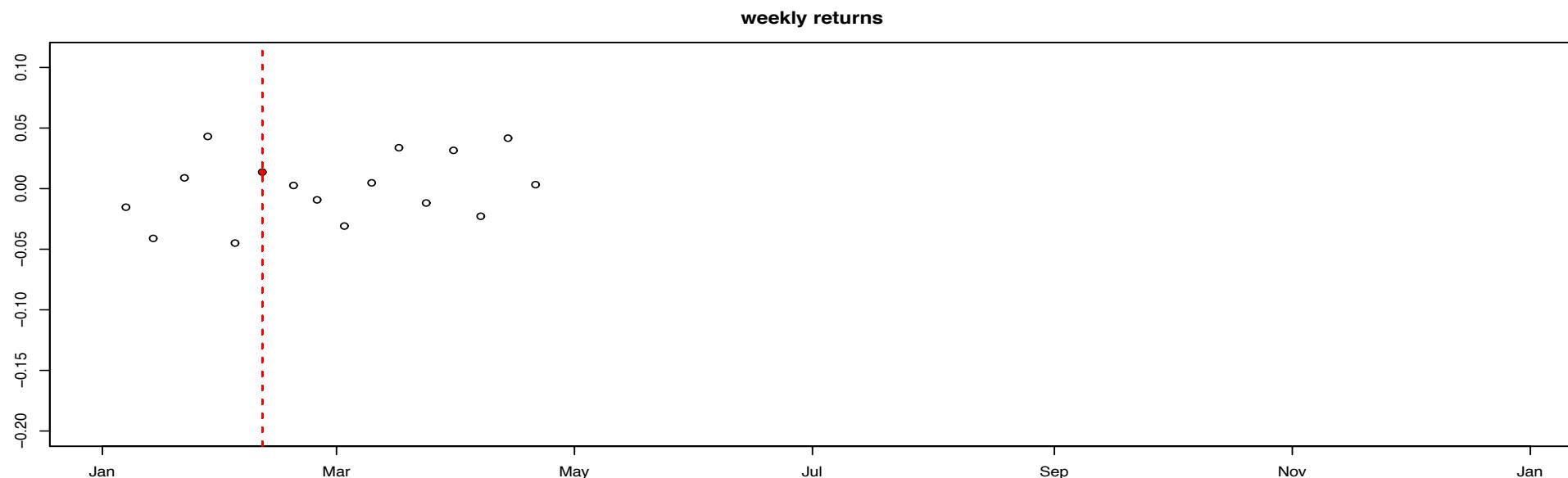
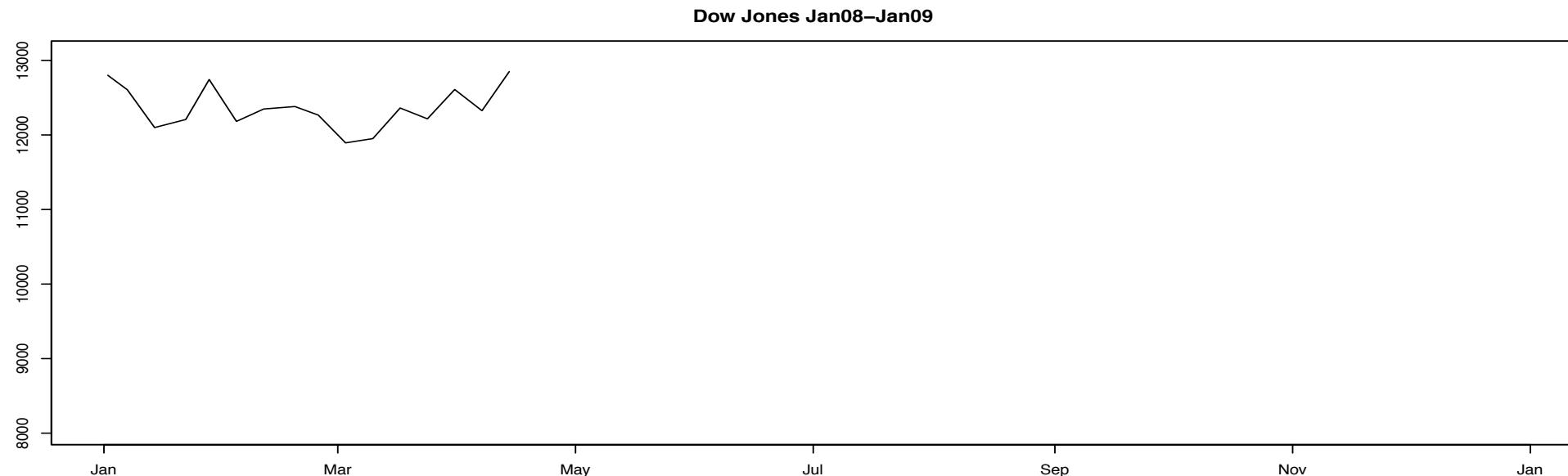
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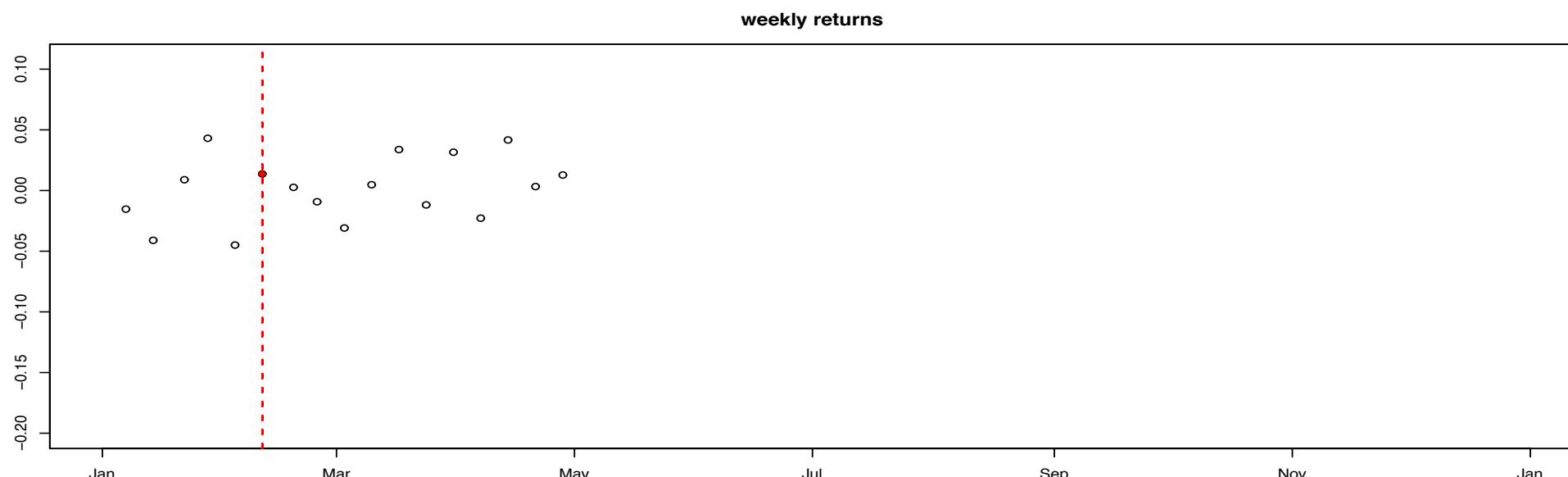
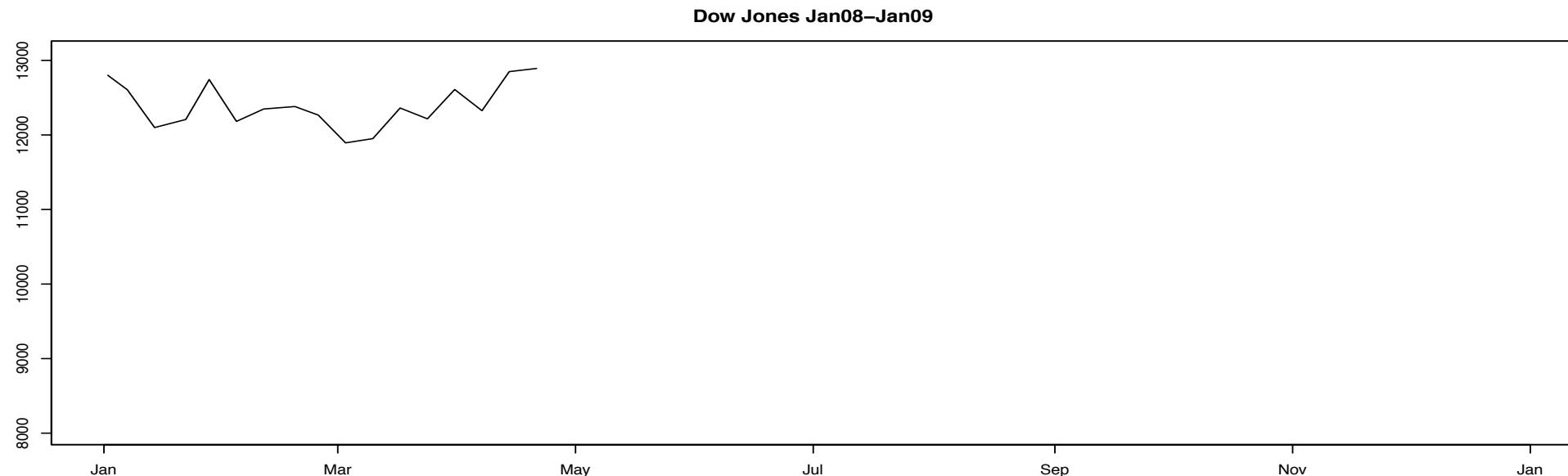
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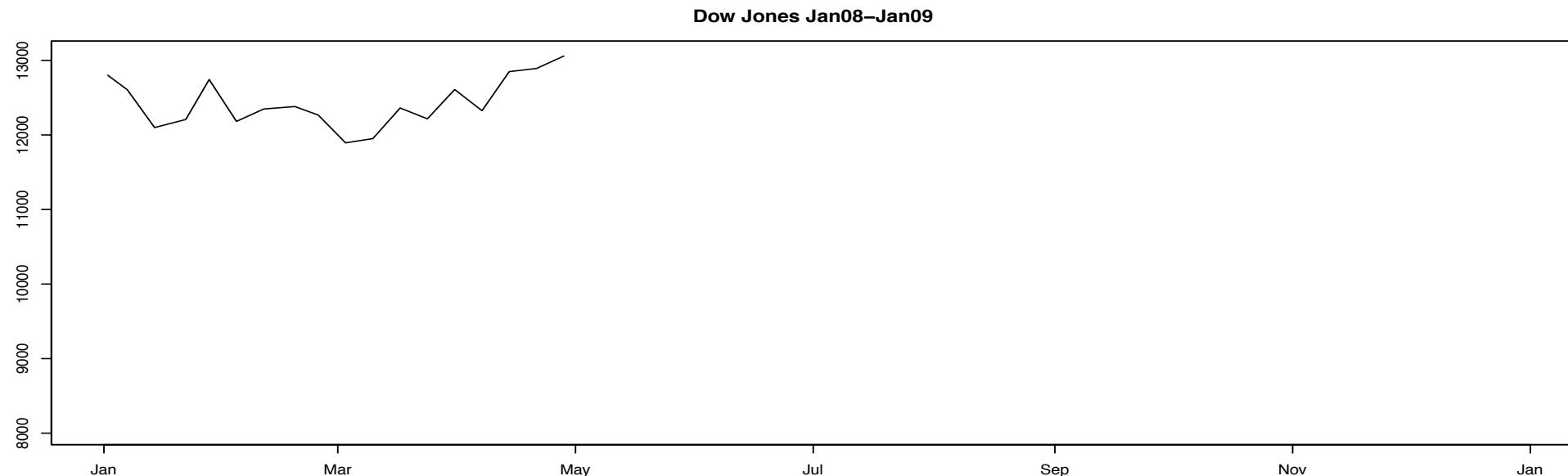
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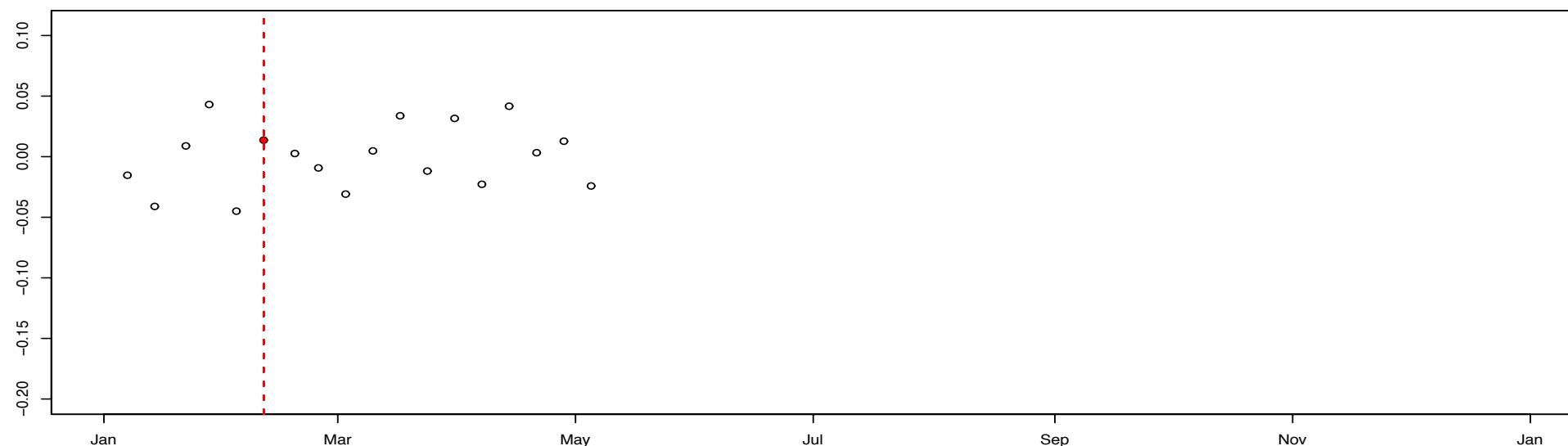
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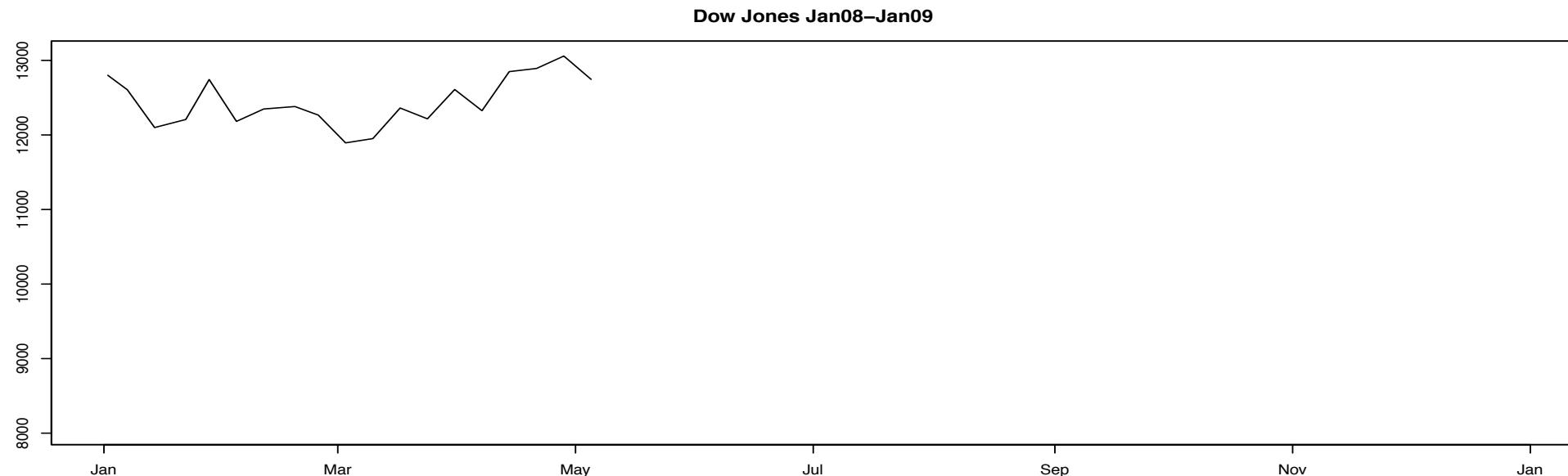
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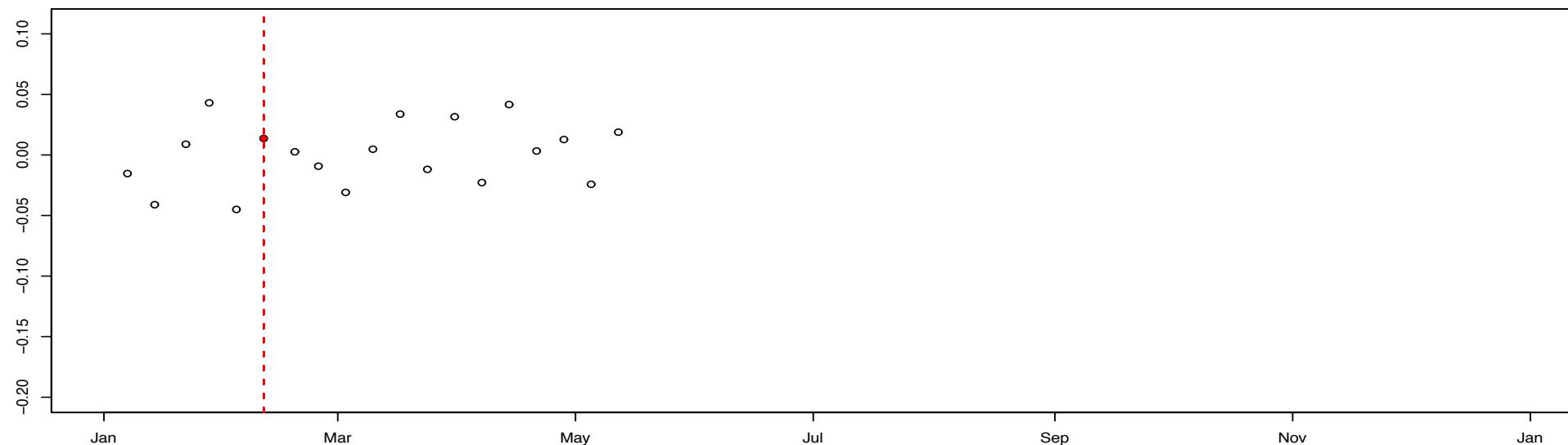
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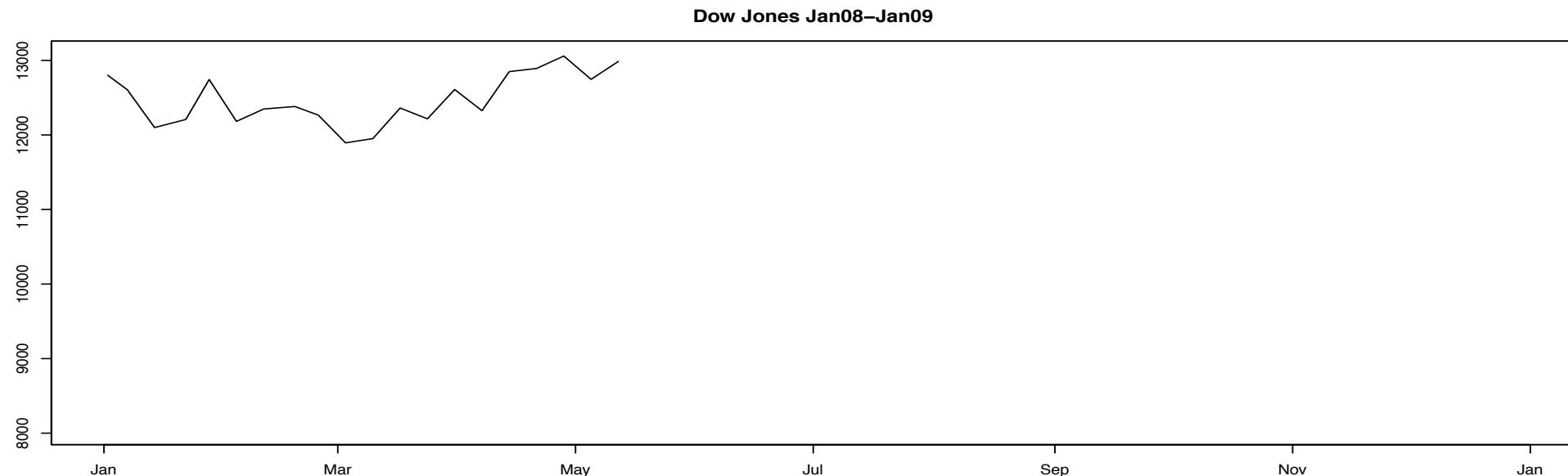
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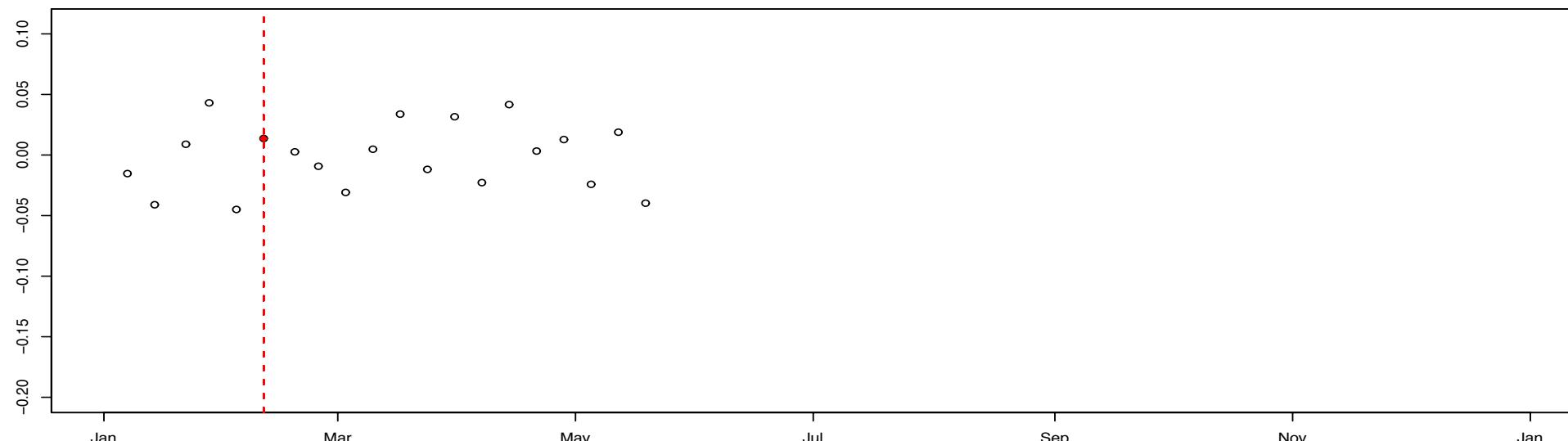
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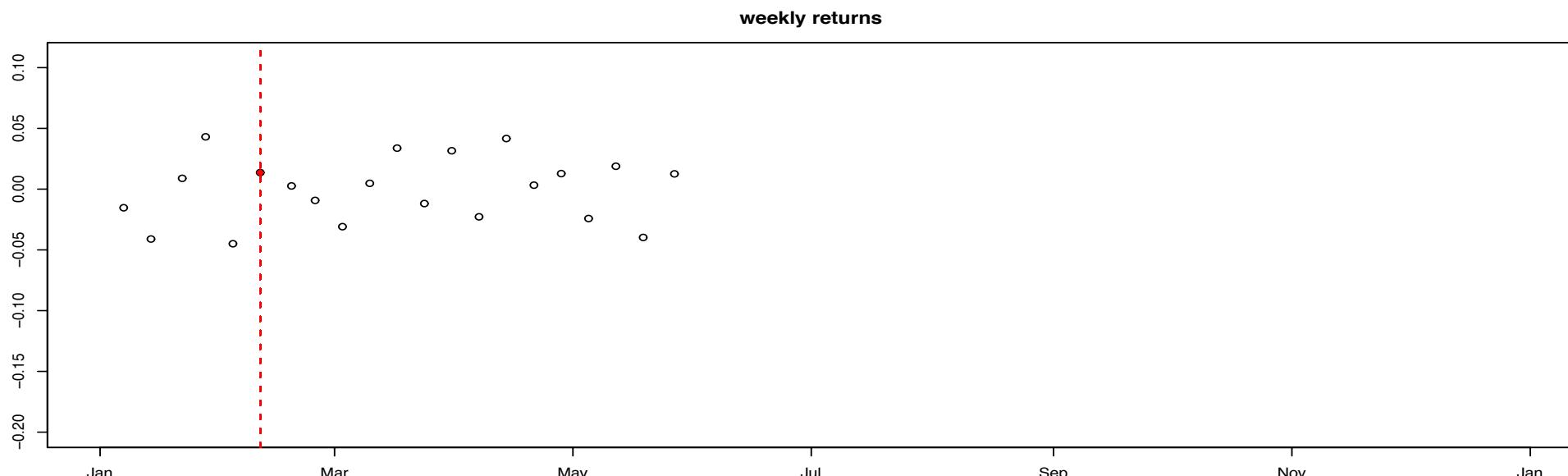
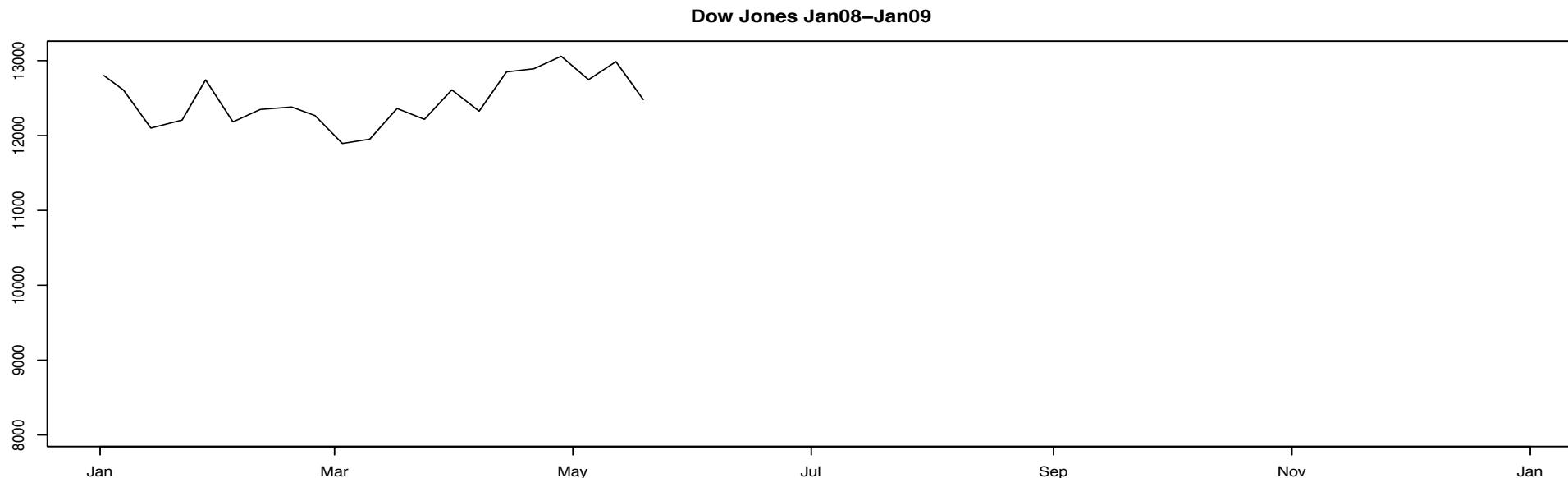
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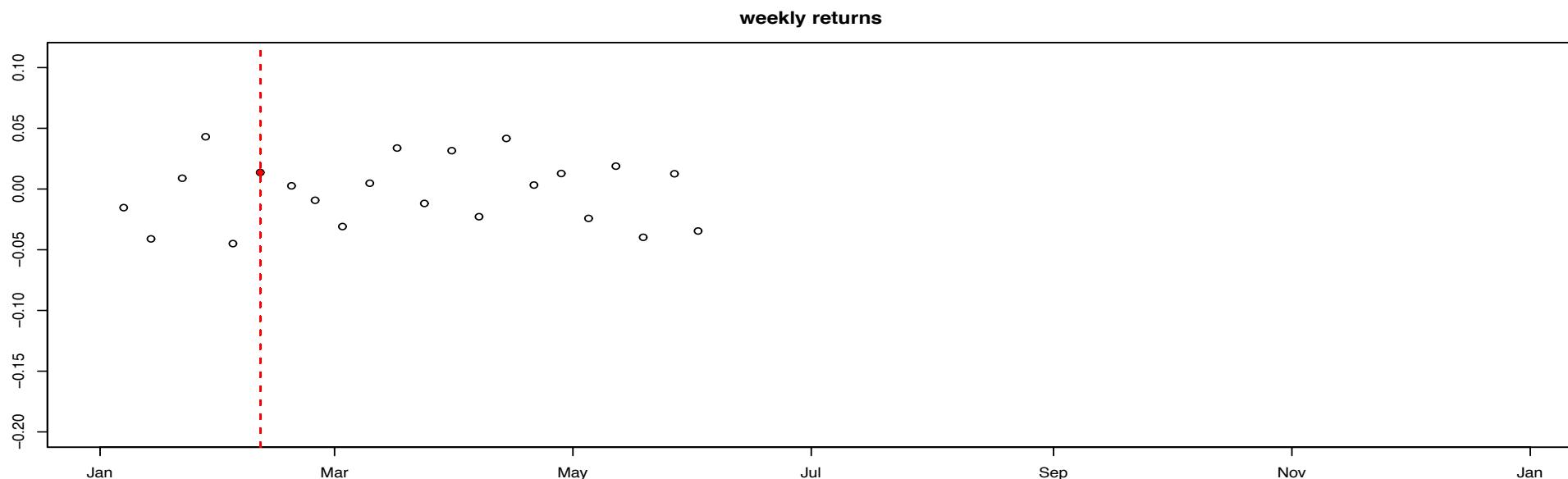
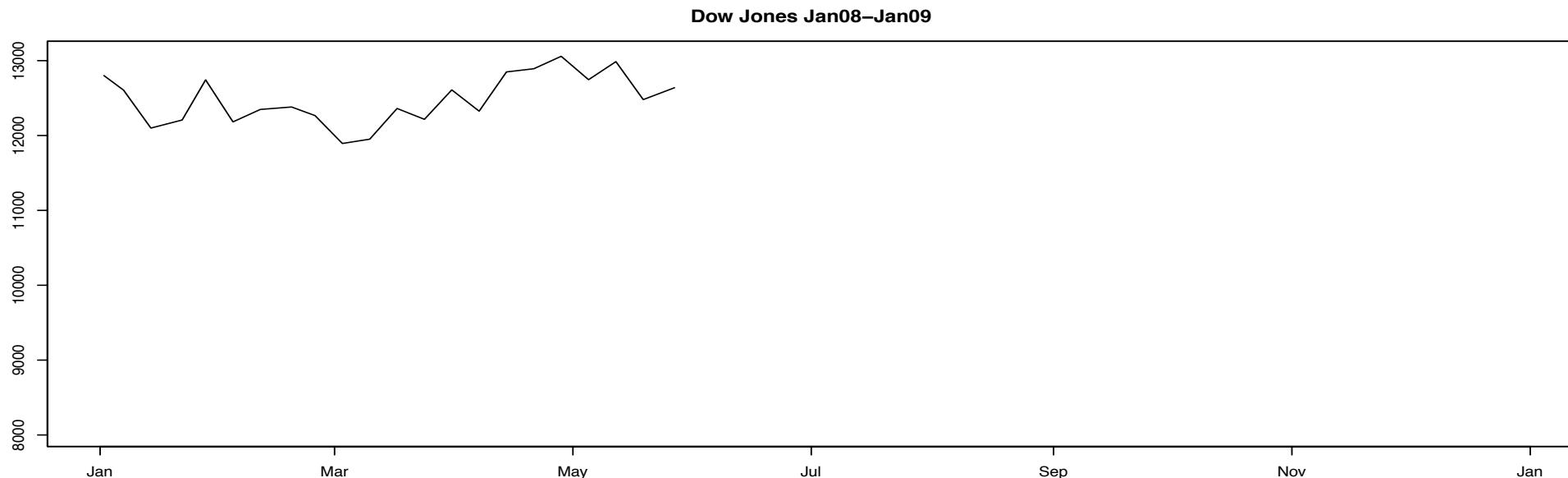
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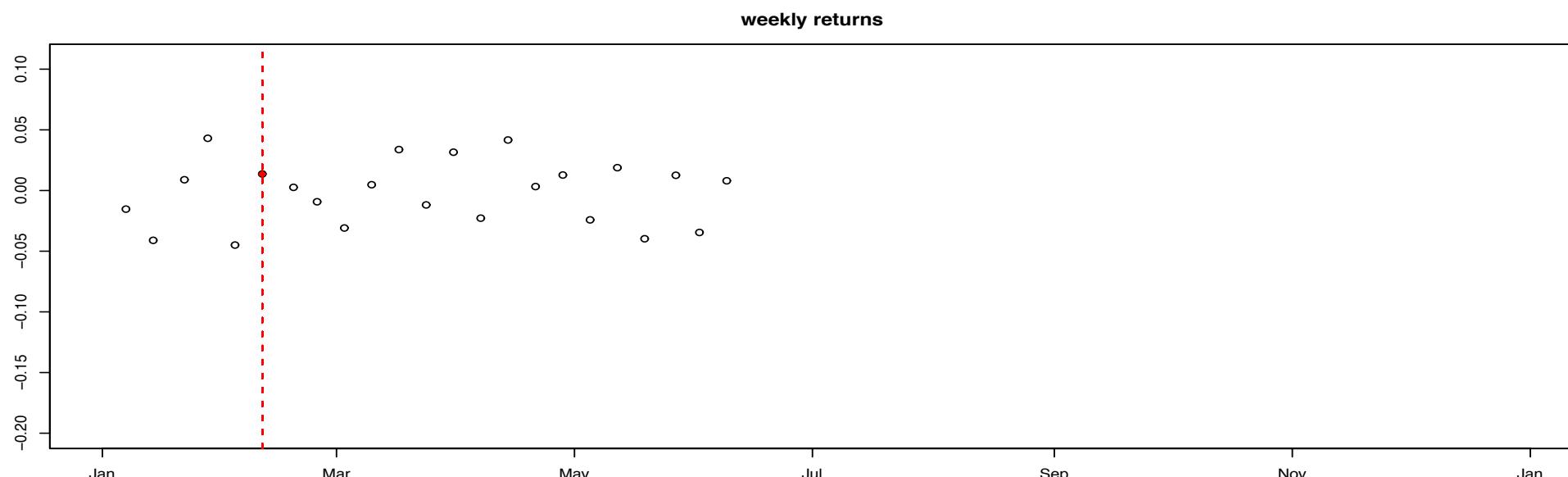
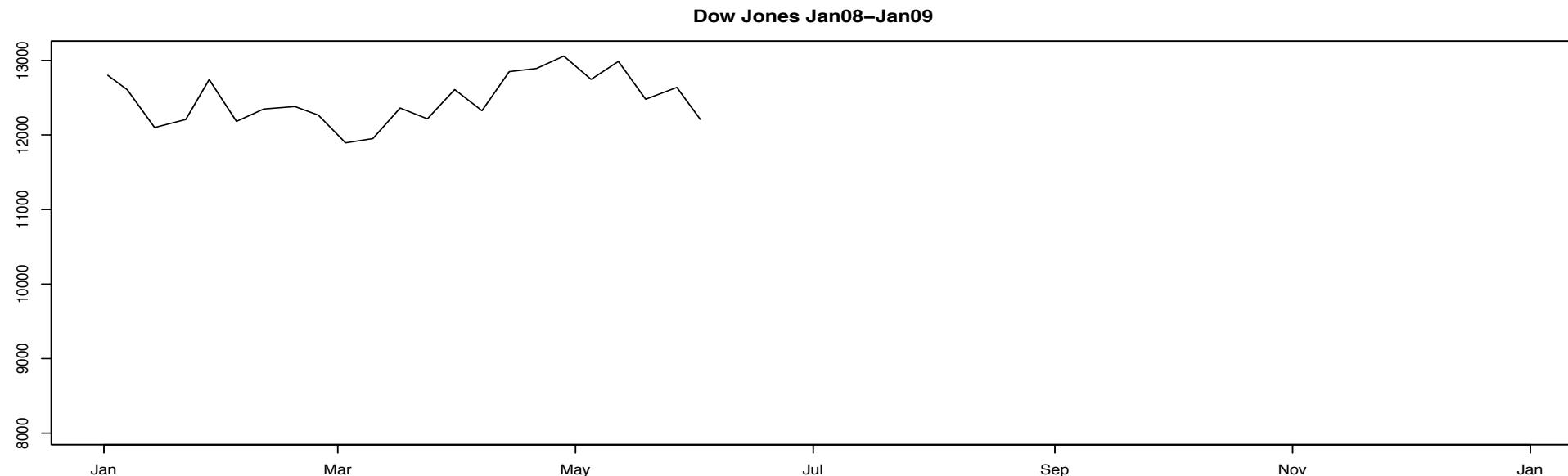
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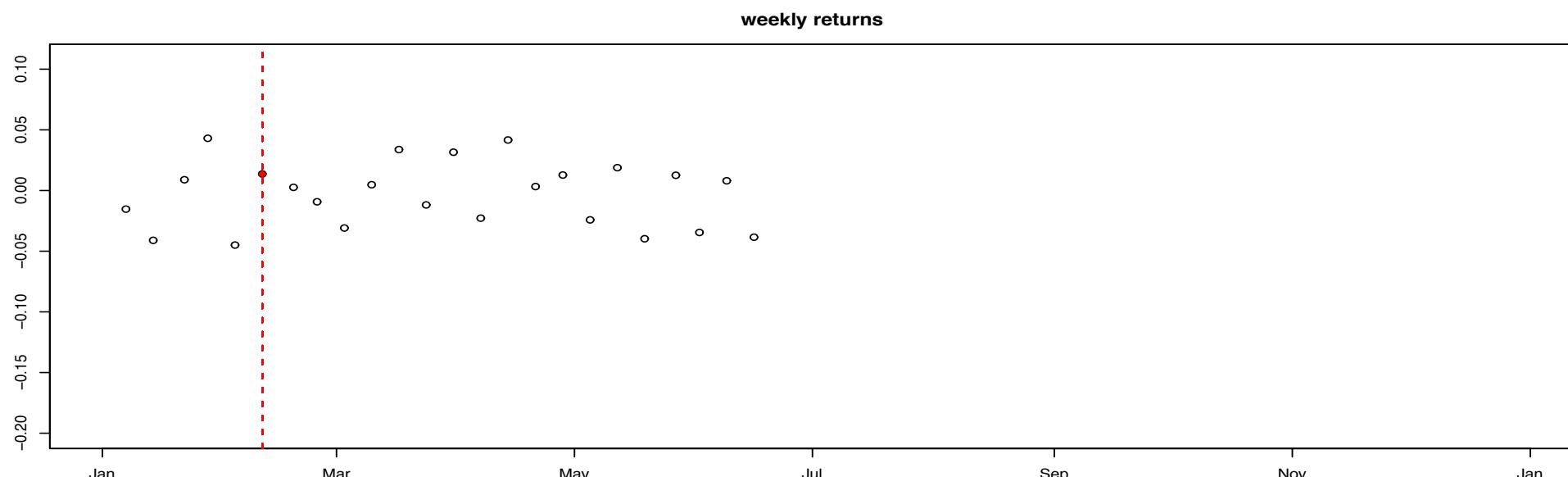
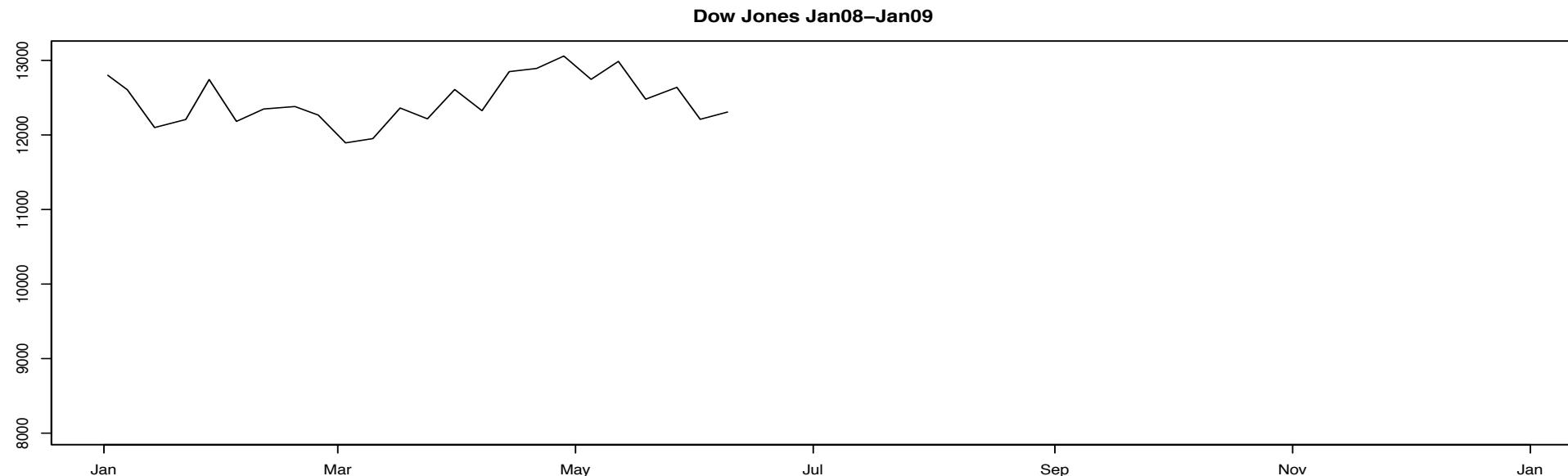
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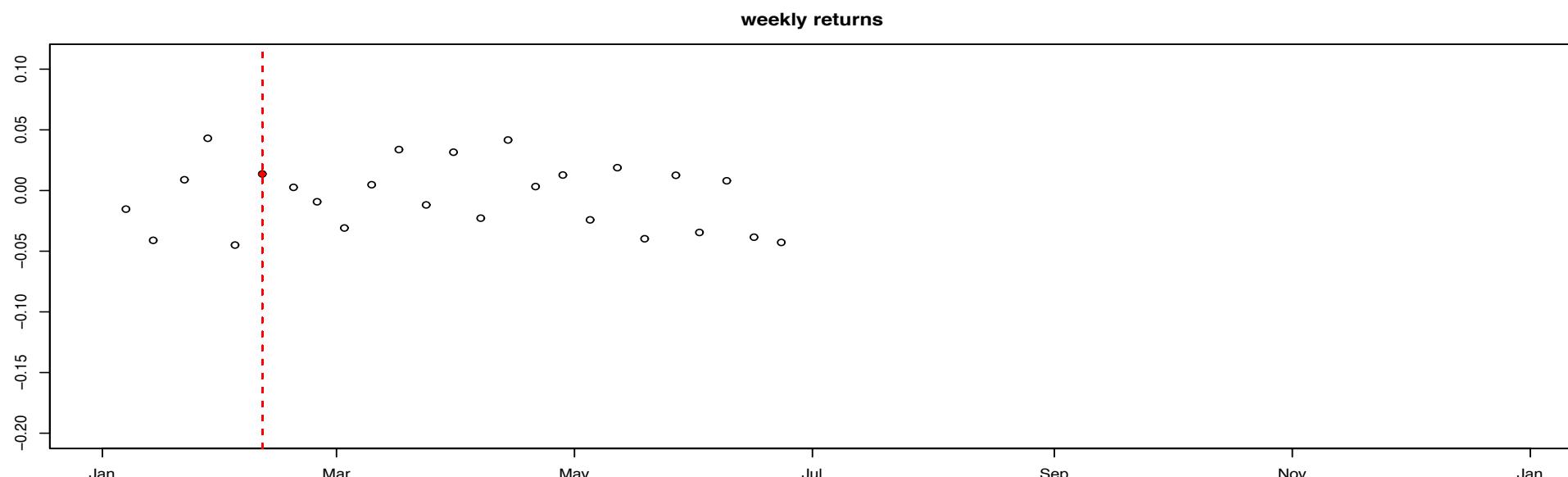
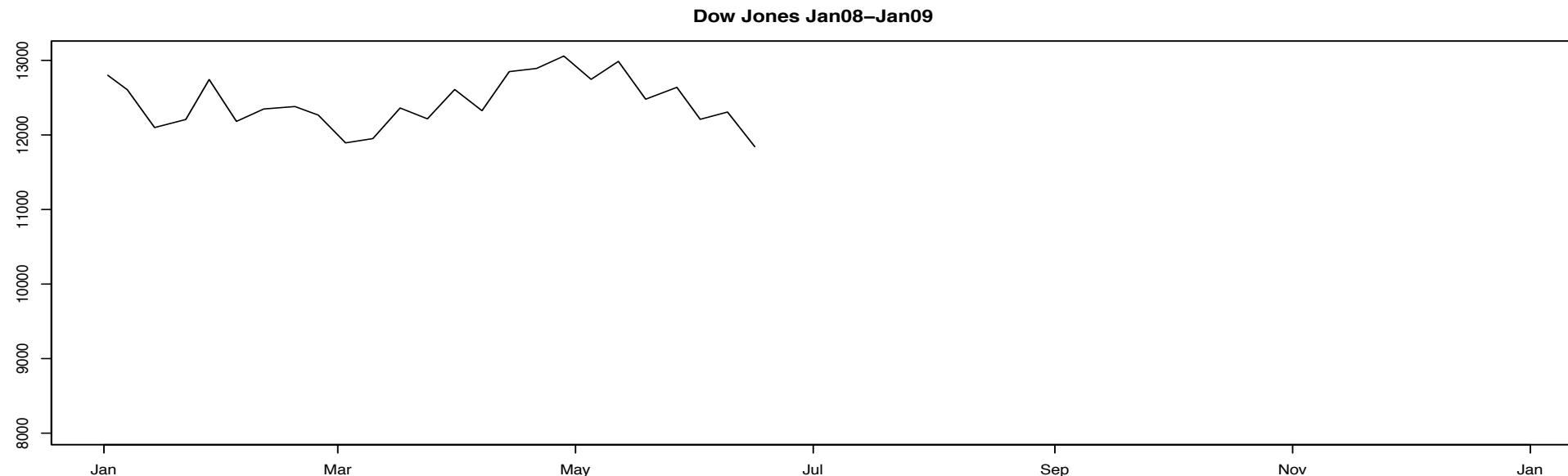
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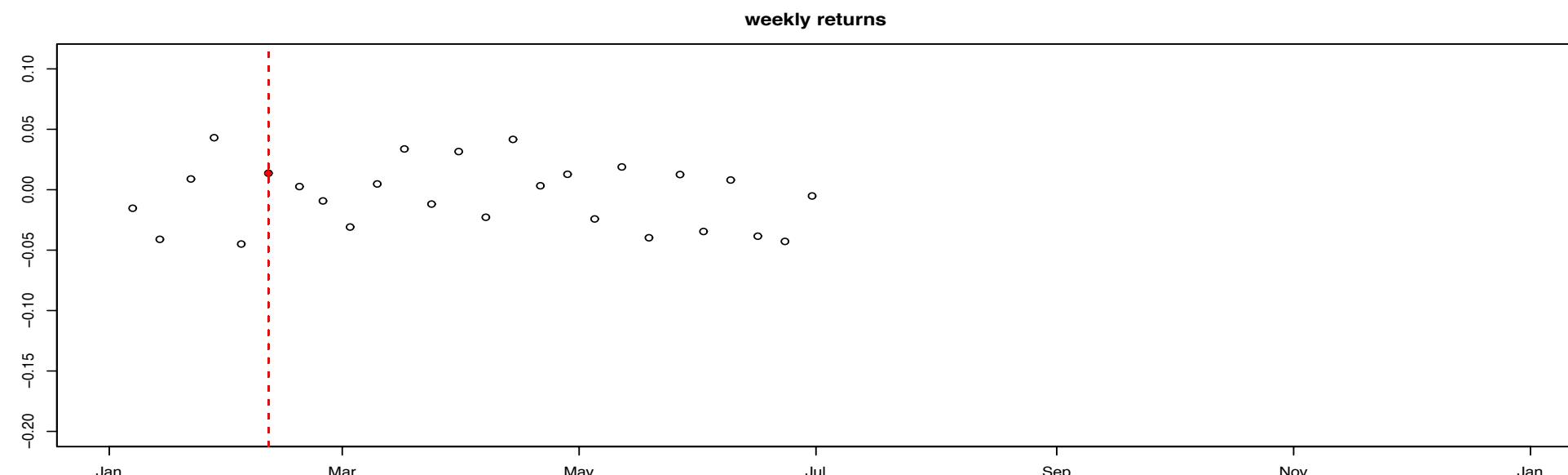
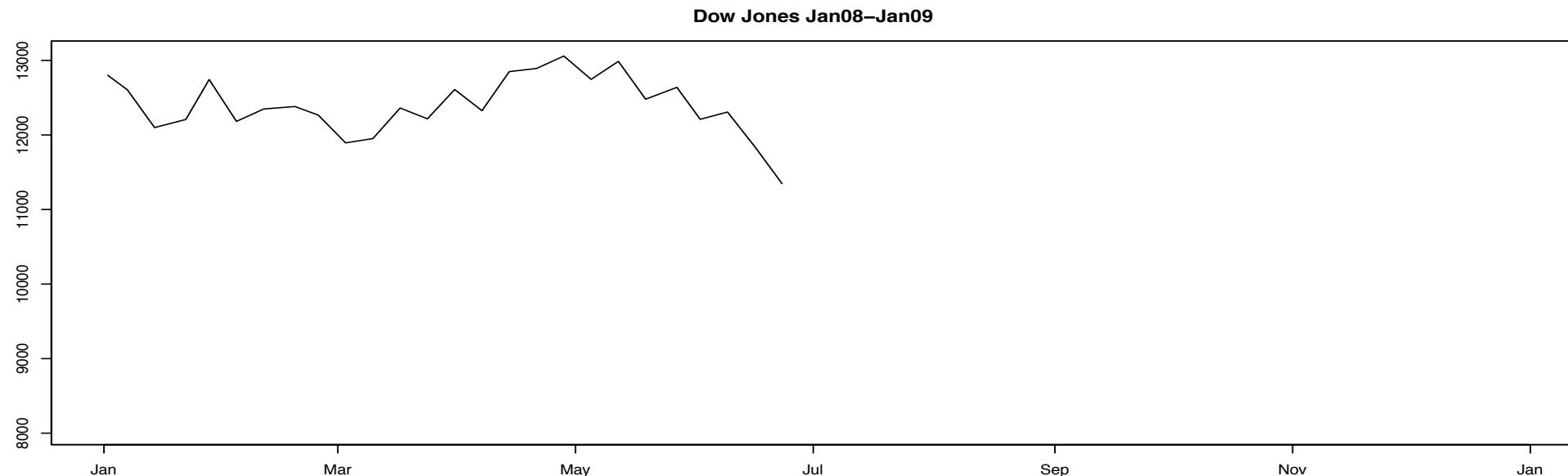
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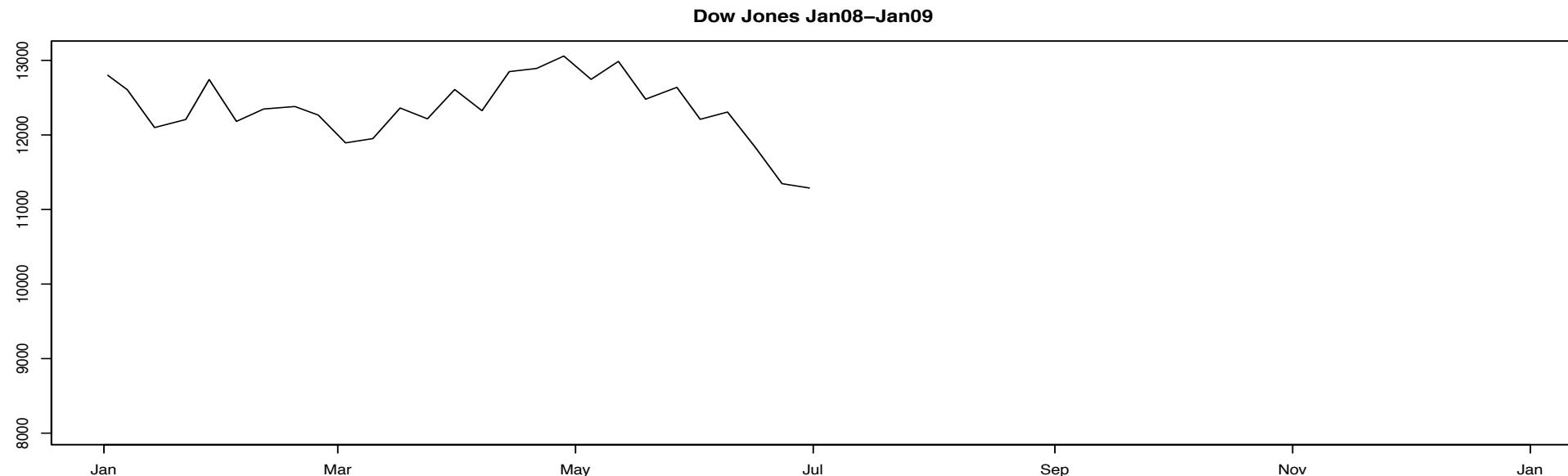
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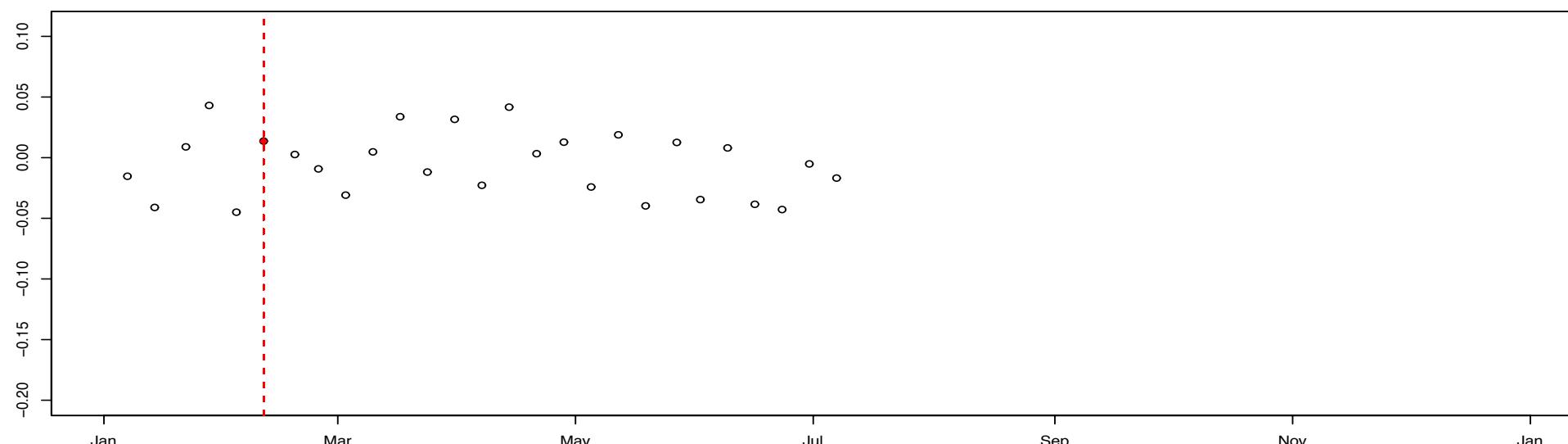
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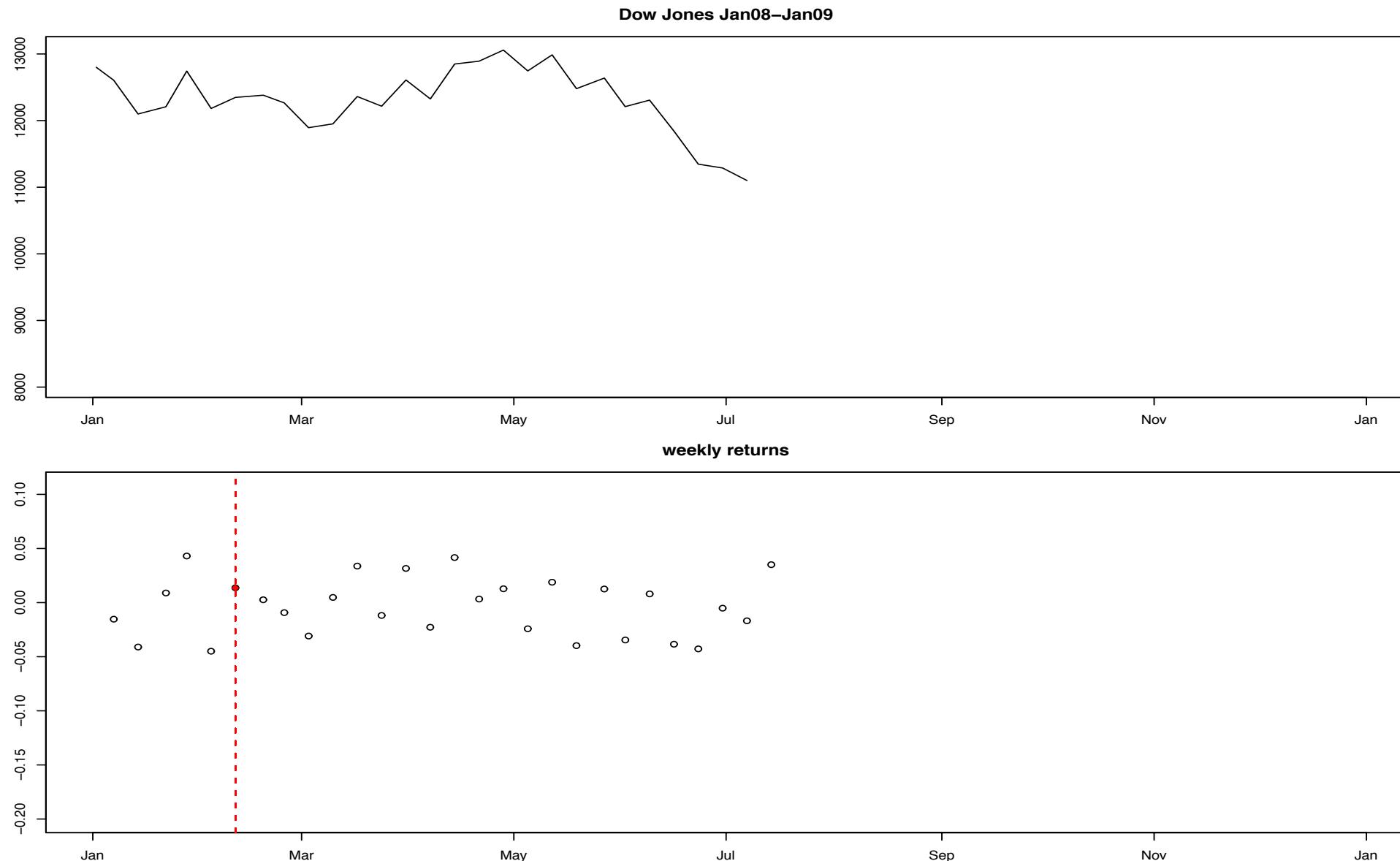
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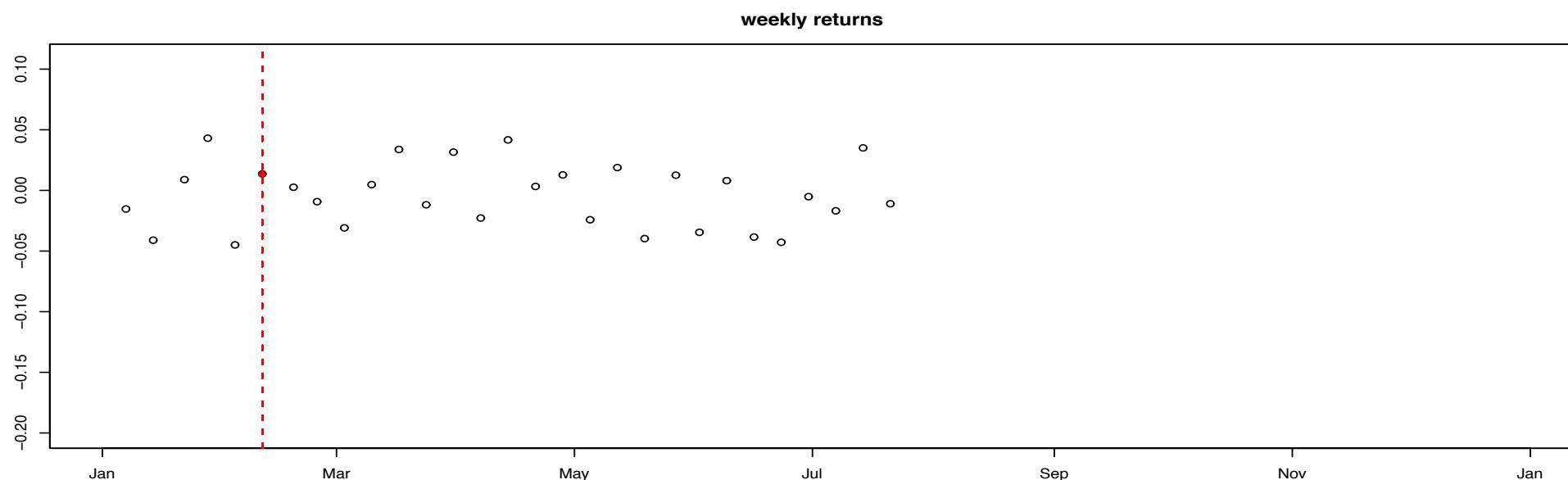
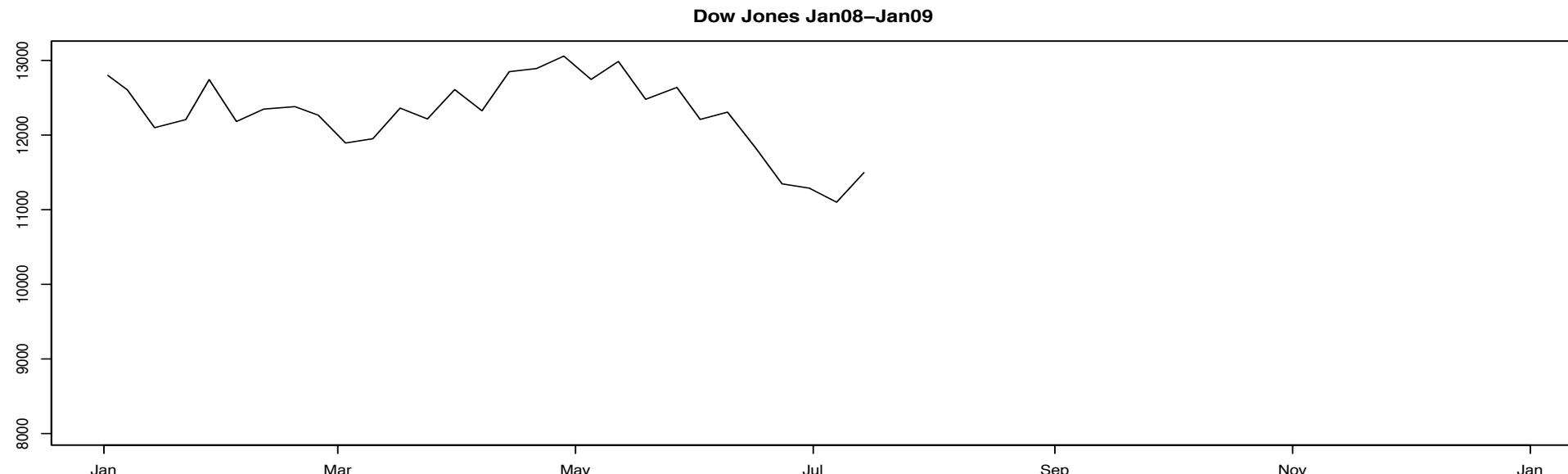
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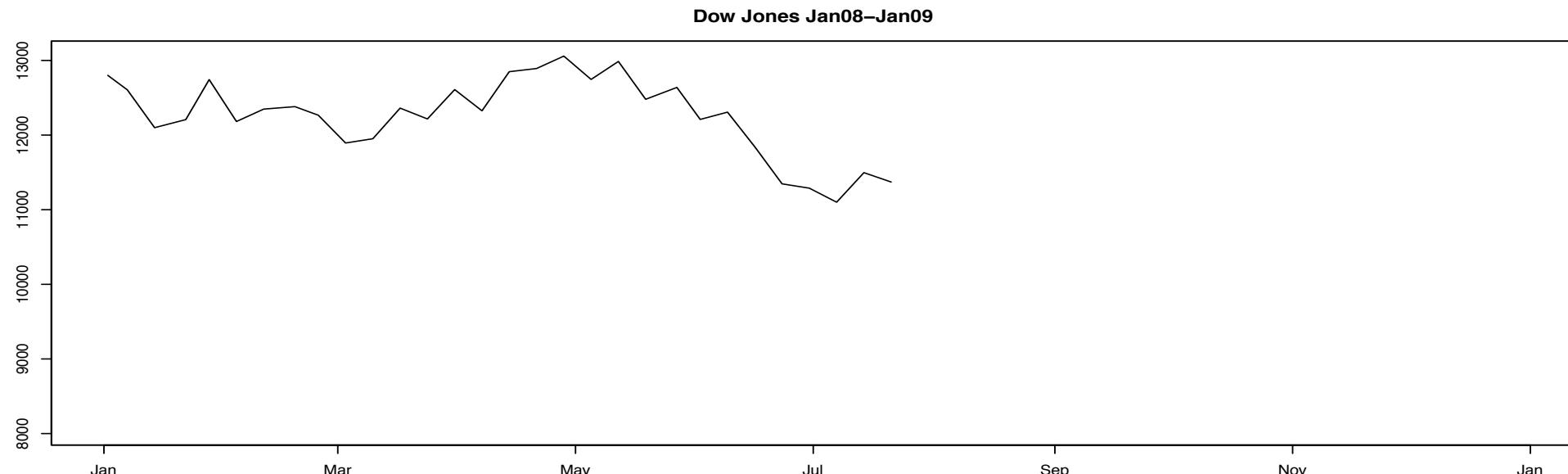
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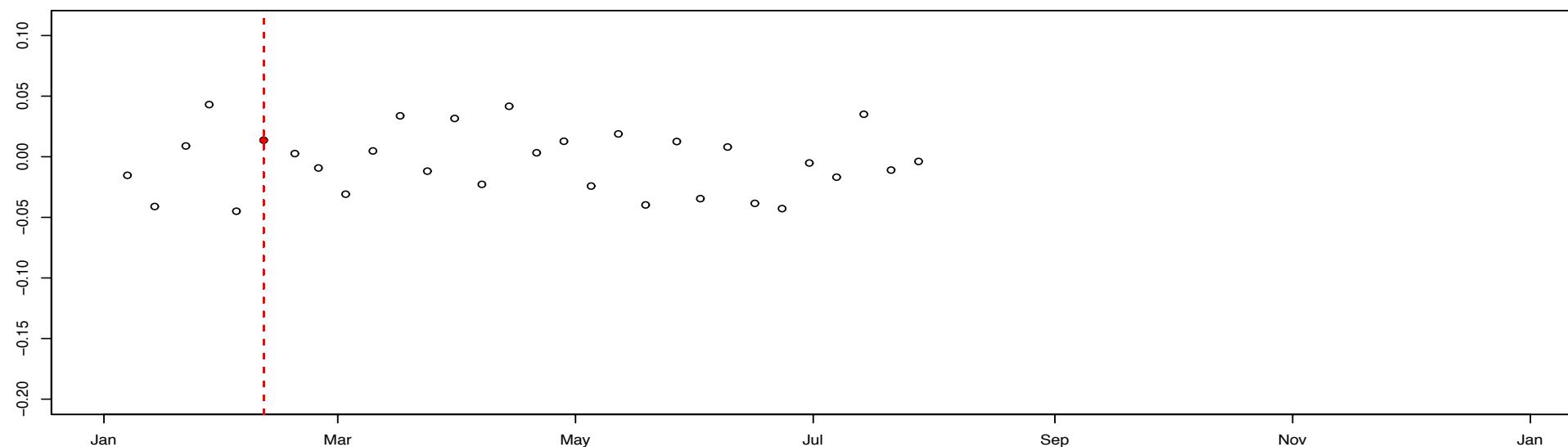
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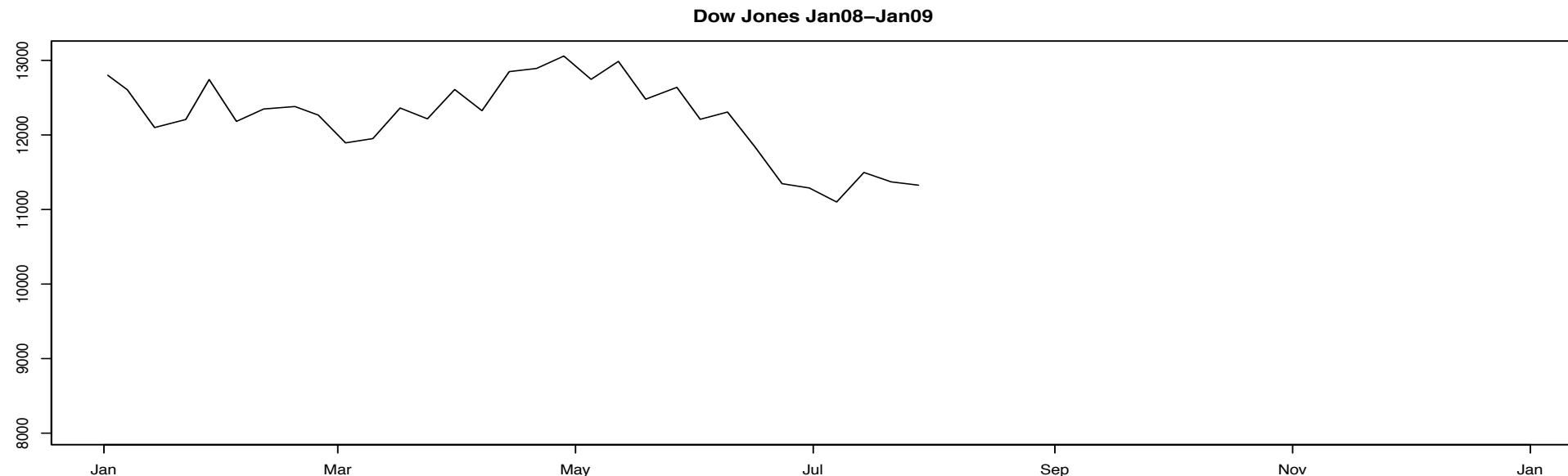
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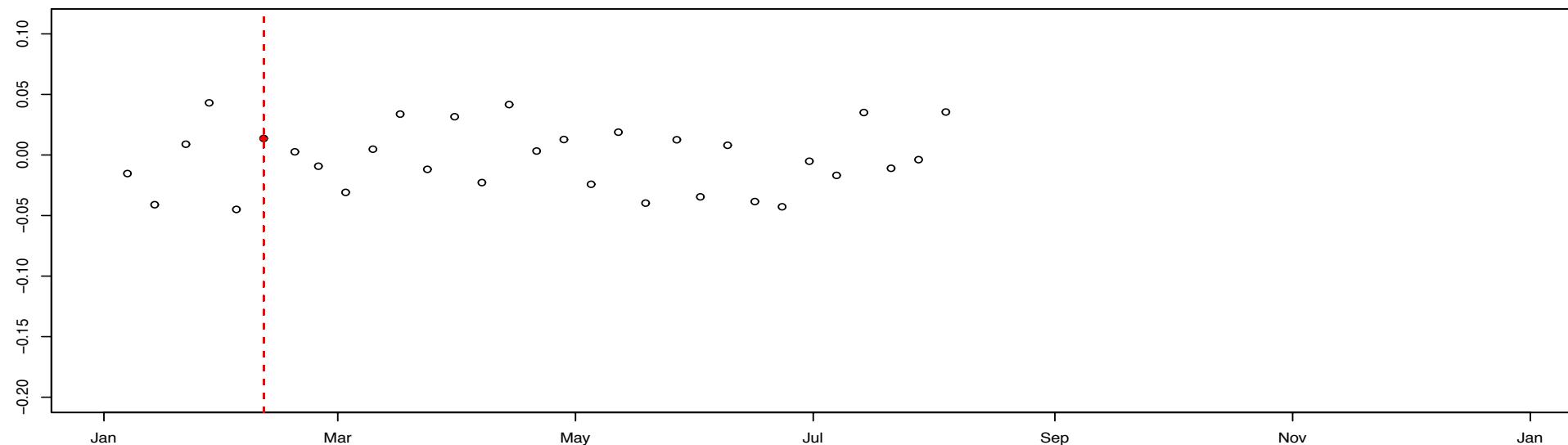
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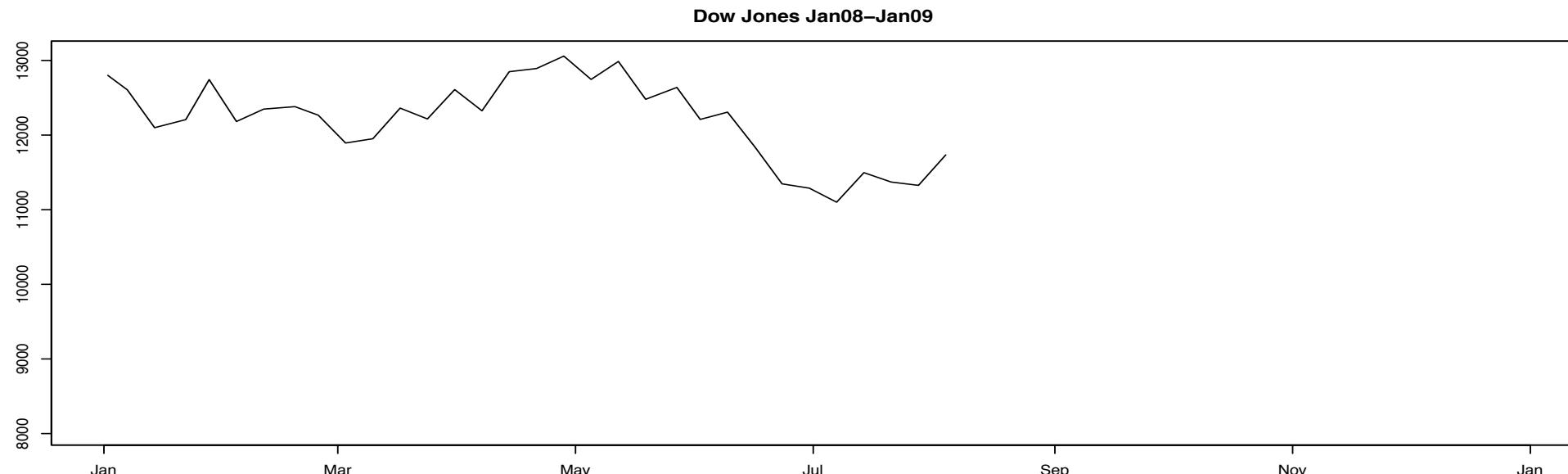
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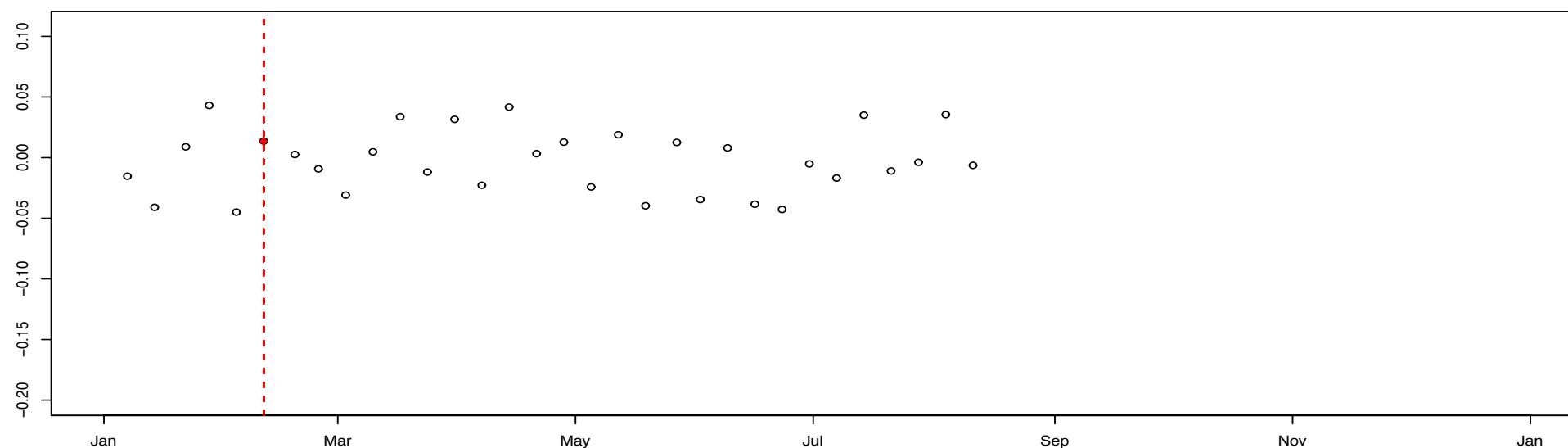
weekly returns



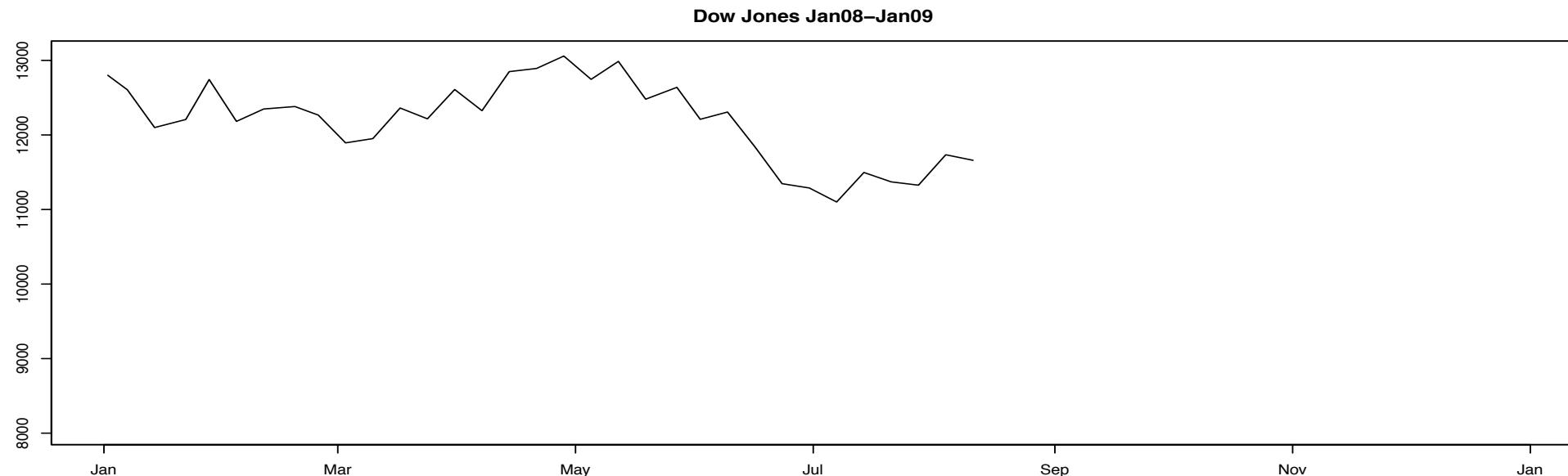
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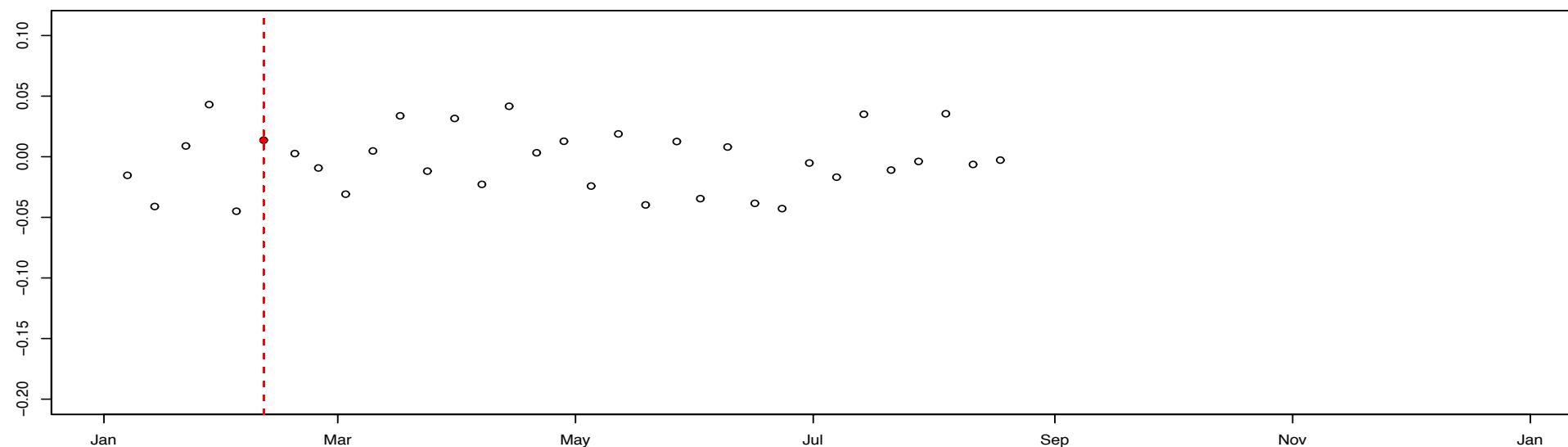
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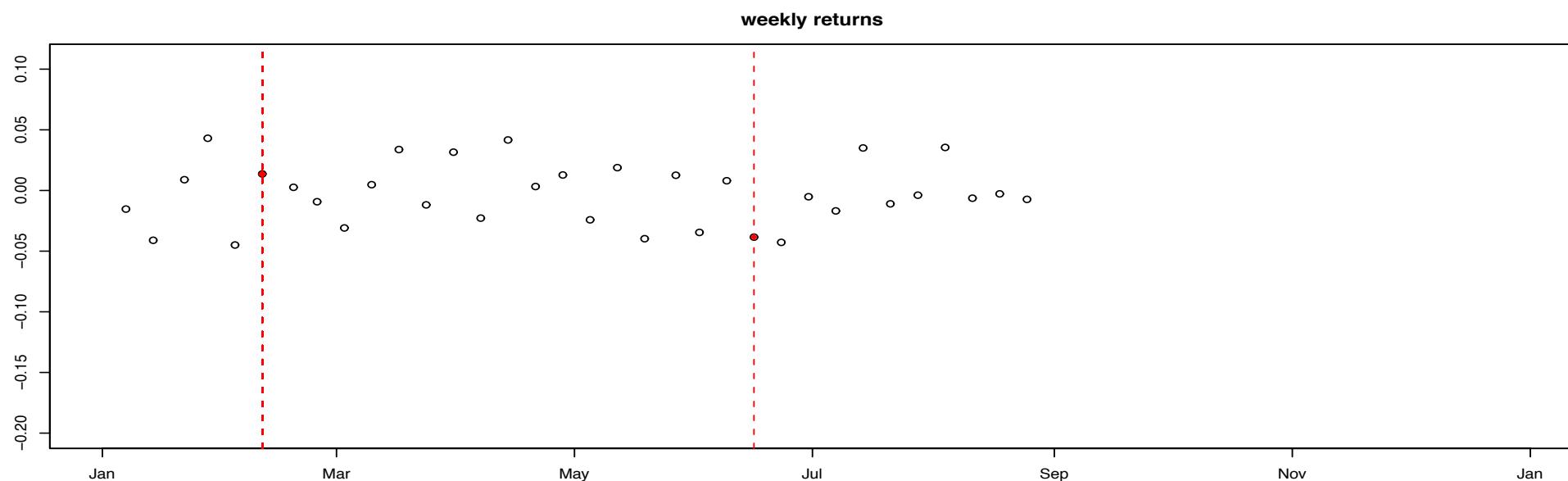
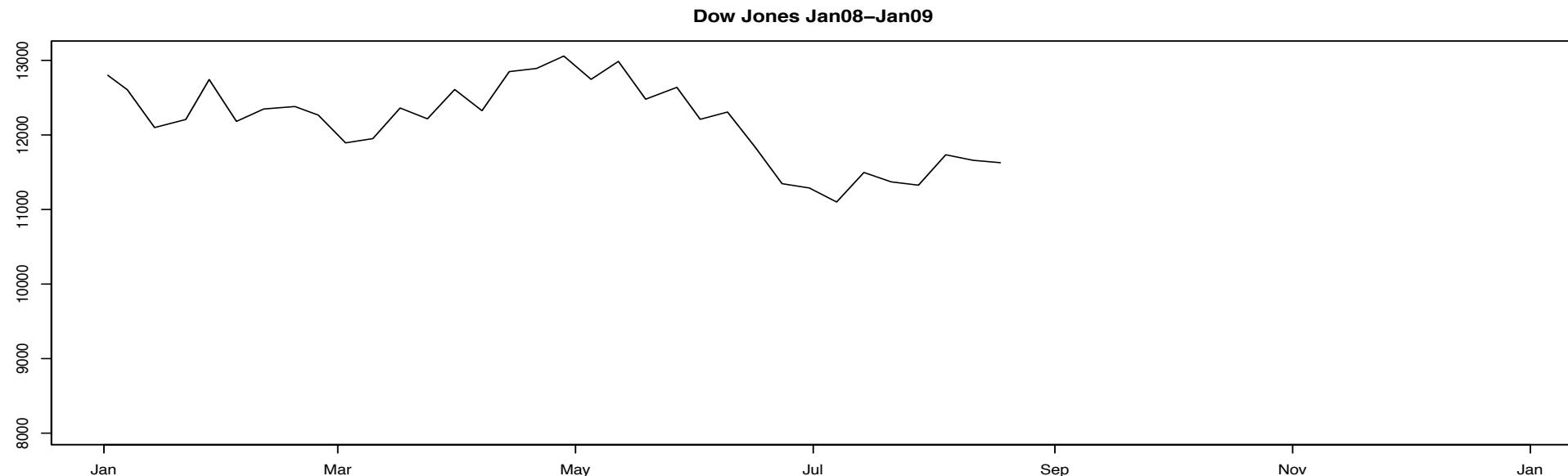
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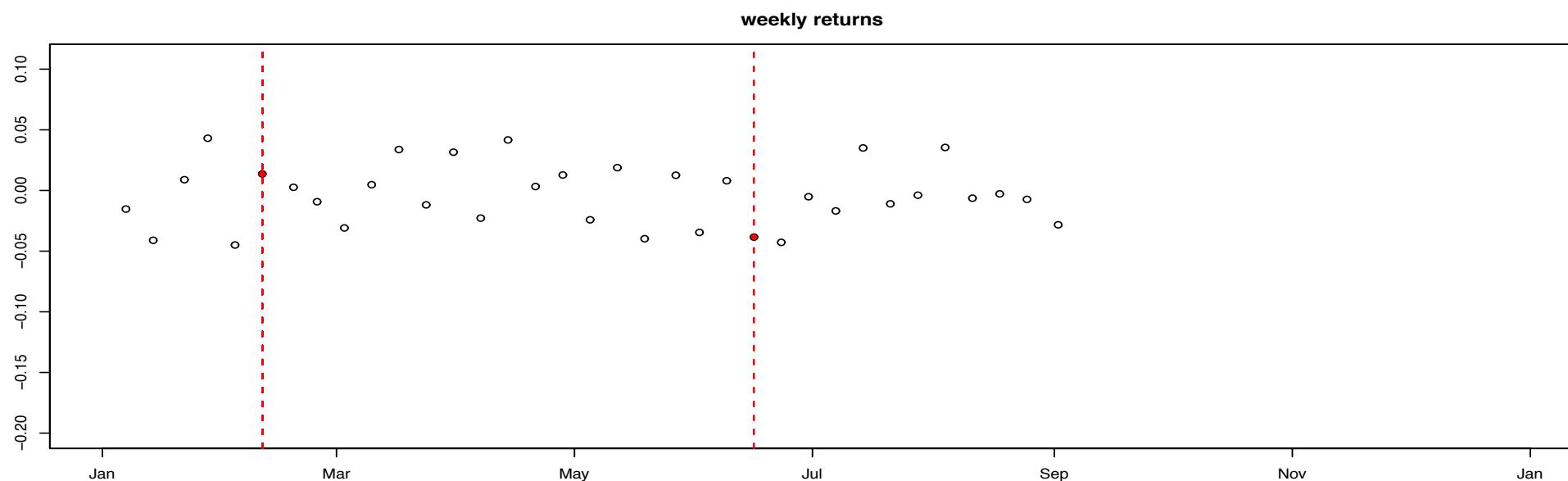
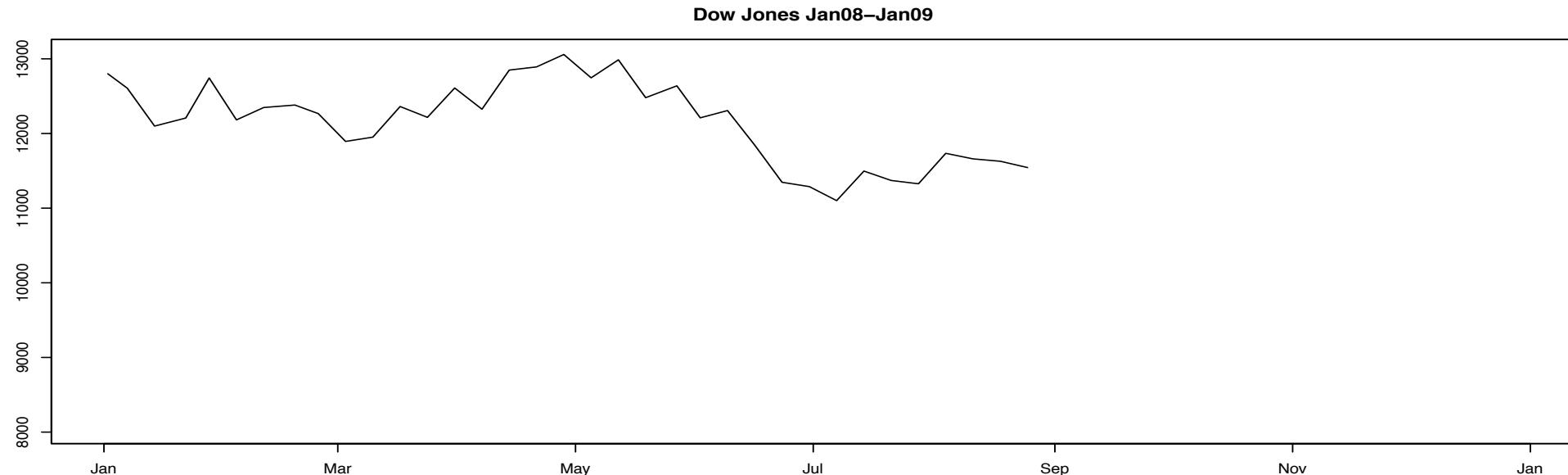
weekly returns



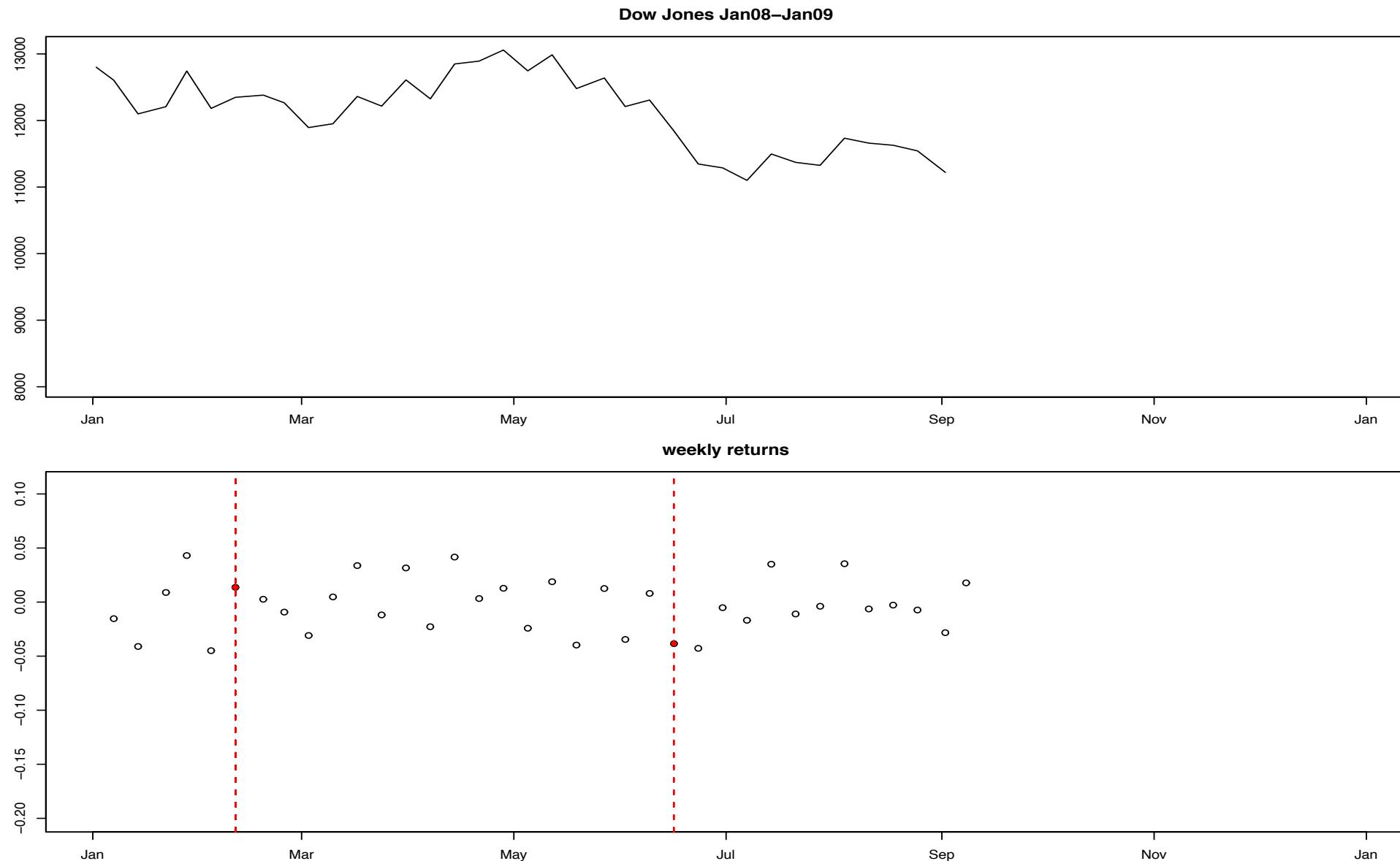
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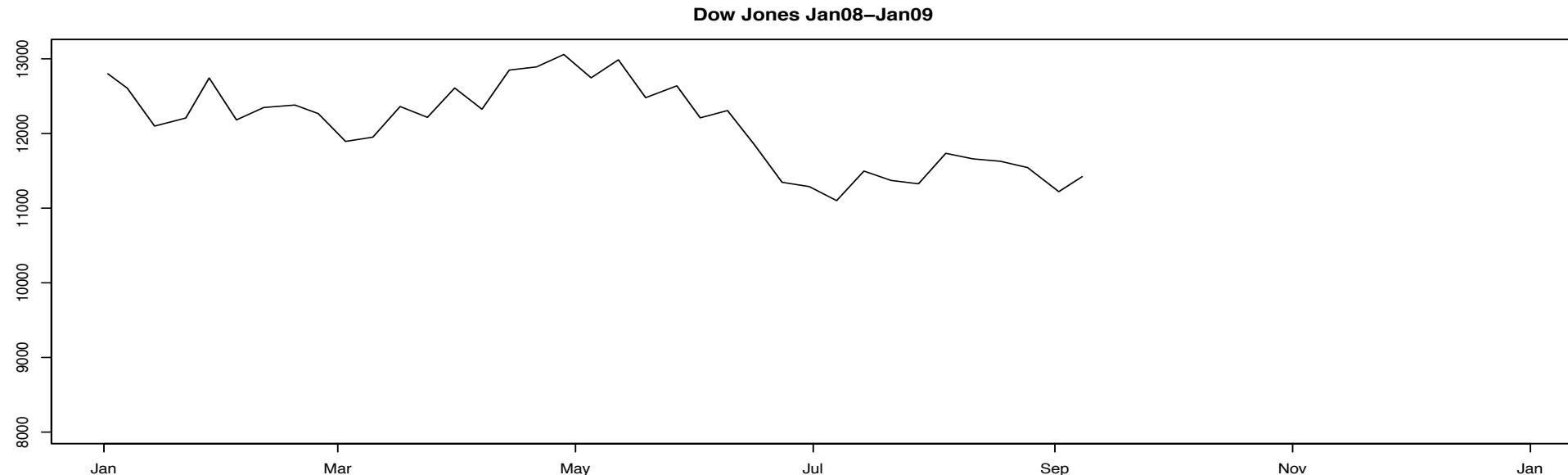
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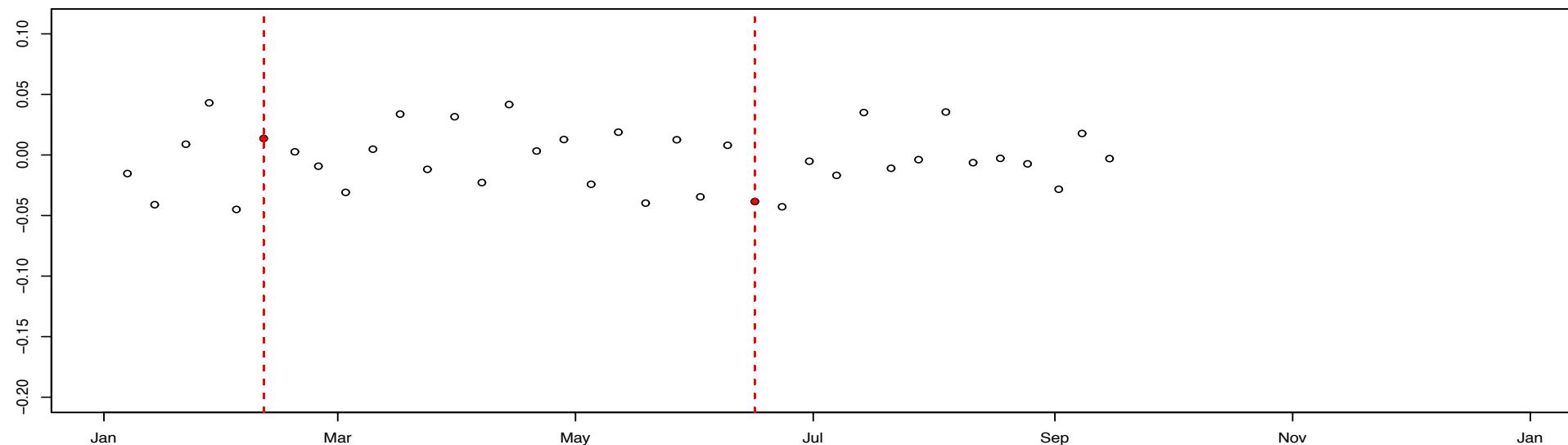
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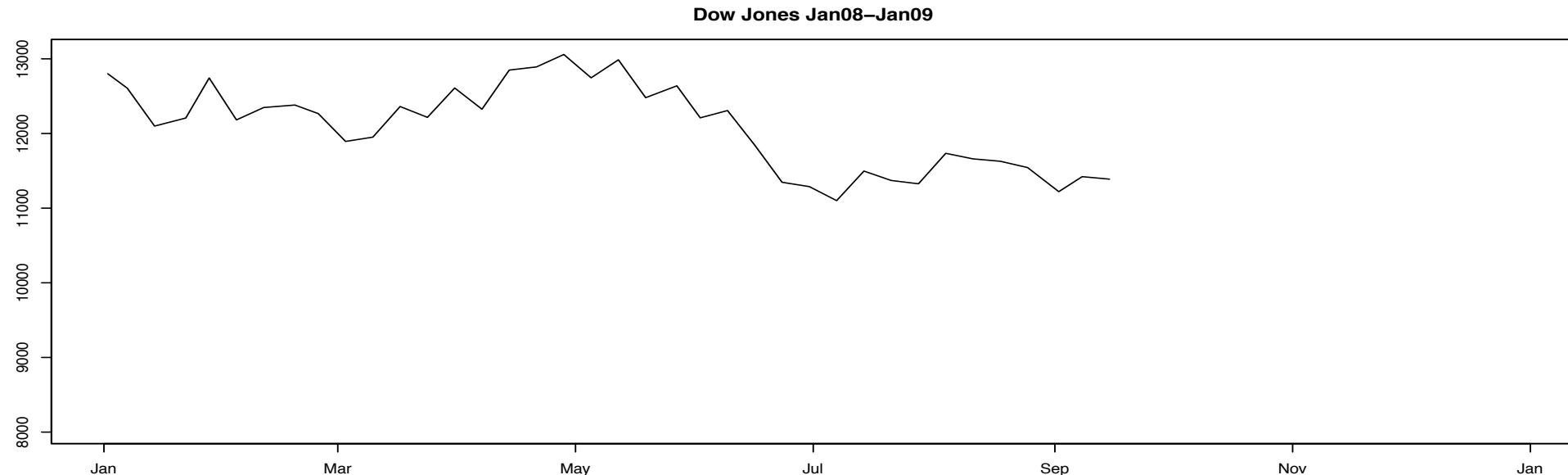
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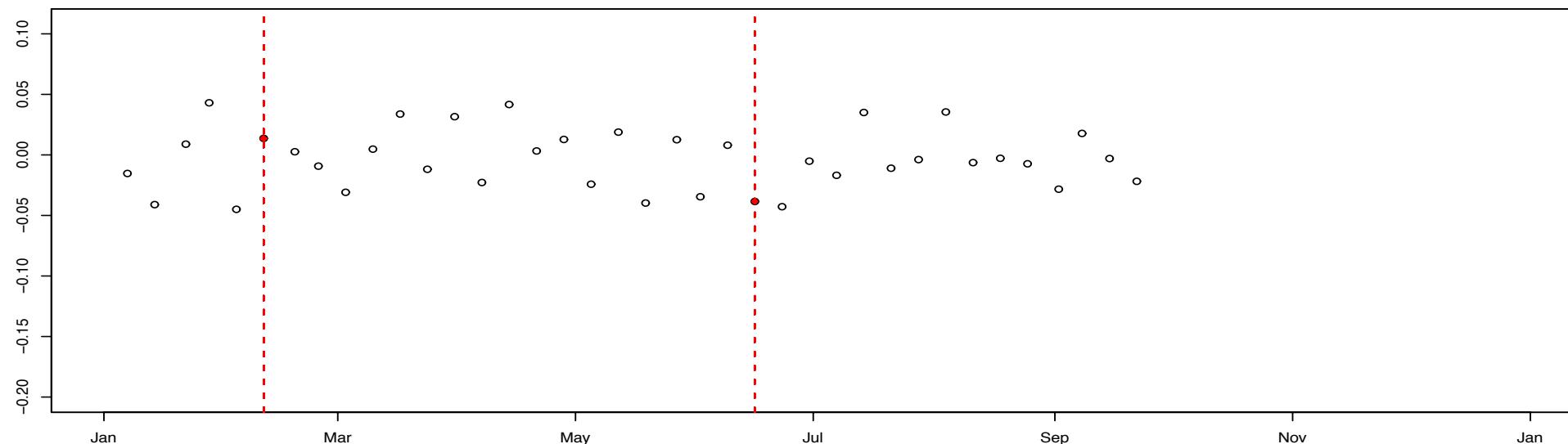
weekly returns



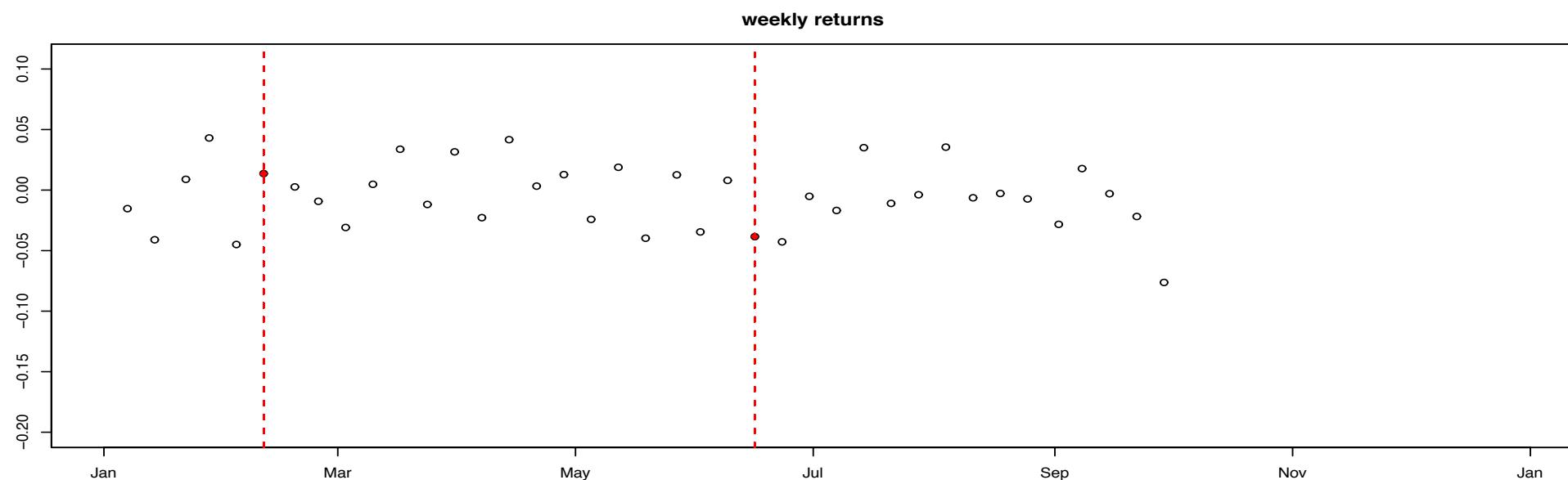
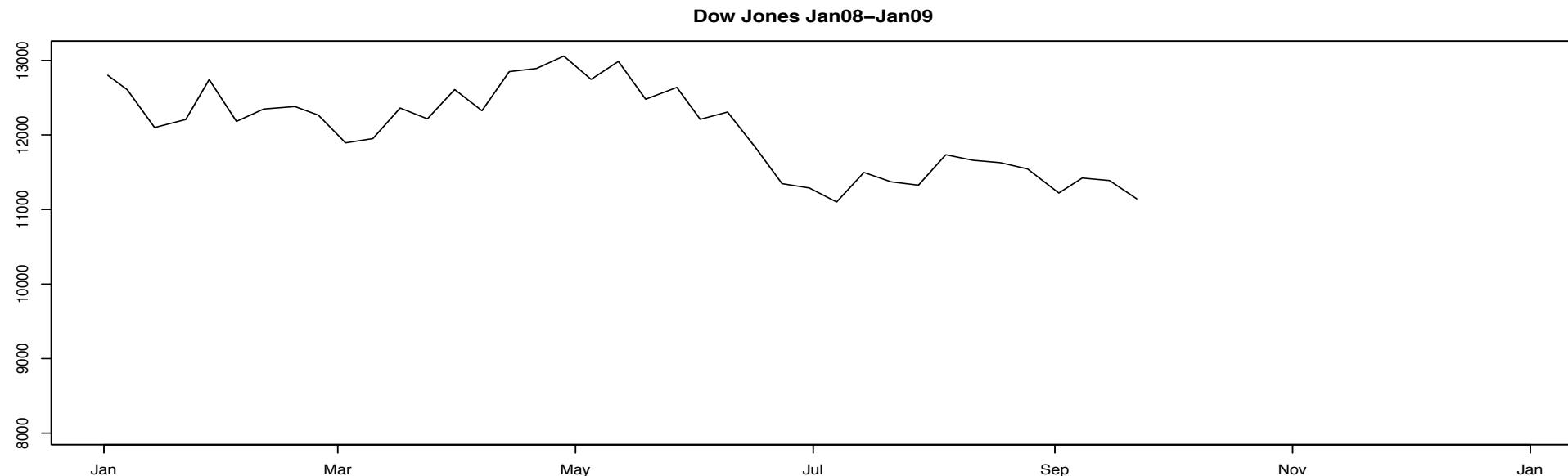
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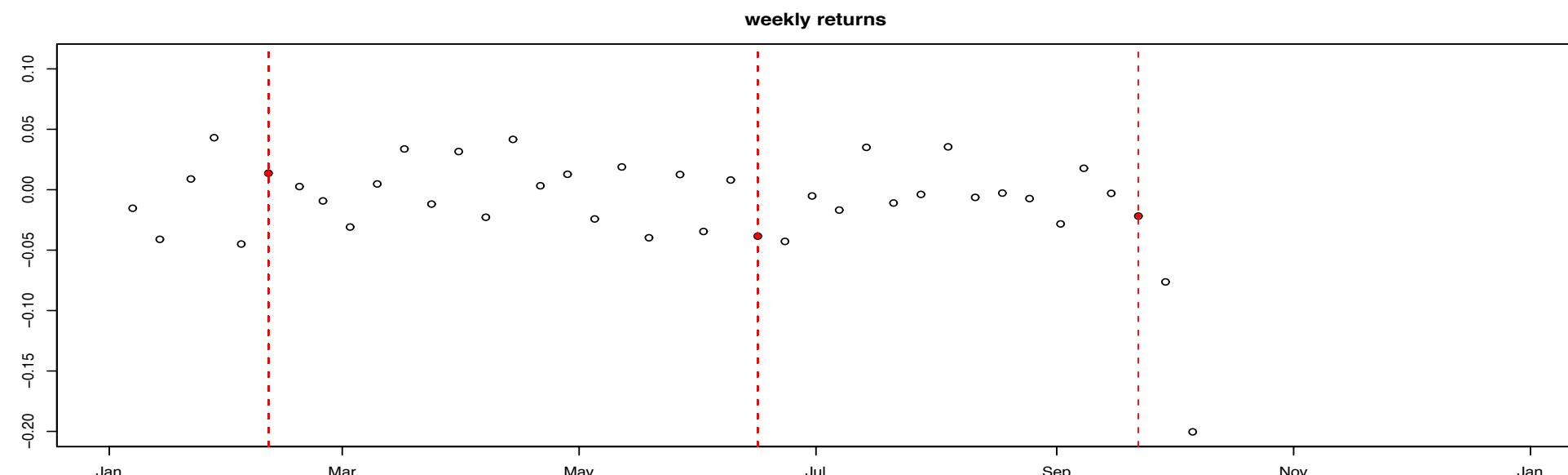
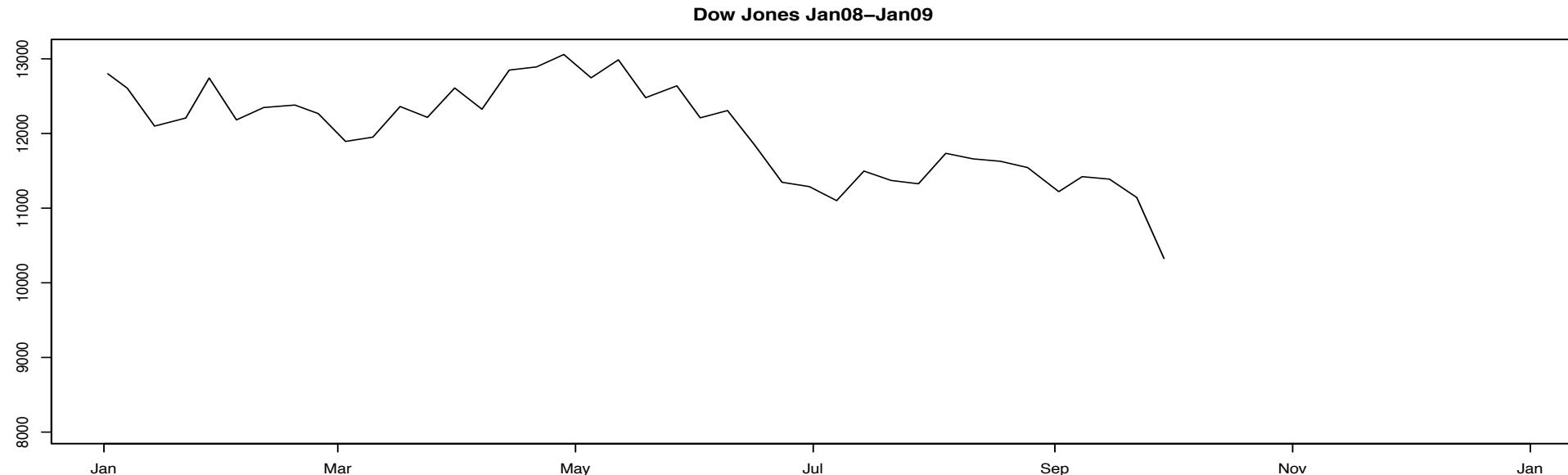
weekly returns



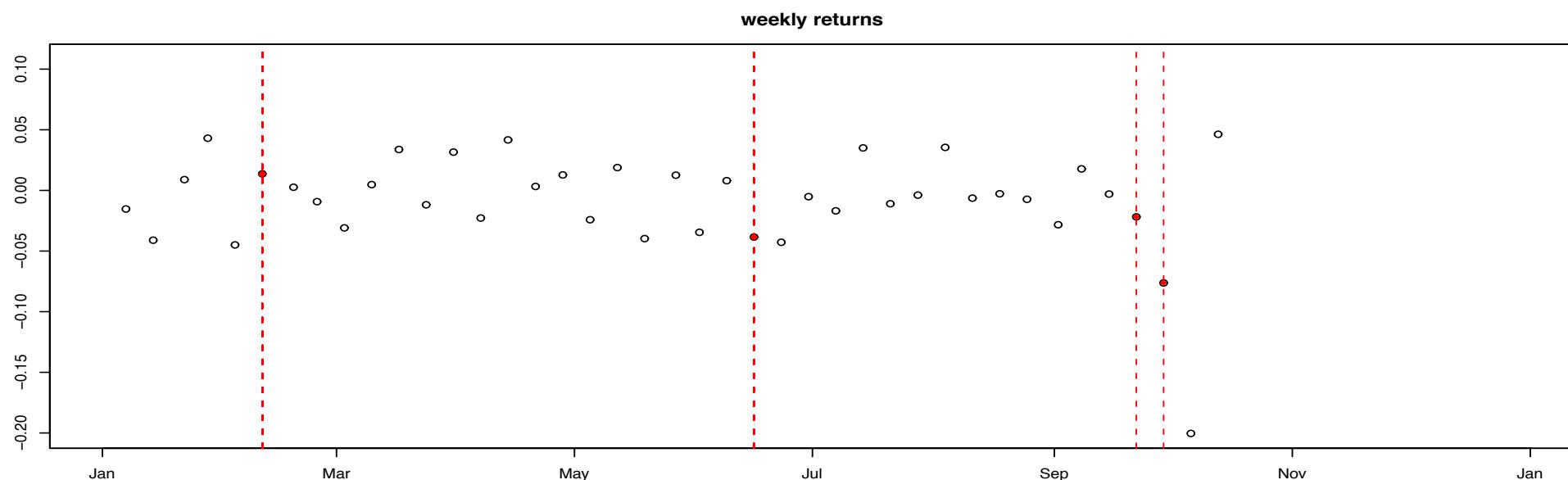
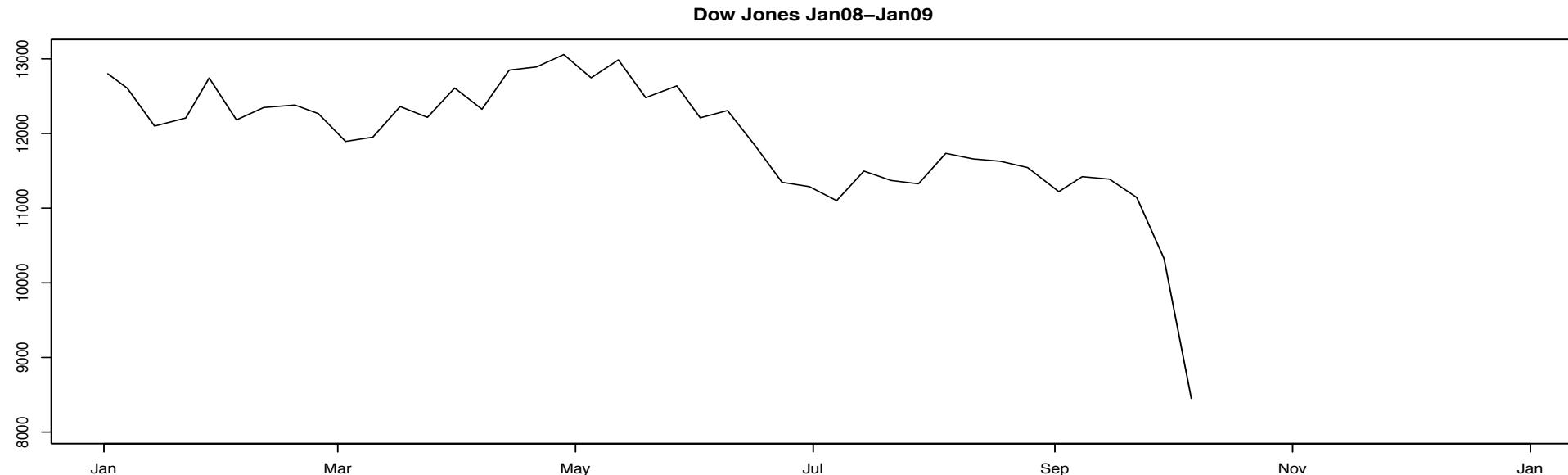
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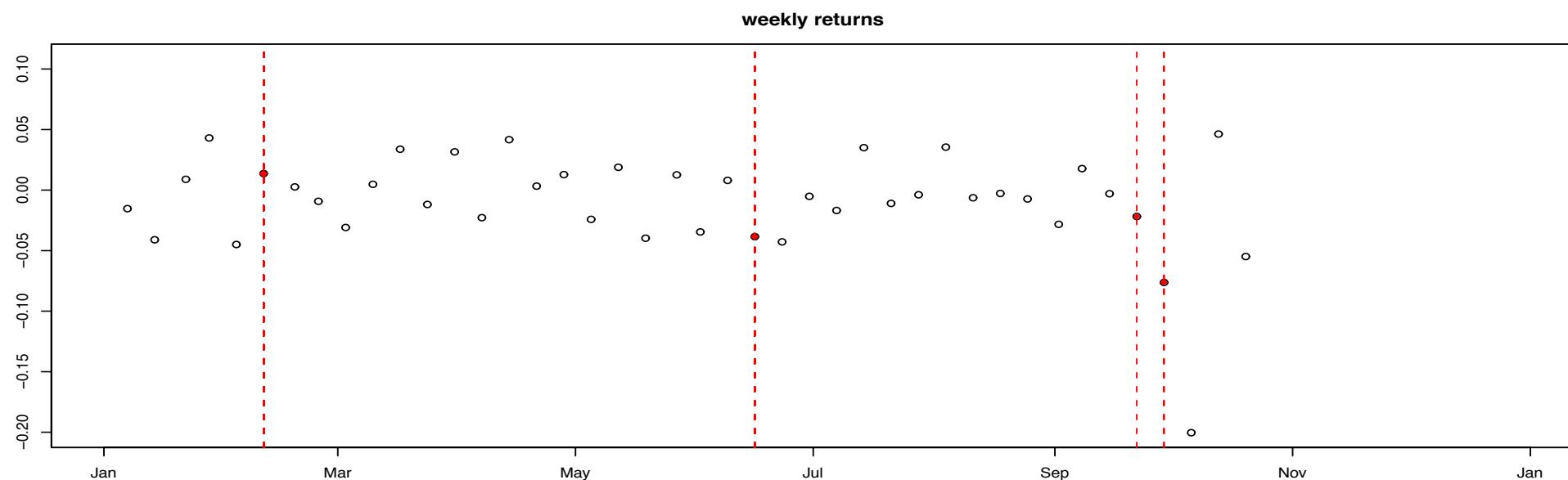
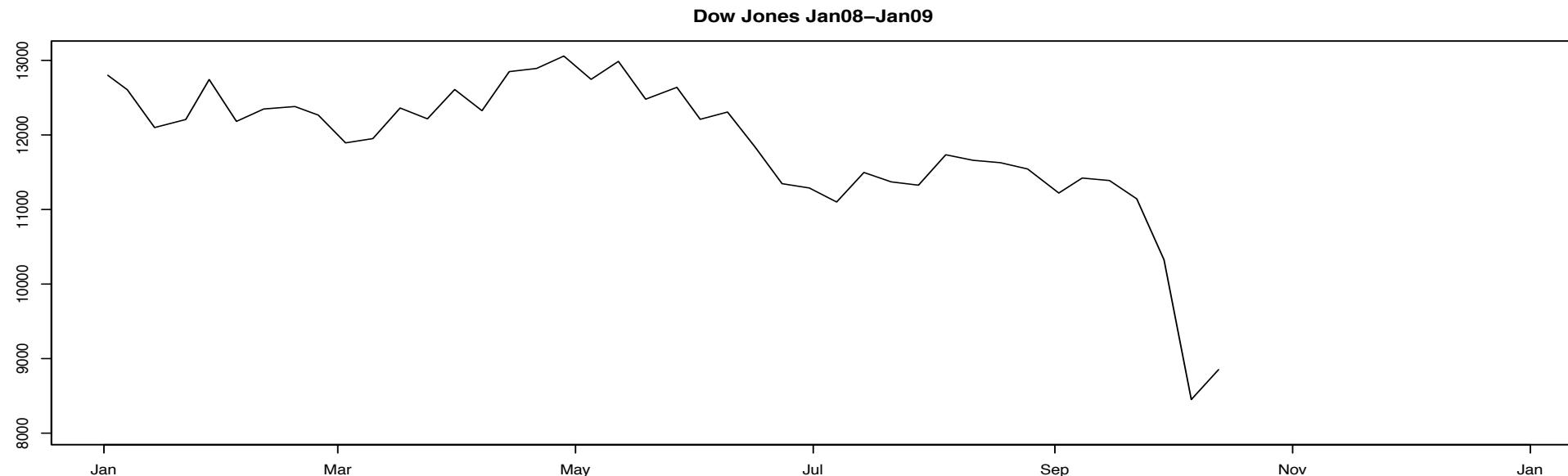
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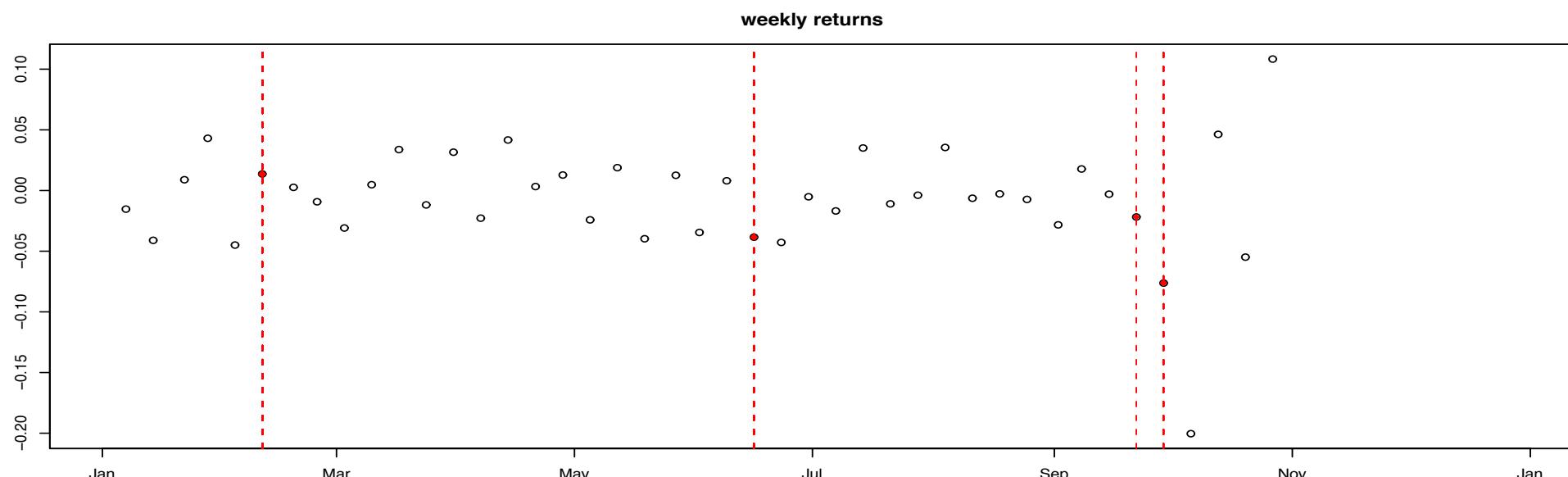
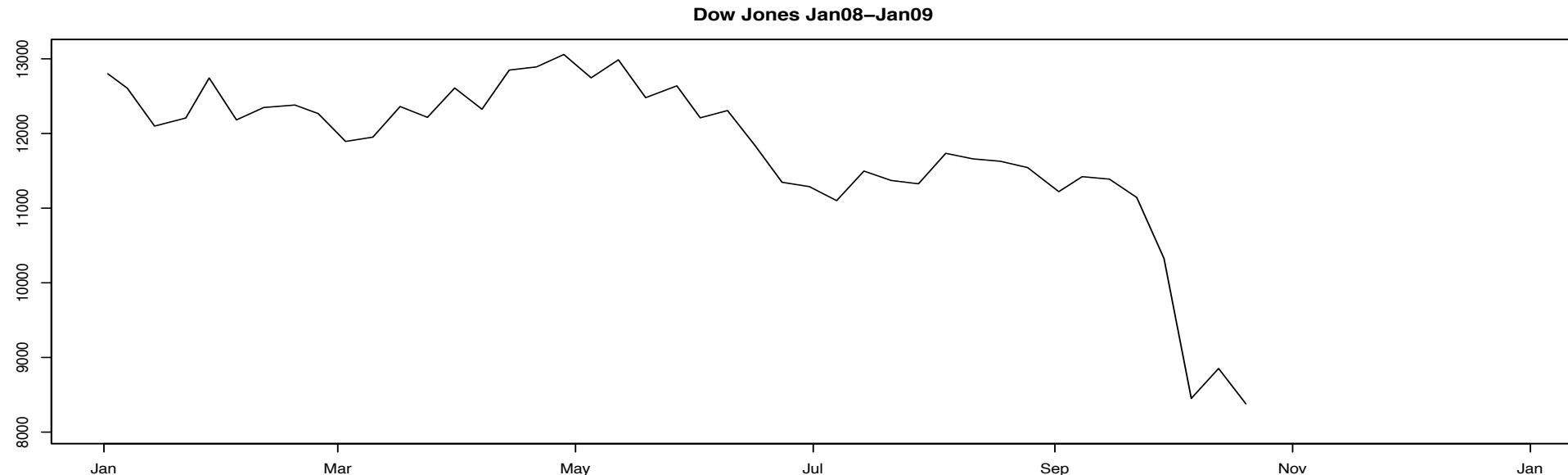
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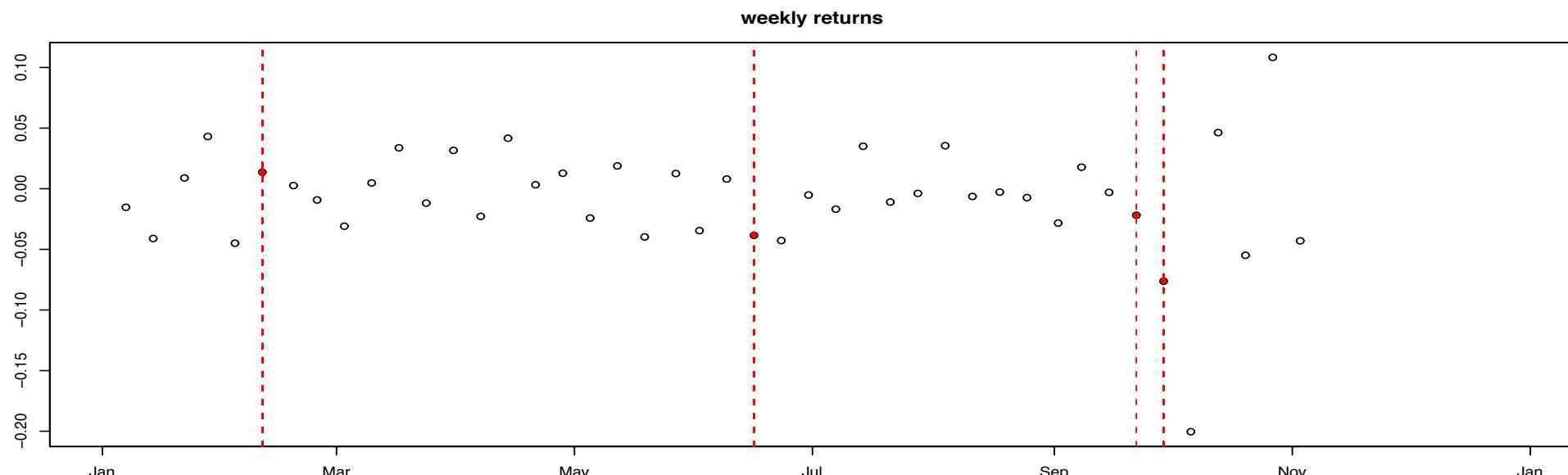
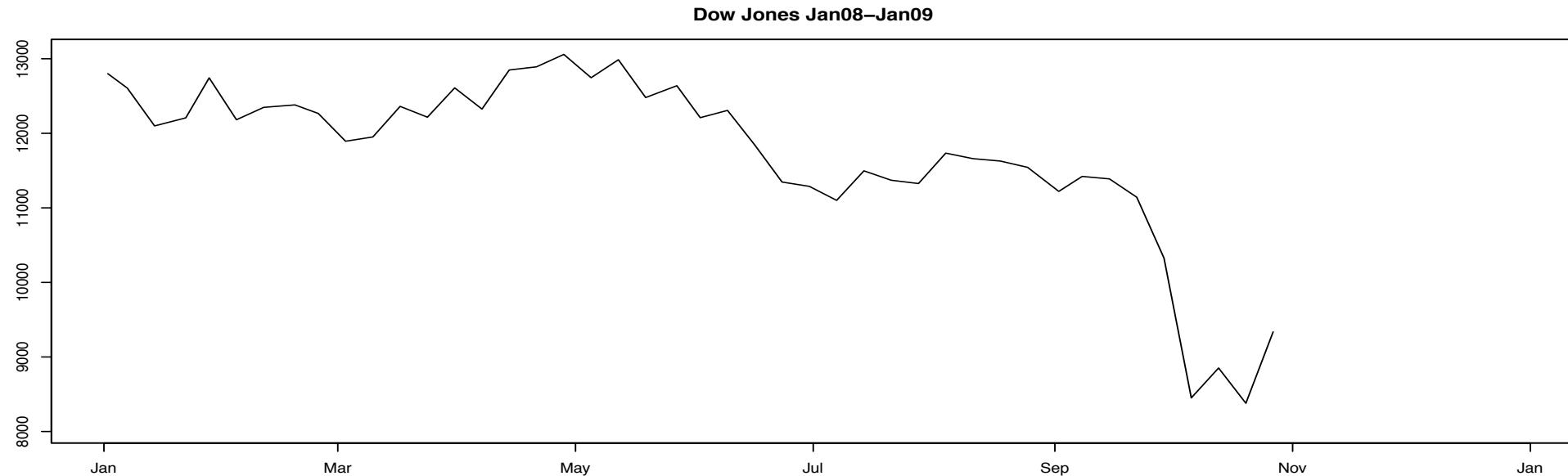
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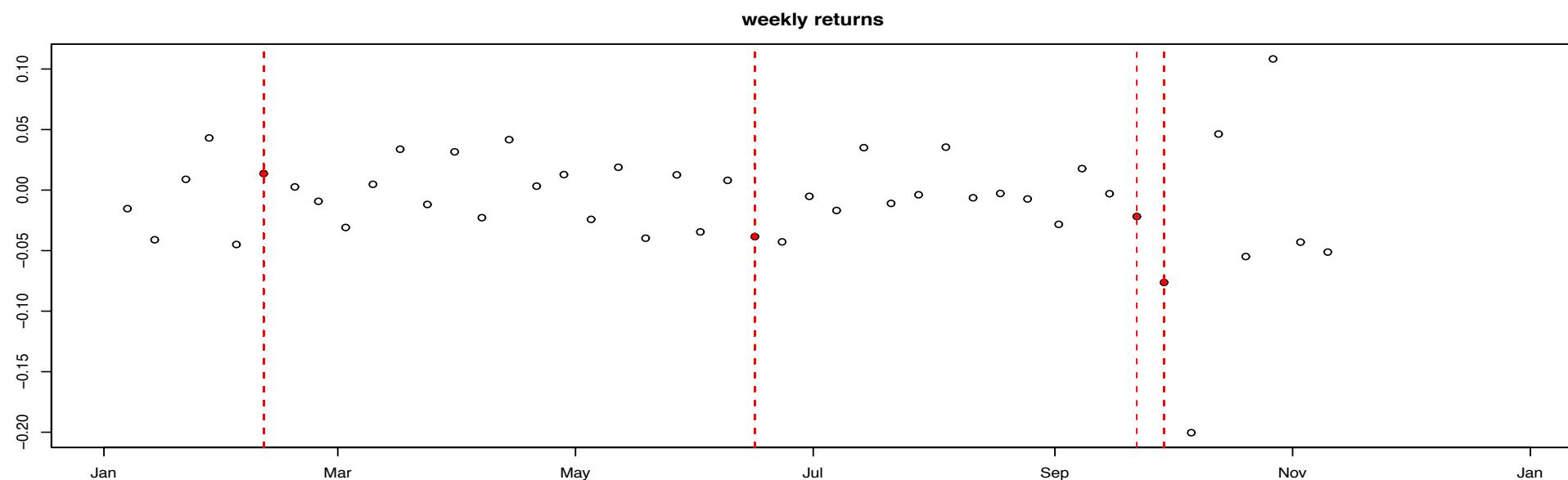
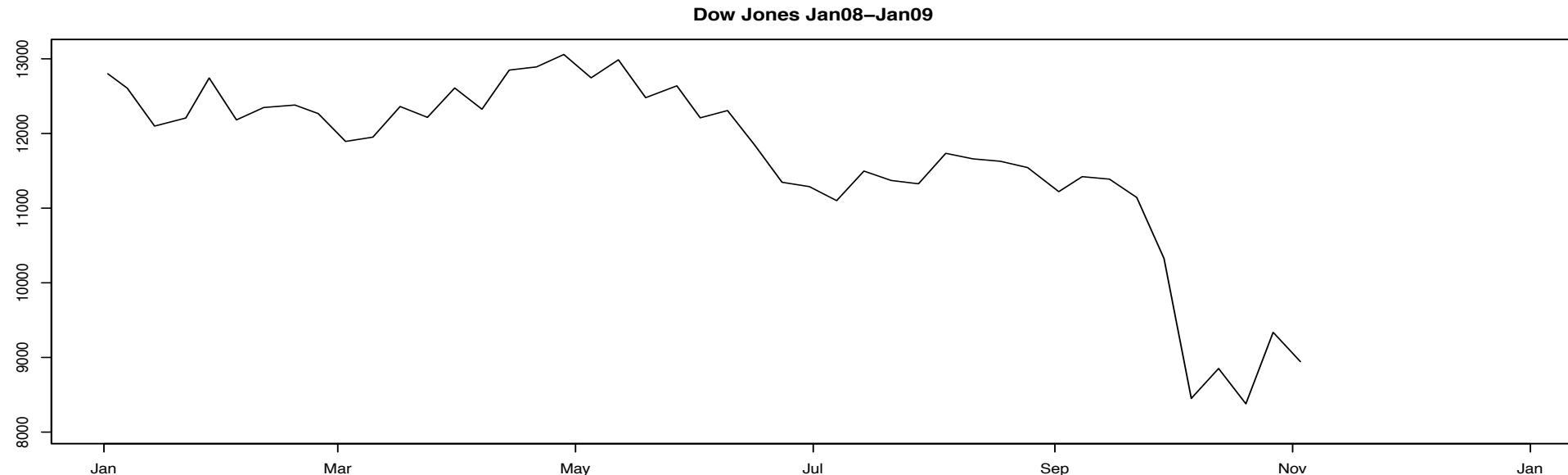
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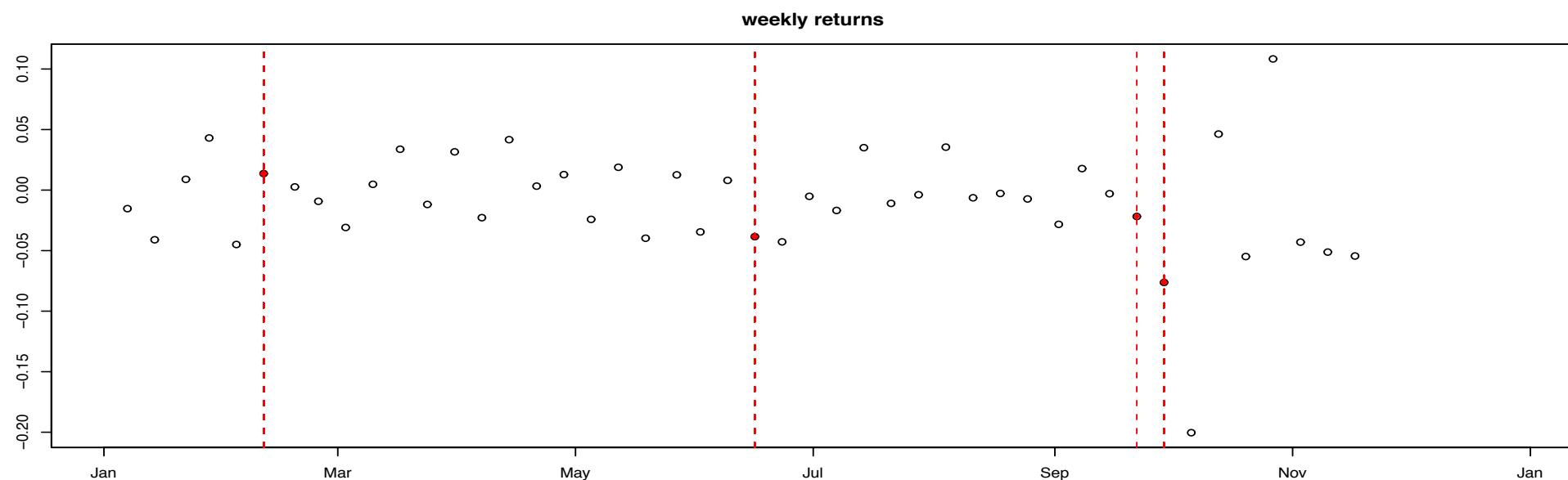
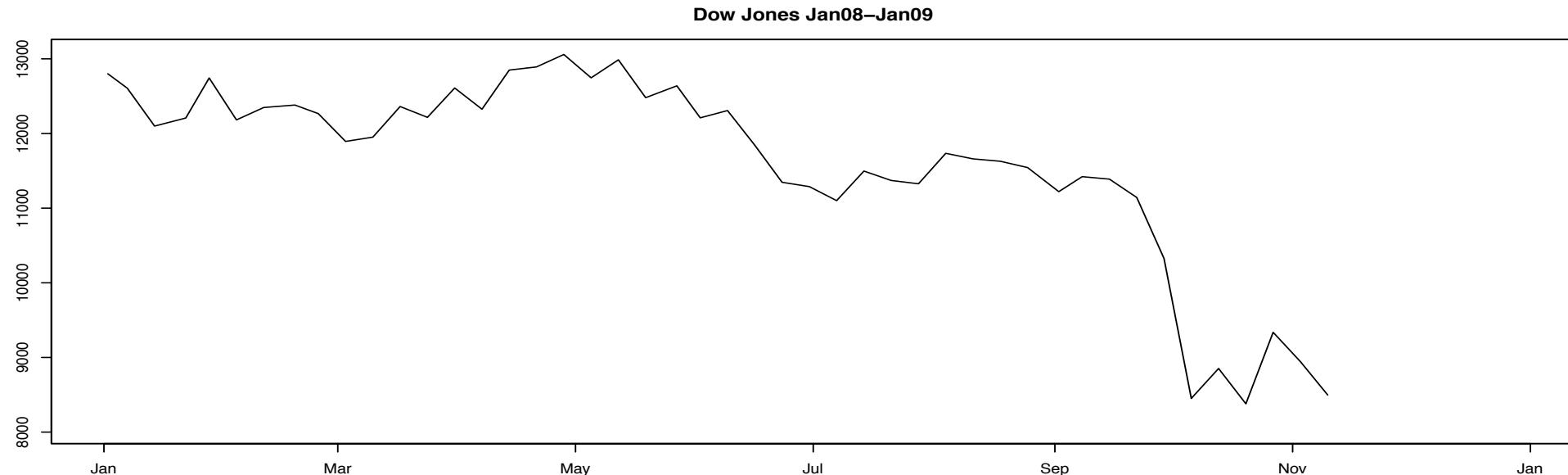
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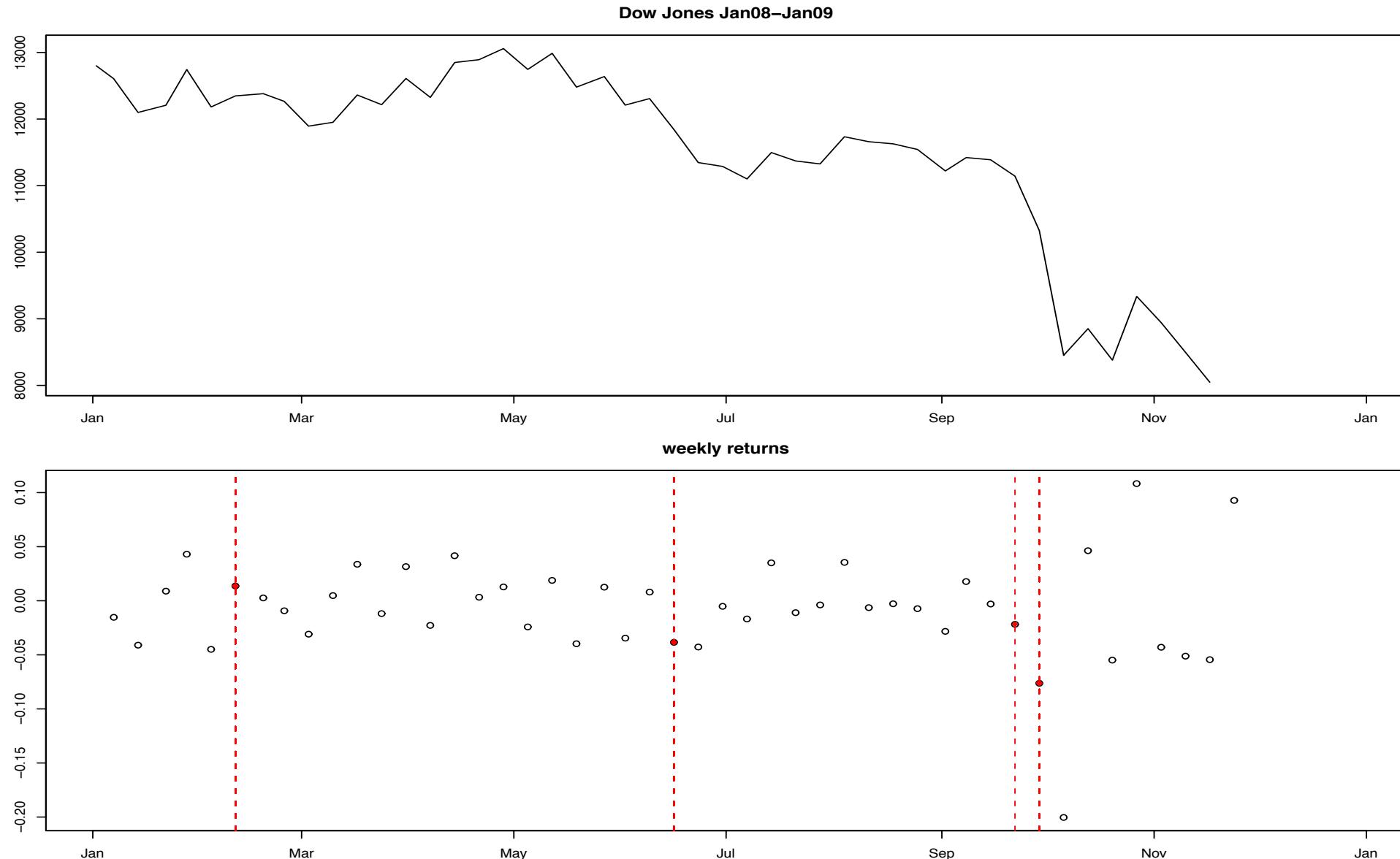
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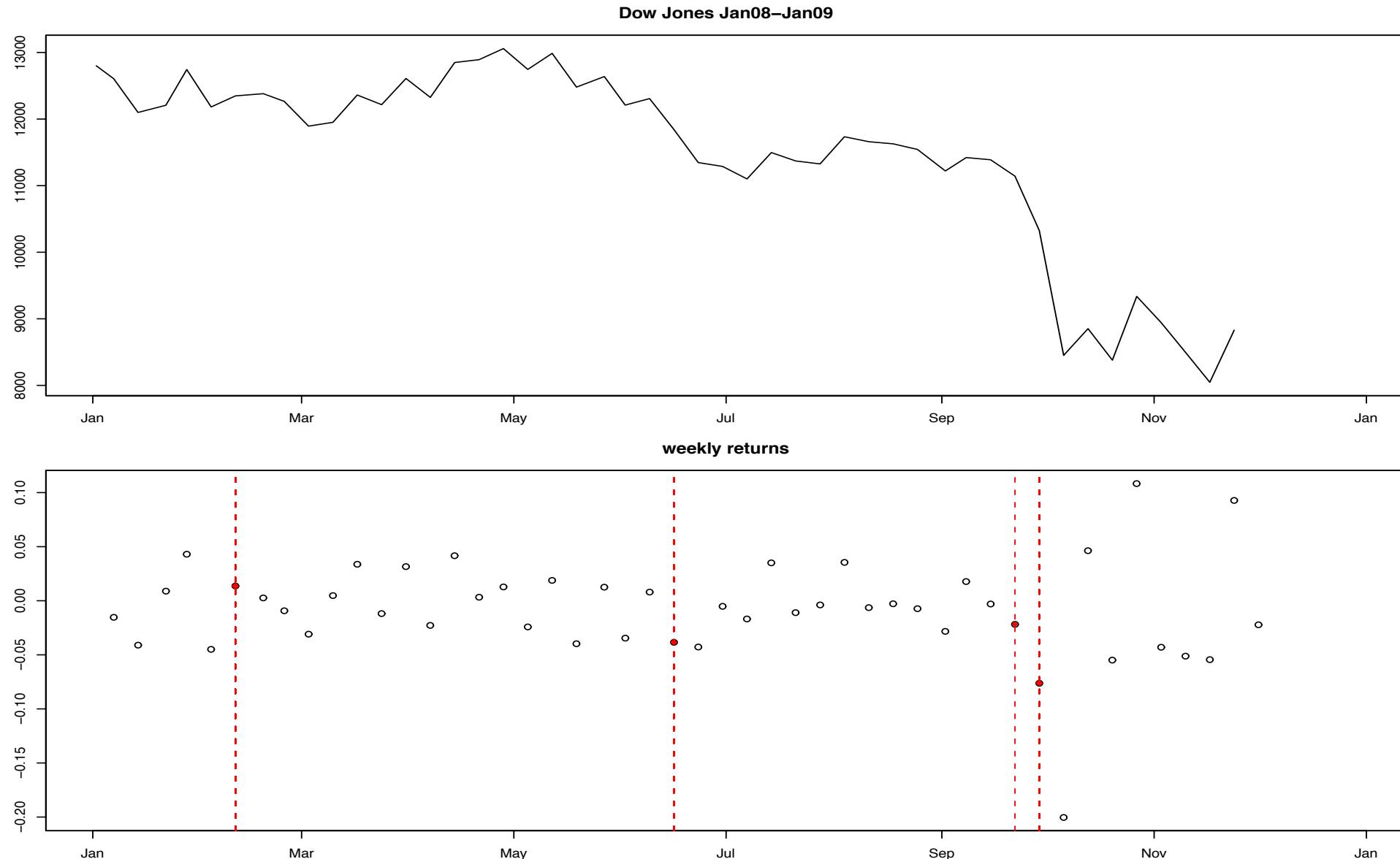
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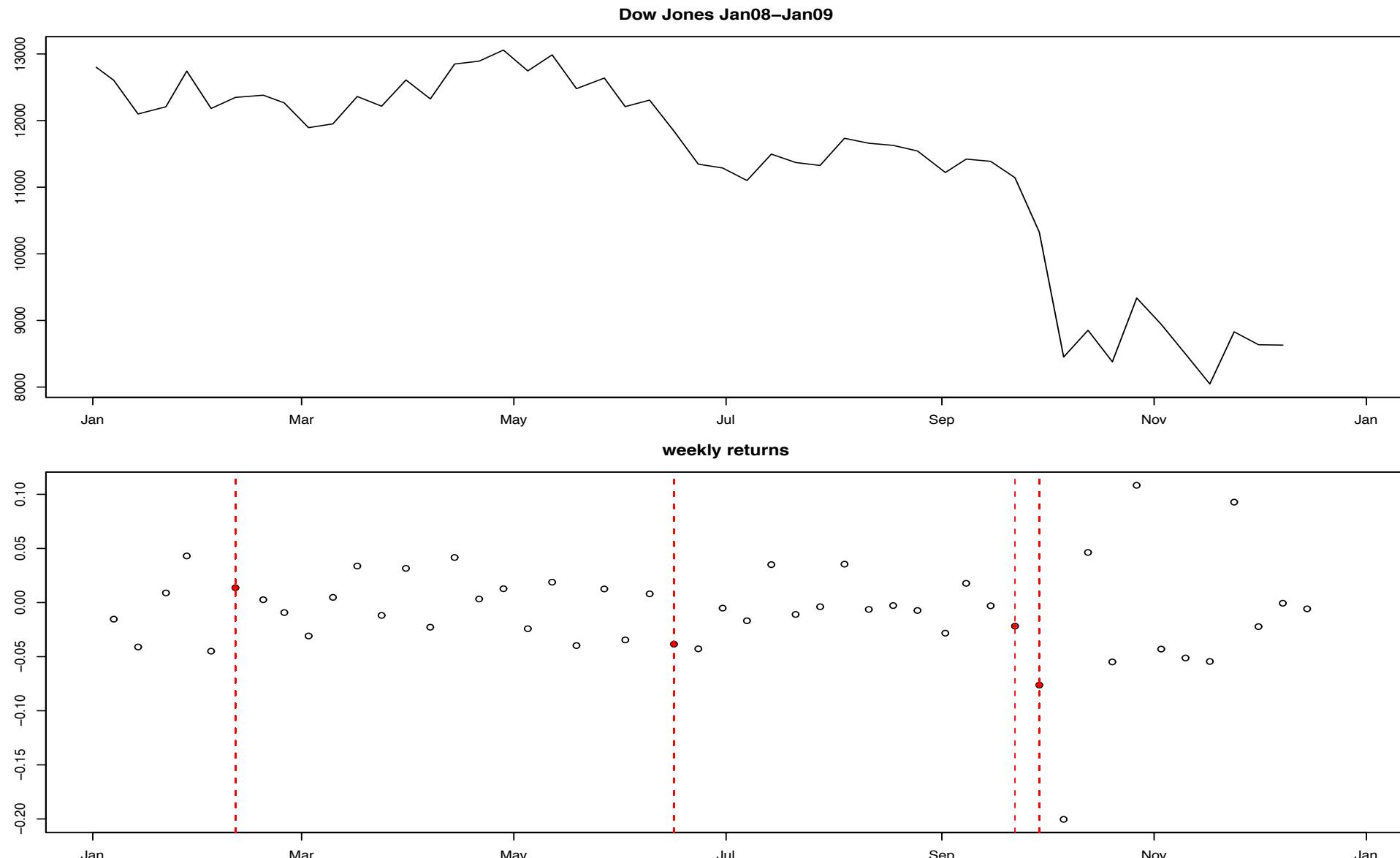
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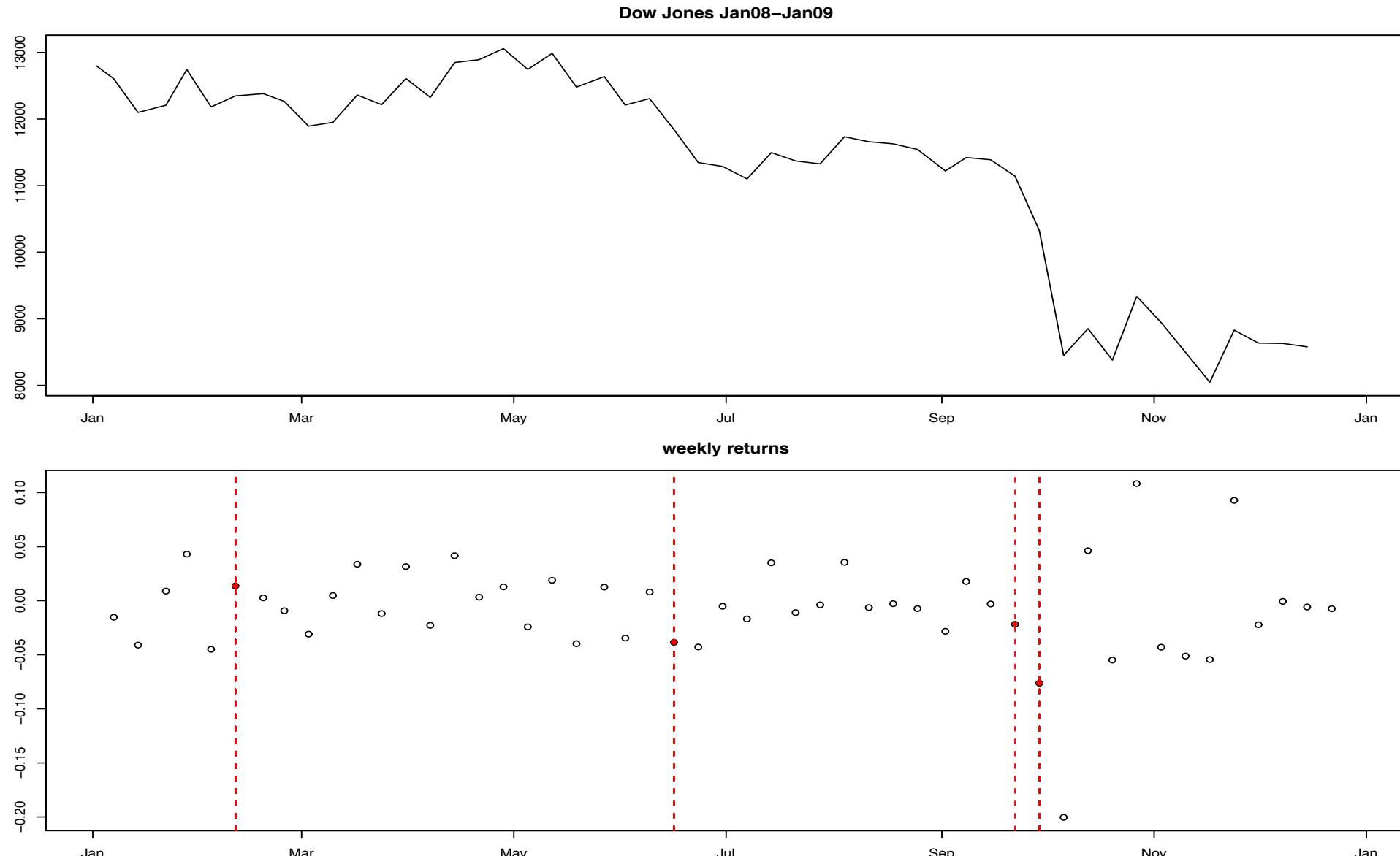
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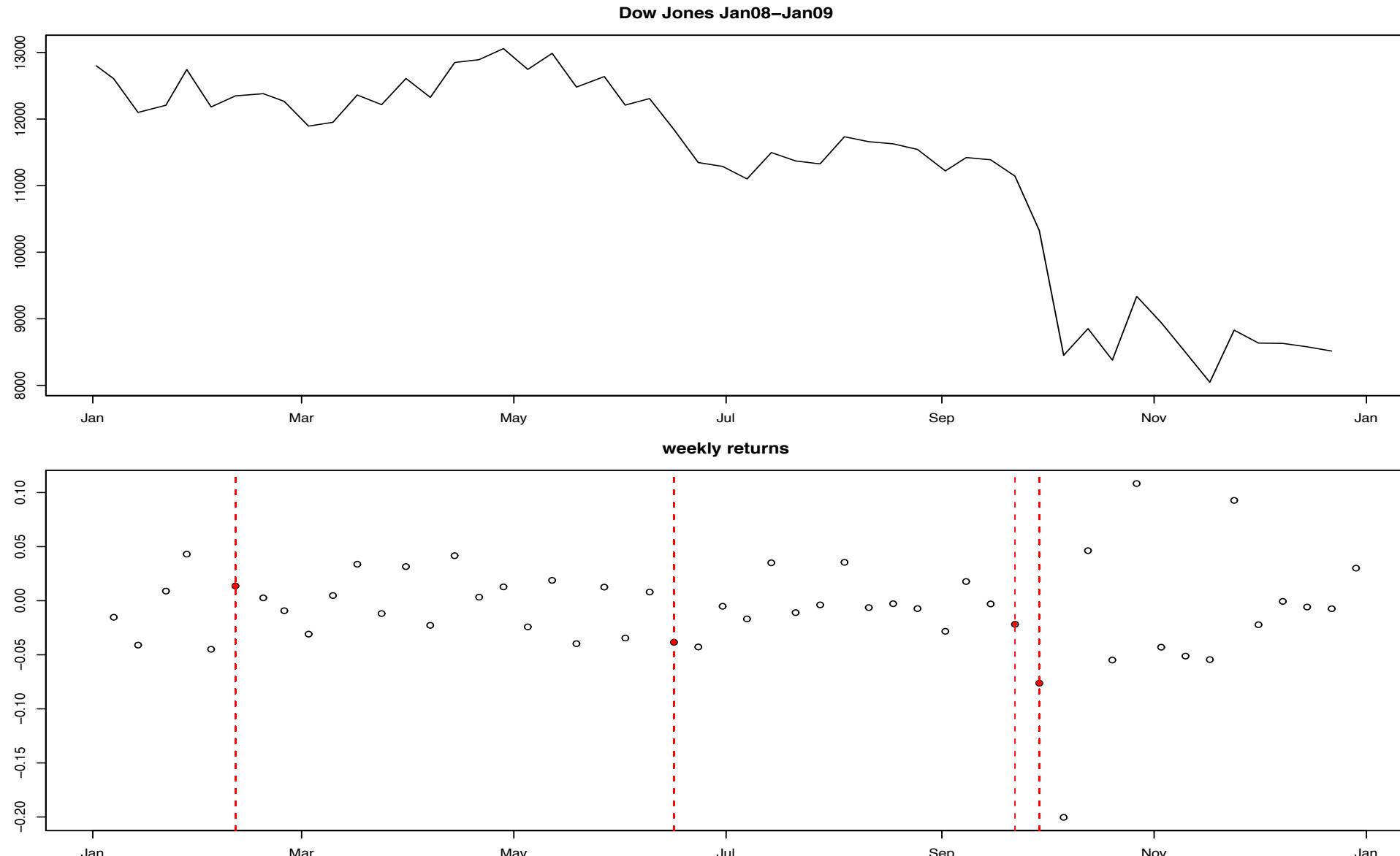
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Financial crisis 2008 - Real time analysis



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cpoint function in the sde package

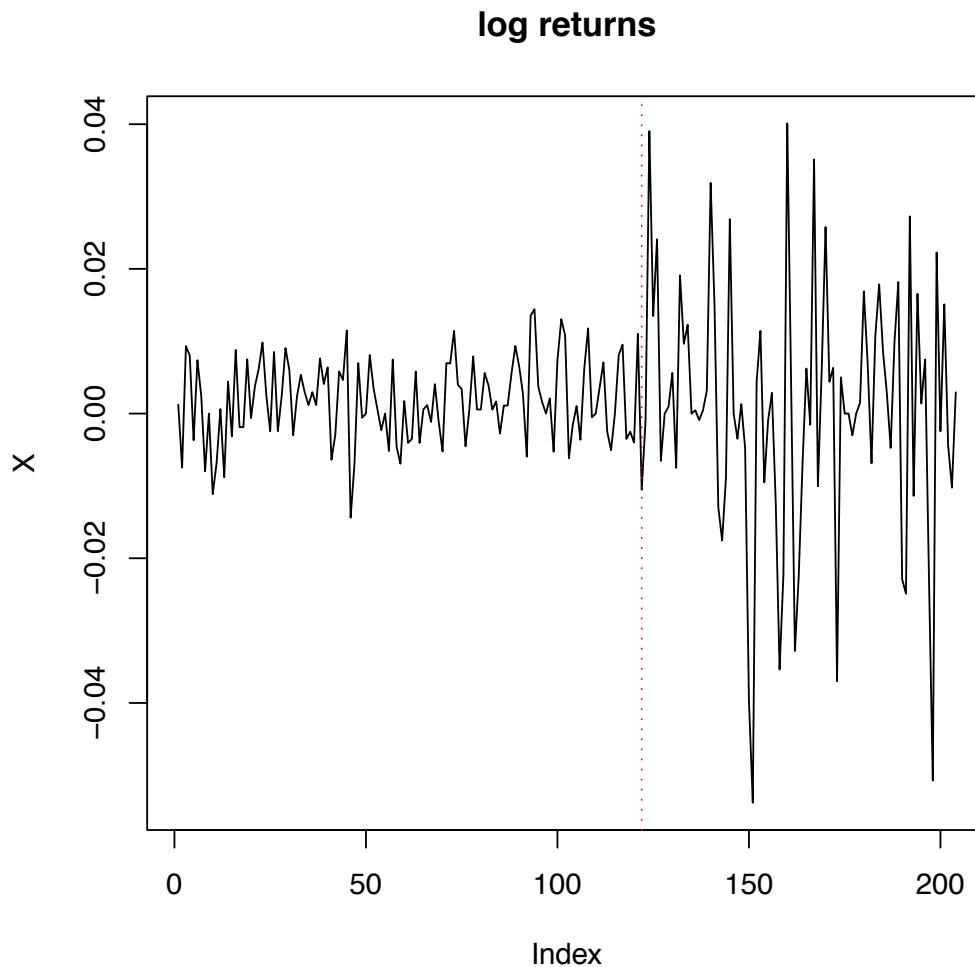
```
> library(tseries)
> S <- get.hist.quote("ATL.MI", start = "2004-07-23", end = "2005-05-05")$Close
> require(sde)
> cpoint(S)

$k0
[1] 123

$tau0
[1] "2005-01-11"


```

cpoint function in the sde package



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Song and Lee (2009) considered the model

$$dX_t = b(X_t)dt + \sigma(\theta, X_t)dW_t, \quad 0 \leq t \leq T, X_0 = x_0,$$

under the sampling scheme $\Delta_n \rightarrow 0$, $n\Delta_n = T \rightarrow \infty$, with b and σ known.

The object is to study a test statistics to verify the presence of the change point rather than estimate the instant of the change point. The results of this paper are based also on Kessler (1997).

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The CUSUM test statistics is given the form

$$T_n = \max_{l \leq k \leq n} \frac{k^2}{n} (\hat{\theta}_{n,k} - \hat{\theta}_{n,n})^2 \hat{\mathcal{J}}_n$$

where $\hat{\theta}_{n,k}$ is the estimator based on the first k observations and $\hat{\theta}_{n,n}$ the one based on all the n observations. $\hat{\mathcal{J}}_n$ is a consistent estimator of

$$\mathcal{J} = 2 \int \left(\frac{\partial_\theta \sigma(\theta_0, x)}{\sigma(\theta_0, x)} \right)^2 d\mu_0(x)$$

μ_0 the invariant law, for $\theta = \theta_0$.

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The main regularity conditions are contained in **Assumption [A]_k**: let k be a positive integer.

- (i) The function $\sigma(\cdot, \cdot)$ is continuously differentiable with respect to x up to order k for all θ and its derivatives are twice differentiable in θ for all x . All derivatives belong to the sets of functions
 $\{f(x, \theta) : |f(\theta, x)| \leq C(1 + |x|)^C \text{ for some } C \text{ not depending on } \theta\}$.
- (ii) The function $b(\cdot)$ is continuously differentiable with respect to x up to order k and its derivatives are at most of polynomial growth in x .

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Under the additional assumptions $n\Delta_n^{\textcolor{red}{p}} \rightarrow 0$, $n\Delta_n^{\textcolor{red}{q}} \rightarrow \infty$, $p > q > 4$ and $[\mathbf{A}]_k$ with $k = 2k_0$, $k_0 = [p/2]$, under $H_0 : \theta_1 = \theta_2 = \theta_0$, i.e. “no change”,

$$T_n \xrightarrow{d} \sup_{0 \leq s \leq 1} B_0^2(s)$$

with B_0 the Brownian bridge and

$$\mathcal{J}^{\frac{1}{2}} \frac{[ns]}{\sqrt{n}} \left(\hat{\theta}_{n,[ns]} - \theta_0 \right) \xrightarrow{d} W_s \quad \text{in} \quad \mathbb{D}[0, 1],$$

No properties of the change point estimator but only on the estimator of parameters.

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Under the above conditions on p and q , CUSUM test may loose efficiency because the time series has to be too long, e.g., for daily data at least 5 years of observations are needed in order to get good rate of convergence.

Recently, S. Lee (2011), considered a new type of CUSUM-type test statistics based on the residuals which provide better performance than his previous result above.

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As in I. and Yoshida (2012), we consider

$$dY_t = \mathbf{b}_t dt + \sigma(\theta, X_t) dW_t, \quad t \in [0, T],$$

where W_t is an r -dimensional standard Wiener process, on a stochastic basis, b_t and X_t are vector valued progressively measurable processes, and $\sigma(\theta, x)$ is a matrix valued function.

The coefficient $\sigma(\theta, x)$ is assumed to be known up to the parameter θ , while \mathbf{b}_t is completely unknown and unobservable, therefore possibly depending on θ and τ^* .

The parameter space Θ of θ is a bounded domain in R^{d_0} , $d_0 \geq 1$, and the parameter θ is a nuisance in estimation of τ^* . Denote by θ_k^* the true value of θ_k for $k = 1, 2$.

$$dY_t = b_t dt + \sigma(\theta, X_t) dW_t$$

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The sample consists of $(X_{t_i}, Y_{t_i}), i = 0, 1, \dots, n$, where $t_i = i\Delta$ for $\Delta = \Delta_n = T/n$. In this work T is fixed. Pure high-frequency setup.

$$dY_t = b_t dt + \sigma(\theta, X_t) dW_t$$

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It will be assumed that the process Y generating the data is an Itô process and satisfies the stochastic integral equation

$$Y_t = \begin{cases} Y_0 + \int_0^t b_s ds + \int_0^t \sigma(\theta_1^*, X_s) dW_s & \text{for } t \in [0, \tau^*] \\ Y_{\tau^*} + \int_{\tau^*}^t b_s ds + \int_{\tau^*}^t \sigma(\theta_2^*, X_s) dW_s & \text{for } t \in [\tau^*, T]. \end{cases}$$

$$dY_t = b_t dt + \sigma(\theta, X_t) dW_t$$

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We assume that consistent estimators for θ exist and afterward propose a way to obtain them. The focus is on the estimation of the change point and its properties.

$$dY_t = b_t dt + \sigma(\theta, X_t) dW_t$$

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$$Y_t = \begin{cases} Y_0 + \int_0^t b_s ds + \int_0^t \sigma(\theta_1^*, X_s) dW_s & \text{for } t \in [0, \tau^*] \\ Y_{\tau^*} + \int_{\tau^*}^t b_s ds + \int_{\tau^*}^t \sigma(\theta_2^*, X_s) dW_s & \text{for } t \in [\tau^*, T]. \end{cases}$$

We assume that consistent estimators for θ exist and afterward propose a way to obtain them. The focus is on the estimation of the change point and its properties.

Remark: clearly this model includes diffusion models by taking, e.g., $Y_t = X_t$ and $b_t = b(X_t)$.

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Write $\Delta_i Y = Y_{t_i} - Y_{t_{i-1}}$ and let

$$\Phi_n(t; \theta_1, \theta_2) = \sum_{i=1}^{[nt/T]} G_i(\theta_1) + \sum_{i=[nt/T]+1}^n G_i(\theta_2),$$

where

$$G_i(\theta) = \log \det S(X_{t_{i-1}}, \theta) + \Delta^{-1} S(X_{t_{i-1}}, \theta)^{-1} [(\Delta_i Y)^{\otimes 2}]$$

with $S(x, \theta) = \sigma(\theta, x)^{\otimes 2}$ with $A[B^{\otimes 2}] = B'AB$. This is a modification of the QL introduced in Genon-Catalot and Jacod (1993).

Change point for multidimensional Itô processes

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Suppose that there exists an estimator $\hat{\theta}_k$ for each θ_k , $k = 1, 2$. In case θ_k^* are known, we define $\hat{\theta}_k$ just as $\hat{\theta}_k = \theta_k^*$.

$$\hat{\tau}_n = \arg \min_{t \in [0, T]} \Phi_n(t; \hat{\theta}_1, \hat{\theta}_2)$$

for the estimation of τ^* .

More precisely, $\hat{\tau}_n$ is any measurable function of $(X_{t_i})_{i=0,1,\dots,n}$ satisfying

$$\Phi_n(\hat{\tau}_n; \hat{\theta}_1, \hat{\theta}_2) = \min_{t \in [0, T]} \Phi_n(t; \hat{\theta}_1, \hat{\theta}_2).$$

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[H]_j (i) $\sigma(\theta, x)$ is a measurable function defined on $\Theta \times \mathcal{X}$ satisfying

- (a) $\inf_{(x, \theta) \in \mathcal{X} \times \Theta} \lambda_1(S(x, \theta)) > 0$, (λ_1 first eigenvalue of S)
- (b) derivatives $\partial_\theta^\ell \sigma$ ($0 \leq \ell \leq j + [d_0/2]$) exist and those functions are continuous on $\Theta \times \mathcal{X}$,
- (c) there exists a locally bounded function $L : \mathcal{X} \times \mathcal{X} \times \Theta \rightarrow \mathbb{R}_+$ such that

$$|\sigma(\theta, x) - \sigma(\theta, x')| \leq L(x, x', \theta) |x - x'|^\alpha \quad (x, x' \in \mathcal{X}, \theta \in \Theta)$$

for some constant $\alpha > 0$.

(ii) $(X_t)_{t \in [0, T]}$ is a progressively measurable process taking values in \mathcal{X} such that

$$w_{[0, T]} \left(\frac{1}{n}, X \right) = o_p(\vartheta_n^{1/\alpha}), \quad (w : \text{modulus of continuity func.})$$

as $n \rightarrow \infty$.

(iii) $(b_t)_{t \in [0, T]}$ is a progressively measurable process taking values in \mathbb{R}^d such that $(b_t - b_0)_{t \in [0, T]}$ is locally bounded.

Identifiability conditions & consistent estimators

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[A] $P [S(X_{\tau^*}; \theta_1^*) \neq S(X_{\tau^*}; \theta_2^*)] = 1;$

[B] $\Xi(X_{\tau^*}, \theta^*)$ is positive-definite a.s., where

$$\Xi(x, \theta) = \left(\text{Tr}((\partial_{\theta^{(i_1)}} S) S^{-1} (\partial_{\theta^{(i_2)}} S) S^{-1})(x, \theta) \right)_{i_1, i_2=1}^{d_0}, \quad \theta = (\theta^{(i)}).$$

[C] Let $\vartheta_n = |\theta_2^* - \theta_1^*|$ and $|\hat{\theta}_k - \theta_k^*| = o_p(\vartheta_n)$ as $n \rightarrow \infty$ for $k = 1, 2$.

In case the parameters are known, $\hat{\theta}_k$ should read θ_k^* , and then Condition [C] requires nothing.

Two kinds of limit theorems

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Let $\vartheta_n = |\theta_2^* - \theta_1^*|$. We will consider the following two different situations.

\mathcal{A} : “Non shrinking case”: θ_1^* and θ_2^* are fixed and do not depend on n .

\mathcal{B} : “Shrinking case” (contiguous alternatives): θ_1^* and θ_2^* depend on n , and as $n \rightarrow \infty$, $\theta_1^* \rightarrow \theta^* \in \Theta$, $\vartheta_n \rightarrow 0$ and $n\vartheta_n^2 \rightarrow \infty$.

In Case \mathcal{A} , ϑ_n is a constant ϑ_0 independent of n .

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Theorem 1: *The family $\{n\vartheta_n^2(\hat{\tau}_n - \tau^*)\}_{n \in \mathbb{N}}$ is tight under any one of the following conditions.*

- (a) $[H]_1$, $[A]$ and $[C]$ hold in Case \mathcal{A} .
- (b) $[H]_2$, $[B]$ and $[C]$ hold in Case \mathcal{B} .

In Case \mathcal{B} , this result gives immediately consistency of $\hat{\tau}_n$ since $n\vartheta_n^2 \rightarrow \infty$ by assumption.

Asymptotic distribution: shrinking case

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Let

$$\mathbb{H}(v) = -2 \left(\Gamma_{\eta}^{\frac{1}{2}} \mathcal{W}(v) - \frac{1}{2} \Gamma_{\eta} |v| \right)$$

for $\Gamma_{\eta} = (2T)^{-1} \Xi(X_{\tau^*}, \theta^*)[\eta^{\otimes 2}]$. Here \mathcal{W} is a two-sided standard Wiener process independent of X_{τ^*} .

Theorem 2: Suppose that the limit $\eta = \lim_{n \rightarrow \infty} \vartheta_n^{-1}(\theta_2^* - \theta_1^*)$ exists. Suppose that [H]₂, [C] and [B] are fulfilled in Case B. Then

$$n\vartheta_n^2(\hat{\tau}_n - \tau^*) \xrightarrow{d} \arg \min_{v \in R} \mathbb{H}(v)$$

as $n \rightarrow \infty$.

Remind that in the simple 1-d case we have

$$\frac{n\vartheta_n^2(\hat{\tau}_n - \tau^*)}{2\hat{\theta}^2} \xrightarrow{d} \arg \max_v \left\{ \mathcal{W}(v) - \frac{|v|}{2} \right\}$$

Asymptotic distribution: non shrinking case

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Let

$$\mathbb{K}(v) = \sum_{i=1}^v \left\{ \text{Tr} \left[{}^t \sigma(\theta_2^*, X_{\tau^*}) (S(X_{\tau^*}, \theta_1^*)^{-1} - S(X_{\tau^*}, \theta_2^*)^{-1}) \sigma(\theta_2^*, X_{\tau^*}) \zeta_i^{\otimes 2} \right] \right. \\ \left. - \log \det \left(S(X_{\tau^*}, \theta_1^*)^{-1} S(X_{\tau^*}, \theta_2^*) \right) \right\},$$

where ζ_i are independent r -dimensional standard normal variables and independent of X_{τ^*} .

Theorem 3: Suppose that $[H]_1$, $[C]$ and $[A]$ are fulfilled in Case A. Then

$$\hat{k}_n - \left[\frac{n\tau^*}{T} \right] \xrightarrow{d} \arg \min_{v \in \mathbb{Z}} \mathbb{K}(v)$$

as $n \rightarrow \infty$.

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When θ_1 and θ_2 are not known, we propose a two stage estimation procedure:

- First θ_1 is estimated with $\hat{\theta}_1$ using the first part of the series $[0, a_n]$ and θ_2 with $\hat{\theta}_2$ using the last part $[T - a_n, T]$.
- Then a first stage change point estimator $\hat{\tau}$ is obtained plugin in the contrast function the first stage estimators $\hat{\theta}_1$ and $\hat{\theta}_2$.
- Then, a second stage estimator of the change point $\check{\tau}$ is obtained, and further refinement of the estimators of θ_1 and θ_2 can be calculated, say $\check{\theta}_1$ and $\check{\theta}_2$.

Two stage procedure

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Under a suitable sequence a_n , Theorems 1-3 hold for $(\check{\tau}, \check{\theta}_1, \check{\theta}_2)$.

For example, we assume that there existst a constant $\beta \in (0, 1/2)$ such that $a_n \geq 1/(n\vartheta_n^{1/\beta})$ and that $|\hat{\theta}_k - \theta_k^*| = o_p((na_n)^{-\beta})$ for $k = 1, 2$.

When $\limsup_{n \rightarrow \infty} \vartheta_n > 0$, we also assume $na_n \rightarrow \infty$.

The first condition implies $n\vartheta_n^2 \rightarrow \infty$, which is required in our theorems.

The second condition is natural because the number of data is proportional to na_n .

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For small sample sizes or when the asymptotic is not realized, it may be the case to modify the QL in the following way

$$\Phi'_n(t; \theta_1, \theta_2) = \sum_{i=1}^{[nt/2T]} G'_i(\theta_1) + \sum_{i=[nt/2T]+1}^n G'_i(\theta_2), \quad (4)$$

with

$$G'_i(\theta) = \log \det S(X_{t_{2i-1}}, \theta) + \Delta_n^{-1} S(X_{t_{2i-1}}, \theta)^{-1} [(\tilde{\Delta}_i Y)^{\otimes 2}]$$

and

$$\tilde{\Delta} Y_i = \frac{Y_{2i+1} - 2Y_{2i} + Y_{2i-1}}{\sqrt{2}}$$

With this approach we use less observation in the estimator of τ^* but we control better for the drift effect. Asymptotically the same results hold. So no change in the proofs and in the construction of the estimators.

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To test the finite sample behaviour of all estimators considered in this review, we set up a simulation study.

We consider different combinations of the sample size $n = 500, 1000, 2000$ and time horizon $T = 1, 2, 5, 10$, which correspond to different $\Delta_n = 0.01, 0.001$. Each experiment is replicated 1000 times.

We consider three different statistical models.

We compare the estimators: LS ($\hat{\tau}_n$), CUSUM ($\tilde{\tau}_n$), QML ($\check{\tau}_n$) and the modified QML ($\check{\tau}'_n$).

Finite sample properties. Driftless model.

The change point problem	n	Δ_n	$T = n\Delta_n$	τ^*	$\hat{\tau}_n$ L.S.	$\check{\tau}_n$ Q.MLE	$\tilde{\tau}_n$ CUSUM	$\check{\tau}'_n$ Modified Q.MLE.
Historical remarks	1000	0.001	1	0.6	0.606 (0.061)	0.602 (0.012)	0.751 (0.150)	0.604 (0.031)
Diffusion processes	2000	0.001	2	1.2	1.202 (0.196)	1.205 (0.095)	1.784 (0.316)	1.214 (0.109)
General Itô processes	500	0.01	5	3	3.020 (0.698)	3.024 (0.512)	4.206 (0.801)	3.075 (0.623)
Numerical experiment	1000	0.01	10	6	5.981 (1.122)	6.087 (1.617)	8.804 (1.547)	6.129 (1.296)
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$$dX_t = (1 + X_t^2)^{\theta^*} dW_t$$

Finite sample properties. Model with drift.

The change point problem	n	Δ_n	$T = n\Delta_n$	τ^*	$\hat{\tau}_n$	$\check{\tau}_n$	$\tilde{\tau}_n$	$\check{\tau}'_n$
				L.S.	Q.MLE	CUSUM	Modified Q.MLE	
Historical remarks	1000	0.001	1	0.6	0.609 (0.018)	0.600 (0.005)	0.692 (0.159)	0.602 (0.011)
Diffusion processes	2000	0.001	2	1.2	1.230 (0.049)	1.200 (0.005)	1.272 (0.223)	1.202 (0.010)
General Itô processes	500	0.01	5	3	3.644 (0.320)	2.977 (0.067)	3.005 (0.012)	3.002 (0.052)
Numerical experiment	1000	0.01	10	6	8.179 (0.221)	3.607 (0.375)	5.983 (0.045)	6.001 (0.011)
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$$dX_t = X_t dt + (1 + X_t^2)^{\theta^*} dW_t$$

Finite sample properties. Mean reverting.

The change point problem	n	Δ_n	$T = n\Delta_n$	τ^*	$\hat{\tau}_n$ L.S.	$\check{\tau}_n$ Q.MLE	$\tilde{\tau}_n$ CUSUM	$\check{\tau}'_n$ Modified Q.MLE
Historical remarks	1000	0.001	1	0.6	0.604 (0.008)	0.601 (0.007)	0.895 (0.136)	0.602 (0.015)
Diffusion processes	2000	0.001	2	1.2	1.204 (0.009)	1.202 (0.009)	1.842 (0.258)	1.204 (0.017)
General Itô processes	500	0.01	5	3	3.030 (0.051)	3.033 (0.084)	4.299 (0.691)	3.013 (0.145)
Numerical experiment	1000	0.01	10	6	6.036 (0.076)	6.027 (0.078)	8.947 (1.367)	6.022 (0.146)
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$$dX_t = (2 - X_t)dt + \theta^* dW_t$$

Empirical vs True Asymptotic Distribution. Model 1

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Due to mixed-normal limit, we studied the distribution of the studentized limiting distribution of $n\theta_n^2(\check{\tau}_n - \tau^*)$ under the true model, i.e.

$$Z = n\theta_n^2(\check{\tau}_n - \tau^*)\hat{\Gamma}(X_{\tau^*}, \theta_0),$$

with $\hat{\Gamma}(X_{\tau^*}, \theta_0) = (\log(1 + X_{\tau^*}^2))^2$. Then Z converges to $\mathcal{W}(v) - \frac{1}{2}|v|$ with density

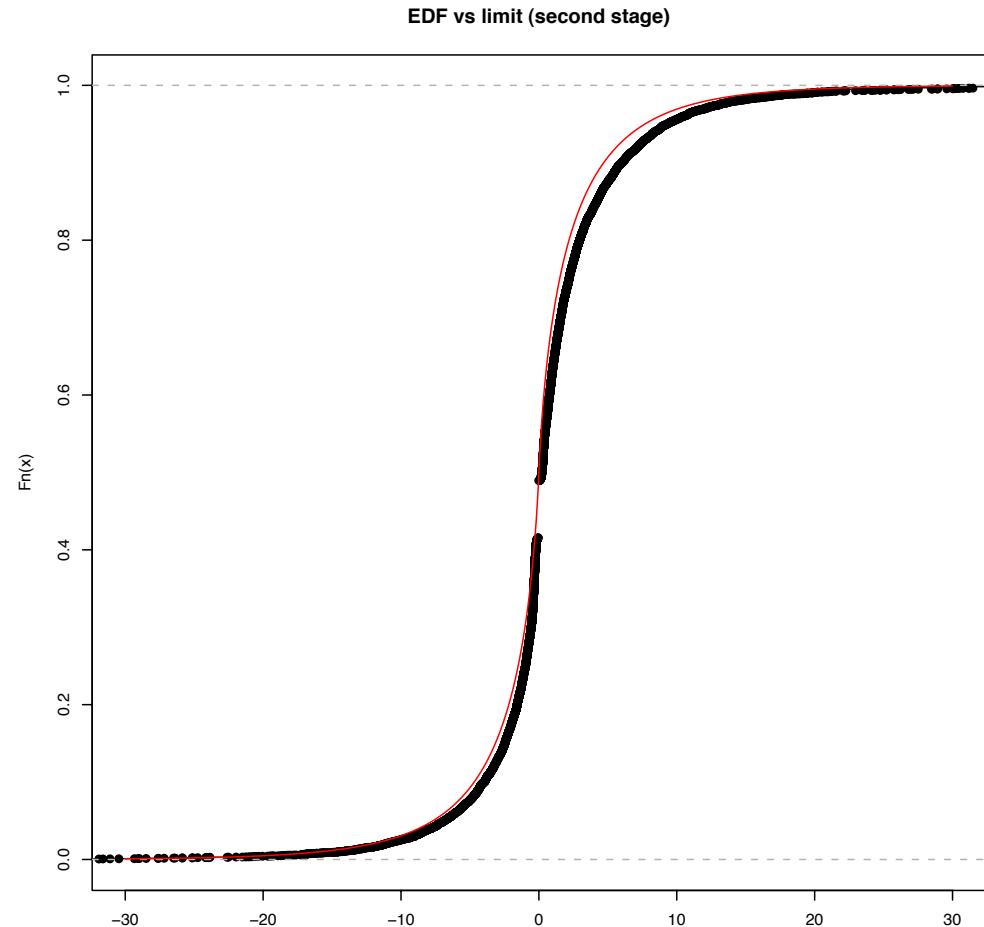
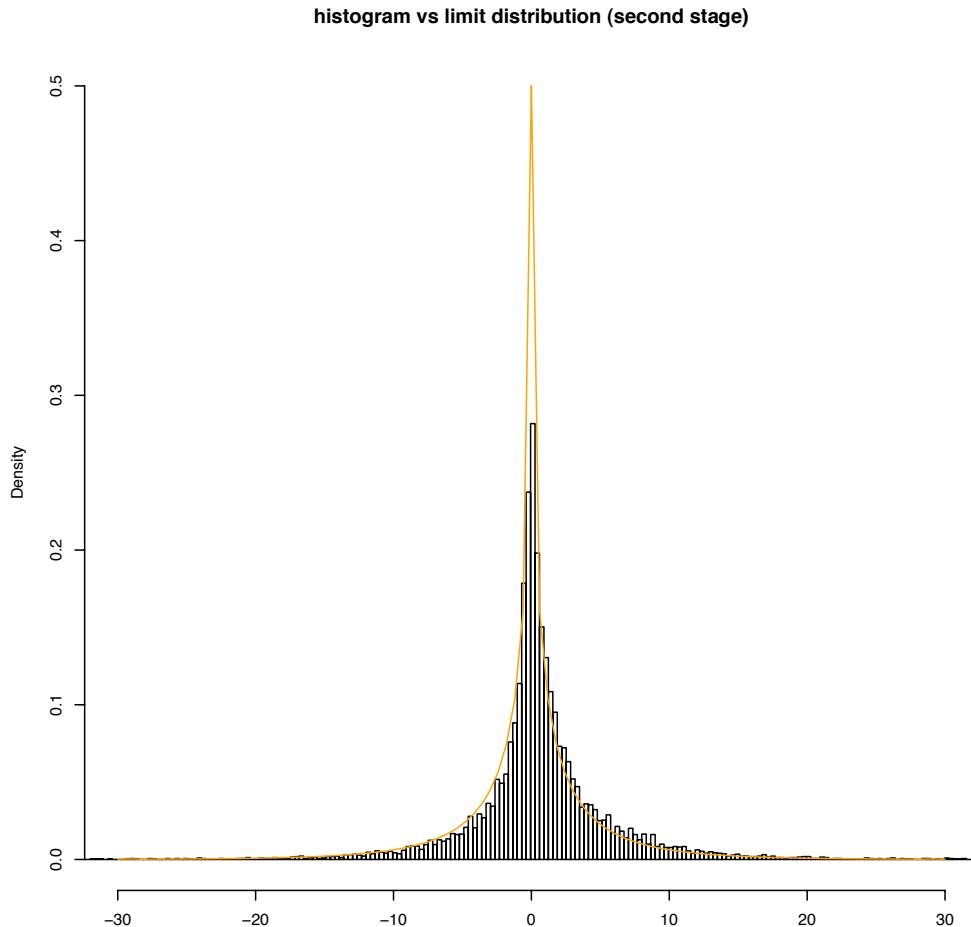
$$f(x) = \frac{3}{2}e^{|x|} \left(1 - \Phi\left(\frac{3}{2}\sqrt{|x|}\right) \right) - \frac{1}{2} \left(1 - \Phi\left(\frac{1}{2}\sqrt{|x|}\right) \right)$$

and distribution function $F(x) = \begin{cases} g(x), & x > 0 \\ 1 - g(-x), & x \leq 0 \end{cases}$, $\Phi(x)$ the distribution function of $\mathcal{N}(0, 1)$, and

$$g(x) = 1 + \sqrt{\frac{x}{2\pi}}e^{-\frac{x}{8}} - \frac{1}{2}(x+5)\Phi\left(-\frac{\sqrt{x}}{2}\right) + \frac{3}{2}e^x\Phi\left(-\frac{3}{2}\sqrt{x}\right)$$

(see e.g. Csörgő and Horváth, 1997).

Empirical vs True Asymptotic Distribution. Driftless model.



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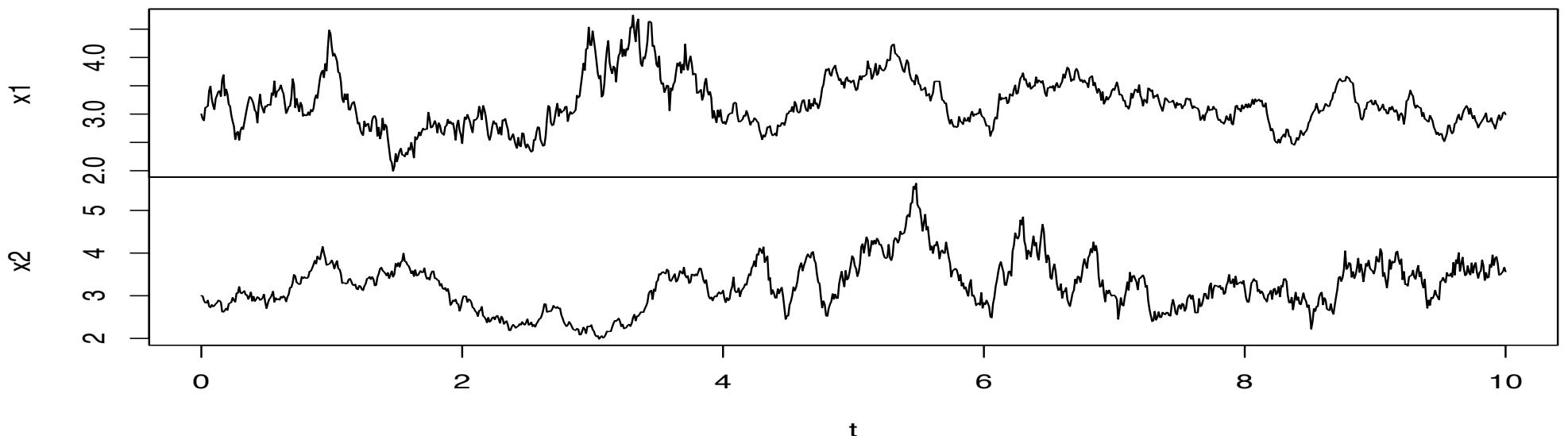
Example of Volatility Change-Point Estimation

Consider the 2-dimensional stochastic differential equation

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \end{pmatrix} = \begin{pmatrix} \sin(X_t^1) \\ 3 - X_t^2 \end{pmatrix} dt + \begin{bmatrix} \theta_{1,k} \cdot X_t^1 & 0 \cdot X_t^1 \\ 0 \cdot X_t^2 & \theta_{2,k} \cdot X_t^2 \end{bmatrix} \begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix}$$

$$X_0^1 = 1.0, \quad X_0^2 = 1.0,$$

with change point instant at time $\tau = 4$



The following model can be specified in the `yuima` package

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \end{pmatrix} = \begin{pmatrix} \sin(X_t^1) \\ 3 - X_t^2 \end{pmatrix} dt + \begin{bmatrix} \theta_{1,k} \cdot X_t^1 & 0 \cdot X_t^1 \\ 0 \cdot X_t^2 & \theta_{2,k} \cdot X_t^2 \end{bmatrix} \begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix}$$

```
> library(yuima)

> diff.matrix <- matrix(c("theta1.k*x1", "0", "0", "theta2.k*x2"), 2, 2)
> drift.c <- c("sin(x1)", "3-x2")
> drift.matrix <- matrix(drift.c, 2, 1)
> ymodel <- setModel(drift=drift.c, diffusion=diff.matrix,
  state.variable=c("x1", "x2"), solve.variable=c("x1", "x2"))

> yuima <- setYuima(model=model, data=mydata)
```

where `mydata` is the time series from previous slide simulated with
 $\theta_{1,k} = 0.5, \theta_{2,k} = 0.3$ before τ^* and $\theta_{1,k} = 0.2, \theta_{2,k} = 0.4$ after the change point

Example with YUIMA package

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We can call CPoint on the yuima object

```
> t1 <- coef(qmleL(yuima, t = 2, start = list(theta1.k = 0.1,
      theta2.k = 0.1), lower = list(theta1.k = 0, theta2.k = 0),
      upper = list(theta1.k = 1, theta2.k = 1), method = "L-BFGS-B" ))
> t2 <- coef(qmleR(yuima, t = 8, start = list(theta1.k = 0.1,
      theta2.k = 0.1), lower = list(theta1.k = 0, theta2.k = 0),
      upper = list(theta1.k = 1, theta2.k = 1), method = "L-BFGS-B" ))

> t.est <- CPoint(yuima,param1=t1,param2=t2)

> t.est$tau
[1] 3.99
> t.est$param1
theta1.k  theta2.k
0.4723067 0.2899005
> t.est$param2
theta1.k  theta2.k
0.2515379 0.5518635
```

<http://R-Forge.R-Project.org/projects/yuima>

The model specification

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We consider here the three main classes of SDE's which can be easily specified. All multidimensional and eventually parametric models.

The model specification

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We consider here the three main classes of SDE's which can be easily specified. All multidimensional and eventually parametric models.

- Diffusions $dX_t = a(t, X_t)dt + b(t, X_t)dW_t$

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We consider here the three main classes of SDE's which can be easily specified. All multidimensional and eventually parametric models.

- Diffusions $dX_t = a(t, X_t)dt + b(t, X_t)dW_t$
- Fractional Gaussian Noise, with H the Hurst parameter

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t^H$$

The model specification

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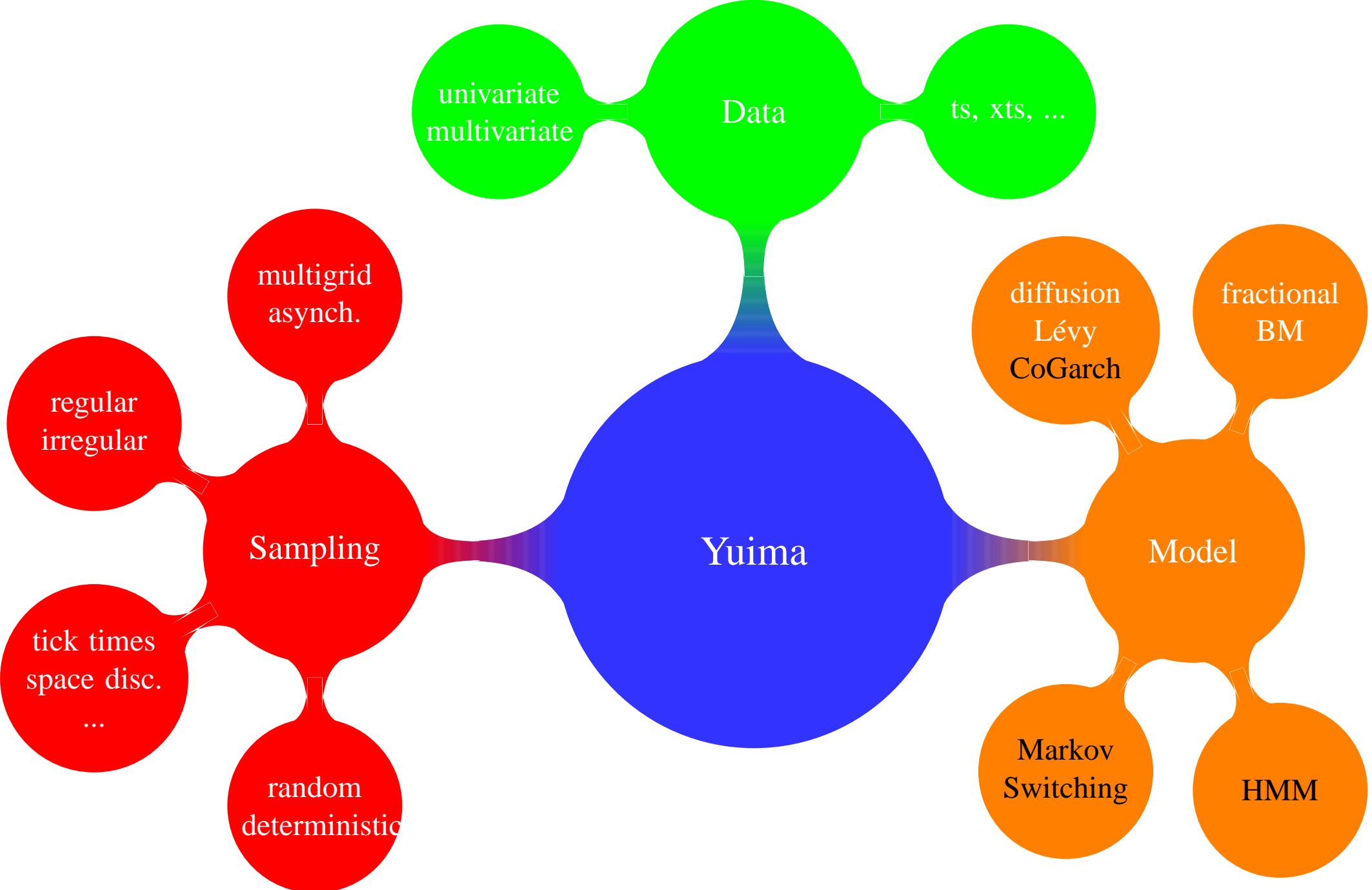
We consider here the three main classes of SDE's which can be easily specified. All multidimensional and eventually parametric models.

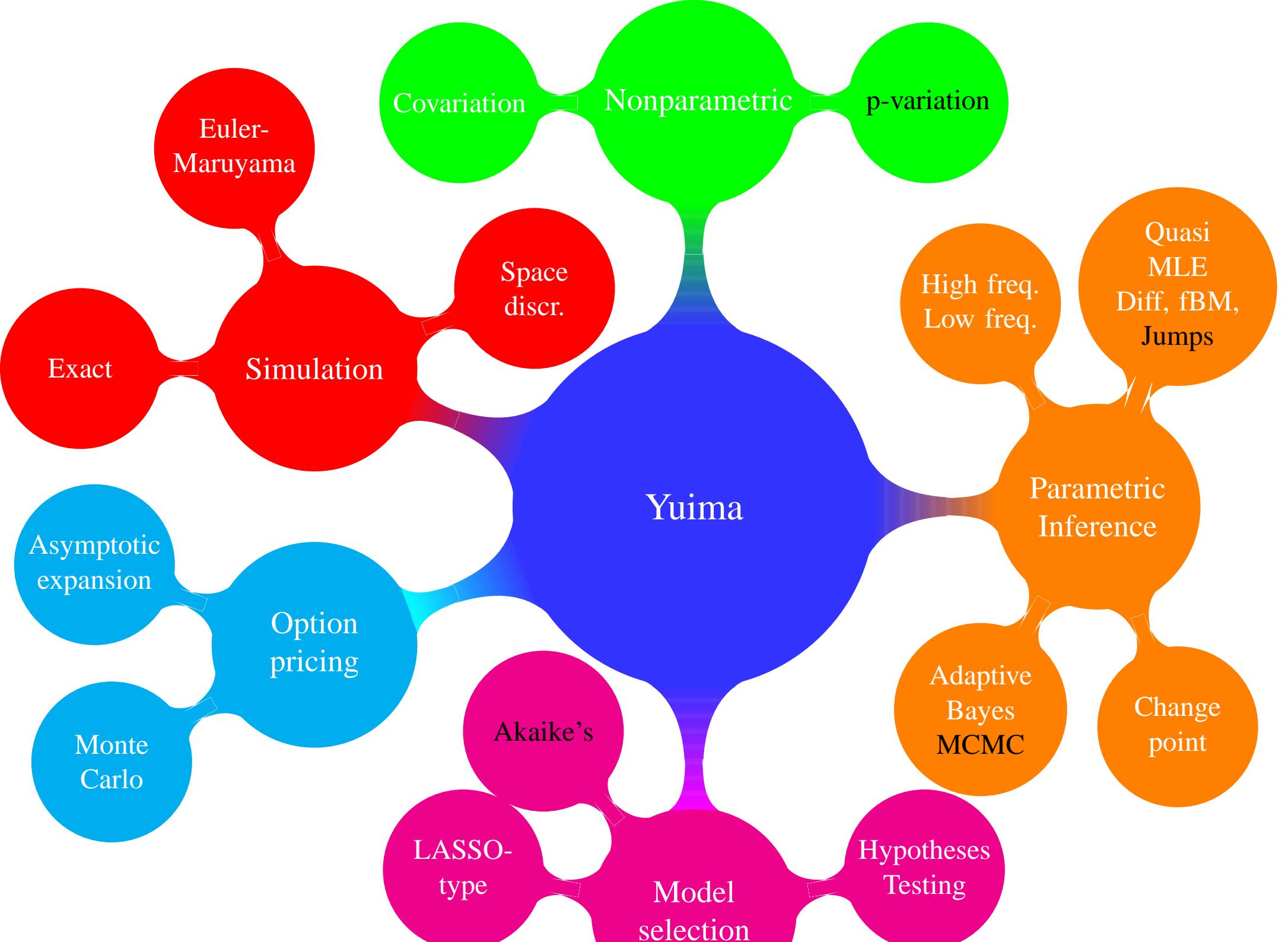
- Diffusions $dX_t = a(t, X_t)dt + b(t, X_t)dW_t$
- Fractional Gaussian Noise, with H the Hurst parameter

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t^H$$

- Diffusions with jumps, Lévy

$$\begin{aligned} dX_t = & \ a(X_t)dt + b(X_t)dW_t + \int_{|z|>1} c(X_{t-}, z)\mu(dt, dz) \\ & + \int_{0<|z|\leq 1} c(X_{t-}, z)\{\mu(dt, dz) - \nu(dz)dt\} \end{aligned}$$





The YUIMA Project

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For more informations and software see

<http://R-Forge.R-Project.org/projects/yuima>

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... more to come



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