

Eurandom – June 2012

Workshop on Parameter Estimation for Dynamical Systems

Martingale estimating functions for stochastic differential equations with jumps

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Diffusions with jumps

$$dX_t = b(X_t; \theta)dt + \sigma(X_t; \theta)dW_t + \gamma(X_{t-}; \theta)dZ_t, \quad \theta \in \Theta \subseteq \mathbb{R}^p,$$

Z is a Lévy process with Lévy measure ν_θ satisfying $\int_{-\infty}^{\infty} |x| \nu_\theta(dx) < \infty$.

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Z is a Lévy process with Lévy measure ν_θ satisfying $\int_{-\infty}^{\infty} |x| \nu_\theta(dx) < \infty$.

Data: X_{t_0}, \dots, X_{t_n} $\Delta_i = t_i - t_{i-1}$

Martingale estimating functions

$$G_n(\theta) = \sum_{i=1}^n g(\Delta_i, X_{t_i}, X_{t_{i-1}}; \theta),$$

$$\begin{aligned} g(\Delta_i, X_{t_i}, X_{t_{i-1}}; \theta) &= \sum_{j=1}^N a_j(X_{t_{i-1}}, \Delta_i; \theta) [f_j(X_{t_i}; \theta) - E_\theta(f_j(X_{t_i}; \theta) \mid X_{t_{i-1}})] \\ &\quad \uparrow \qquad \qquad \uparrow \\ &\text{p-dimensional} \quad \text{real valued} \end{aligned}$$

$G_n(\theta)$ is a P_θ -martingale:

$$E_\theta(a_j(X_{t_{i-1}}, \Delta_i; \theta) [f_j(X_{t_i}; \theta) - E_\theta(f_j(X_{t_i}; \theta) \mid X_{t_{i-1}})] \mid X_{t_1}, \dots, X_{t_{i-1}}) = 0$$

G_n -estimator(s): $G_n(\hat{\theta}_n) = 0$

Martingale estimating functions

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- Easy asymptotics by martingale limit theory

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- Simple expression for Godambe-Heyde optimal estimating function

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- Easy asymptotics by martingale limit theory
- Simple expression for Godambe-Heyde optimal estimating function
- Approximates the score function, which is a P_θ -martingale
- Particular and most efficient instance of GMM

A simple jump diffusion

$$dX_t = \alpha dt + \sigma dW_t + dZ_t$$

$$Z_t = \sum_{j=0}^{N_t} Y_j$$

N is a Poisson process with intensity λ

Y_j , $j = 1, 2, \dots$, are i.i.d. normal with mean μ and variance τ^2

W , N and the Y_j s are independent

A simple jump diffusion

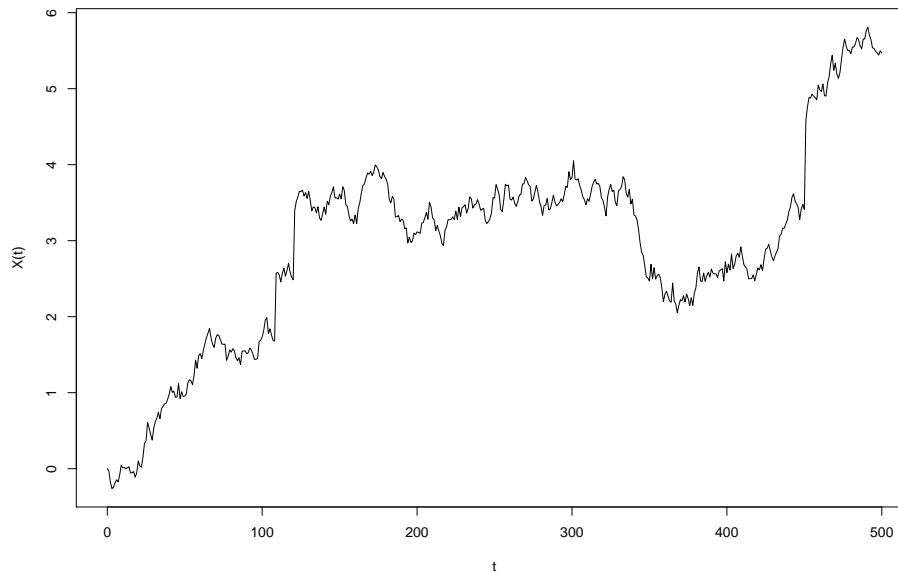
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$\alpha = 0.0001$, $\sigma = 0.1$, $\lambda = 0.01$, $\mu = 1$, and $\tau = 0.1$.

A simple jump diffusion

$$\Delta = 1$$

$$G_n(\theta) = \sum_{i=1}^n A(X_{i-1}, \theta) h(X_i, X_{i-1}; \theta)$$

$$h(x, y; \theta) = \begin{pmatrix} y - F(x; \theta) \\ (y - F(x; \theta))^2 - \Phi(x; \theta) \\ e^y - \kappa(x; \theta) \end{pmatrix}$$

$$F(x; \theta) = \mathsf{E}_\theta(X_i | X_{i-1} = x) = x + \alpha + \lambda\mu$$

$$\Phi(x; \theta) = \mathsf{Var}_\theta(X_i | X_{i-1} = x) = \sigma^2 + \lambda(\mu^2 + \tau^2)$$

$$\kappa(x; \theta) = \mathsf{E}_\theta(e^{X_i} | X_{i-1} = x) = \exp\left(x + \alpha + \frac{1}{2}\sigma^2 + \lambda(e^{\mu + \frac{1}{2}\tau^2} - 1)\right)$$

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Explicit expression for the optimal weight matrix $A^*(x, \theta)$

A simple jump diffusion

Parameter	True value	Mean	Standard error
α	0.0001	-0.0009	0.0070
σ	0.1	0.0945	0.0180
λ	0.01	0.0155	0.0209
μ	1	0.9604	0.5126
τ	0.1	0.2217	0.3156

500 observations (500 simulated estimates)

Bibby, Jacobsen and Sørensen (2010)

Explicit martingale estimating functions

Kessler and Sørensen (1999)

$$dX_t = b(X_t; \theta)dt + \sigma(X_t; \theta)dW_t$$

Generator:

$$L_\theta = \frac{1}{2}\sigma^2(x; \theta)\frac{d^2}{dx^2} + b(x; \theta)\frac{d}{dx},$$

φ eigenfunction for L_θ :

$$L_\theta\varphi = -\lambda_\theta\varphi$$

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Under weak regularity conditions $E_\theta(\varphi(X_{t_i})|X_{t_{i-1}}) = e^{-\lambda_\theta\Delta_i}\varphi(X_{t_{i-1}})$

$$G_n(\theta) = \sum_{i=1}^n g(\Delta_i, X_{t_i}, X_{t_{i-1}}; \theta),$$

$$g(\Delta_i, X_{t_i}, X_{t_{i-1}}; \theta) = \sum_{j=1}^N a_j(X_{t_{i-1}}, \Delta_i; \theta) \left[\varphi_j(X_{t_i}; \theta) - e^{-\lambda_j(\theta)\Delta_i} \varphi_j(X_{t_{i-1}}; \theta) \right]$$

Pearson diffusions

Wong (1964), Forman & Sørensen (2008)

$$dX_t = -\beta(X_t - \alpha)dt + \sqrt{2\beta(aX_t^2 + bX_t + c)}dW_t, \quad \beta > 0$$

$$L\varphi = \beta(ax^2 + bx + c)\varphi'' + \beta(x - \alpha)\varphi'$$

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Thus we can find eigenfunctions that are explicit polynomials

$$\varphi_n(x) = \sum_{j=0}^n p_{n,j}x^j, \quad p_{n,n} = 1$$

$$(a_j - a_n)p_{n,j} = b_{j+1}p_{n,j+1} + c_{j+2}p_{n,j+2}, \quad j = 0, \dots, n-1, \quad p_{n,n+1} = 0$$

$$a_j = j\{1 - (j-1)a\}\beta, \quad b_j = j\{\alpha + (j-1)b\}\beta, \quad c_j = j(j-1)c\beta$$

Example: Jacoby diffusions

$$dX_t = -\beta[X_t - \gamma]dt + \sigma\sqrt{1 - X_t^2}dW_t$$

State space: the interval $(-1, 1)$, $\beta, \sigma > 0$, $\gamma \in (-1, 1)$

Stationary distribution: Beta-distribution

Eigenfunctions: $P_n^{(\beta(1-\gamma)\sigma^{-2}-1, \beta(1+\gamma)\sigma^{-2}-1)}(x)$

$P_n^{(a,b)}(x)$ denotes the Jacobi polynomial of order n

Jump diffusions

$$dX_t = b(X_t; \theta)dt + \sigma(X_t; \theta)dW_t + \gamma(X_{t-})dZ_t, \quad \theta \in \Theta \subseteq \mathbb{R}^p,$$

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By Ito's formula

$$\begin{aligned} e^{\lambda t} \varphi(X_t) &= \varphi(X_0) + \int_0^t e^{\lambda s} [L_\theta \varphi(X_s) + \lambda \varphi(X_s)] ds + \int_0^t e^{\lambda s} \varphi'(X_s) \sigma(X_s) dW_s \\ &\quad + \sum_{s \leq t} e^{\lambda s} [\varphi(X_{s-} + \gamma(X_{s-}) \Delta Z_s) - \varphi(X_{s-})] \\ &\quad - \int_0^t \int_{-\infty}^{\infty} e^{\lambda s} [\varphi(X_{s-} + \gamma(X_{s-})y) - \varphi(X_{s-})] \nu_\theta(dy) ds \end{aligned}$$

Generator:

$$L_\theta \varphi = \frac{1}{2} \sigma^2(x) \varphi'' + b(x) \varphi' + \int_{-\infty}^{\infty} [\varphi(x + \gamma(x)y) - \varphi(x)] \nu_\theta(dy)$$

Jump diffusions

$$e^{\lambda t} \varphi(X_t) = \varphi(X_0) + \int_0^t e^{\lambda s} [L_\theta \varphi(X_s) + \lambda \varphi(X_s)] ds + \text{a local martingale}$$

Under regularity conditions, the local martingale is a martingale, and if moreover φ_θ is an eigenfunction of the generator, i.e.

$$L_\theta \varphi_\theta(x) = -\lambda_\theta \varphi_\theta(x)$$

then

$$E_\theta(\varphi_\theta(X_t) \mid X_0) = e^{-\lambda_\theta t} \varphi_\theta(X_0)$$

and

$$G_n(\theta) = \sum_{i=1}^n a(X_{t_{i-1}}; \theta) \left[\varphi_\theta(X_{t_i}) - e^{-\lambda_\theta(t_i - t_{i-1})} \varphi_\theta(X_{t_{i-1}}) \right]$$

is a martingale estimating function

Pearson diffusion with jump - example

Schmidt and Sørensen (2012)

An example:

$$dX_t = -\beta(X_t - \alpha)dt + \sqrt{2\beta(aX_t^2 + bX_t + c)}dW_t + \gamma(X_{t-})dZ_t$$

Lévy measure ν_ξ satisfies $\int_{-\infty}^{\infty} |y|^k \nu_\xi(dy) < \infty$, $k = 1, \dots, K$,

$$\nu_{\xi,k} = \int_{-\infty}^{\infty} y^k \nu_\xi(dy)$$

Polynomial eigenfunction? $\varphi_n(x) = \sum_{i=0}^n p_{n,i}x^i$, $p_{n,n} = 1$, $n \leq K$

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$$\begin{aligned} \varphi_n(x + \gamma(x)y) - \varphi_n(x) &= \sum_{i=0}^n p_{n,i}(x + \gamma(x)y)^i - \sum_{i=0}^n p_{n,i}x^i \\ &= \sum_{i=0}^n p_{n,i} \sum_{k=0}^i \binom{i}{k} x^{i-k} (\gamma(x)y)^k - \sum_{j=0}^n p_{n,j}x^j \\ &= \sum_{i=1}^n p_{n,i} \sum_{k=1}^i \binom{i}{k} x^{i-k} \gamma(x)^k y^k \end{aligned}$$

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$$dX_t = -\beta(X_t - \alpha)dt + \sqrt{2\beta(aX_t^2 + bX_t + c)}dW_t + \gamma(X_{t-})dZ_t$$

$$\int_{-\infty}^{\infty} [\varphi_n(x) + \gamma(x)y) - \varphi_n(x)]\nu_{\xi}(dy) = \sum_{i=1}^n p_{n,i} \sum_{k=1}^i \binom{i}{k} x^{i-k} \gamma(x)^k \nu_{\xi,k}$$

Polynomial for $\gamma(x) = 1$ and $\gamma(x) = x$

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Polynomial for $\gamma(x) = 1$ and $\gamma(x) = x$

For $\gamma(x) = x$:

$$\int_{-\infty}^{\infty} [\varphi_n(x) + \gamma(x)y) - \varphi_n(x)]\nu_{\xi}(dy) = \sum_{i=1}^n p_{n,i} \pi_i x^i$$

$$\pi_i = \sum_{k=1}^i \binom{i}{k} \nu_{\xi,k}$$

Pearson diffusion with jump - example

$$dX_t = -\beta(X_t - \alpha)dt + \sqrt{2\beta(aX_t^2 + bX_t + c)}dW_t + X_{t-} dZ_t$$

$$\varphi_n(x) = \sum_{i=0}^n p_{n,i}x^i, \quad p_{n,n} = 1$$

The equation

$$L\varphi_n = -\lambda_n\varphi_n$$

is satisfied if

$$\lambda_n = a_n - \pi_n \quad \text{and} \quad p_{n,i} = \frac{b_{i+1} p_{n,i+1} + c_{i+2} p_{n,i+2}}{a_i - \pi_i - \lambda_n}$$

where

$$\pi_0 = 0, \quad \pi_i = \sum_{k=1}^i \binom{i}{k} \nu_{\xi,k}, \quad i = 1, \dots, n$$

$$a_i = \beta i(1 - a(i-1)), \quad b_i = \beta i(b(i-1) + \alpha) \quad \text{and} \quad c_i = \beta ci(i-1)$$

A diffusion with jumps and its generator

$$dX_t = -\beta(X_t - \alpha)dt + \sqrt{2\beta(aX_t^2 + bX_t + c)}dW_t + \int_{\mathbb{R}^d} g_\gamma(y, X_{t-})\mu_\xi(dt, dy)$$

W is a Wiener process, and μ_ξ is a Poisson random measure on $(0, \infty) \times \mathbb{R}^d$ with intensity measure

$$\nu_\xi(dt, dy) = F_\xi(dy)dt.$$

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Generator:

$$L\varphi = \beta(ax^2 + bx + c)\varphi'' - \beta(x - \alpha)\varphi' + \int_{\mathbb{R}^d} [\varphi(x + g_\gamma(y, x)) - \varphi(x)]F_\xi(dy)$$

Polynomial eigenfunctions

$$\varphi_n(x) = \sum_{i=0}^n p_{n,i} x^i, \quad p_{n,n} = 1$$

$$\begin{aligned}\varphi_n(x + g_\gamma(y, x)) - \varphi_n(x) &= \sum_{i=0}^n p_{n,i} (x + g_\gamma(y, x))^i - \sum_{i=0}^n p_{n,i} x^i \\ &= \sum_{i=0}^n p_{n,i} \sum_{k=0}^i \binom{i}{k} x^{i-k} g_\gamma(y, x)^k - \sum_{j=0}^n p_{n,j} x^j \\ &= \sum_{i=1}^n p_{n,i} \sum_{k=1}^i \binom{i}{k} x^{i-k} g_\gamma(y, x)^k\end{aligned}$$

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Assume that (Zhou (2003))

$$\int_{\mathbb{R}^d} g_\gamma(y, x)^k F_\xi(dy) = \sum_{j=0}^k \kappa_{j,k} x^j, \quad k = 1, \dots, K$$

Polynomial eigenfunctions

$$\int_{\mathbb{R}^d} g_\gamma(y, x)^k F_\xi(dy) = \sum_{j=0}^k \kappa_{j,k} x^j, \quad k = 1, \dots, K$$

Then for $n \leq K$

$$\begin{aligned} \int_{\mathbb{R}^d} [\varphi(x + g_\gamma(y, x)) - \varphi(x)] F_\xi(dy) &= \sum_{i=1}^n p_{n,i} \sum_{k=1}^i \binom{i}{k} x^{i-k} \int_{\mathbb{R}^d} g_\gamma(y, x)^k F_\xi(dy) \\ &= \sum_{i=1}^n \sum_{k=1}^i \sum_{j=0}^k p_{n,i} \binom{i}{k} \kappa_{j,k} x^{i-k+j} \\ &= \sum_{i=0}^n x^i \sum_{r=i \vee 1}^n p_{n,r} \pi_{i,r}, \end{aligned}$$

where

$$\pi_{i,r} = \sum_{k=(i-r) \vee 1}^i \binom{i}{k} \kappa_{k-i+r,k}$$

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The equation

$$L\varphi_n = -\lambda_n \varphi_n$$

is satisfied if

$$\lambda_n = a_n - \pi_{n,n} = a_n - \sum_{k=1}^n \binom{n}{k} \kappa_{k,k},$$

where $a_i := \beta i(1 - a(i-1))$, and

$$p_{n,i} = \frac{b_{i+1} p_{n,i+1} + c_{i+2} p_{n,i+2} + \sum_{k=i+1}^n p_{n,k} \pi_{i,k}}{a_i - \pi_{i,i} - \lambda_n},$$

where $b_i := \beta i(b(i-1) + \alpha)$ and $c_i := \beta ci(i-1)$.

Example 1

$$dX_t = -\beta(X_t - \alpha)dt + \sigma dW_t + dZ_t.$$

Z is a Lévy process with Lévy measure F_ξ satisfying that

$$\int_{-\infty}^{\infty} |y|^k F_\xi(dy) < \infty, \quad k = 1, \dots, K,$$

$$g(y, x) = y$$

$$\int_{-\infty}^{\infty} g(y, x)^k F_\xi(dy) = \int_{-\infty}^{\infty} y^k F_\xi(dy)$$

Example 2

$$dX_t = -\beta(X_t - \alpha)dt + \sigma\sqrt{X_t}dW_t + X_{t-}dZ_t.$$

Here

$$Z_t = \sum_{i=1}^{N_t} J_i$$

where N is a Poisson process with intensity λ , and $J_i \sim \text{Exp}(\xi)$,
 $i = 1, 2, \dots$ are i.i.d. (and independent of N and W)

$$g(y, x) = yx \quad F_{\lambda, \xi}(dy) = \lambda \xi e^{-\xi y} dy$$

$$\int_0^\infty g(y, x)^k F_{\lambda, \xi}(dy) = x^k \lambda \int_0^\infty y^k e^{-\xi y} dy = x^k \frac{\lambda k!}{\xi^k}$$

Example 3

$$g_\gamma(y, x) = y + \gamma x \quad F_{\lambda, \xi}(dy) = \lambda \frac{e^{-y^2/(2\xi^2)}}{\sqrt{2\pi\xi^2}} dy \quad \text{state-space: } \mathbb{R}$$

When k is even

$$\begin{aligned} \int_{-\infty}^{\infty} (y + \gamma x)^k F_{\lambda, \xi}(dy) &= \sum_{i=0}^k \binom{k}{i} \gamma^i x^i \lambda \int_{-\infty}^{\infty} y^{k-i} \frac{e^{-y^2/(2\xi^2)}}{\sqrt{2\pi\xi^2}} dy \\ &= \sum_{i=0}^{k/2} \kappa_{k,i}(\gamma, \xi) x^{2i} \end{aligned}$$

Example 4

$$dX_t = -\beta(X_t - \alpha)dt + \sigma\sqrt{X_t(1 - X_t)} dW_t + \int_{\mathbb{R}} g(y, X_{t-})\mu_\xi(dt, dy)$$

1) $X_{t-} dZ_t : \quad g(y, x) = yx, \quad F_{\lambda, \xi}(y) = \lambda f_\xi(y)$

f_ξ concentrated on $(-1, 0)$.

2) $g_\gamma(y, x) = \gamma(y - x), \quad F_{\lambda, \xi} = \lambda f_\xi(y)$

f_ξ concentrated on $(0, 1)$, $\gamma \in (0, 1]$

$$\int_0^1 g_\gamma(y, x)^k f_\xi(y) dy = \gamma^k \sum_{i=0}^k \binom{k}{i} (-x)^i \int_0^1 y^{k-i} f_\xi(y) dy$$

Example 5

$$dX_t = -\beta(X_t - \alpha)dt + \sigma\sqrt{X_t(X_t + 1)} dW_t + (X_{t-} + 1)dZ_t.$$

Here

$$Z_t = \sum_{i=1}^{N_t} J_i$$

where N is a counting process with intensity $\lambda X_{t-}/(1 + X_{t-})$ and $J_i \sim f_\xi$, $i = 1, 2, \dots$ are i.i.d. (and independent of N and W) with f_ξ concentrated on $(0, \infty)$.

$$g(y_1, y_2, x) = (x+1)y_1 1_{(0, x/(x+1))}(y_2), \quad F_{\lambda, \xi}(dy_1, dy_2) = \lambda f_\xi(y_1) 1_{(0, 1)}(y_2) dy_1 dy_2$$

$$\begin{aligned} & \int_0^\infty \int_0^1 g(y_1, y_2, x)^k F_{\lambda, \xi}(dy_1, dy_2) \\ &= (x+1)^k \lambda \int_0^\infty y_1^k f_\xi(y_1) dy_1 \int_0^1 1_{(0, x/(x+1))}(y_2) dy_2 \\ &= \lambda \kappa_k(\xi) (x+1)^k x / (1+x) = \lambda \kappa_k(\xi) x (x+1)^{k-1} \end{aligned}$$

Transformations of Pearson diffusions

X_t : $\varphi(x)$ eigenfunction with eigenvalue λ

T : twice continuously differentiable, and 1-1

$T(X_t)$: $\varphi(T^{-1}(x))$ eigenfunction with eigenvalue λ

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Jacobi diffusion: state space $(-1, 1)$, $\beta, \sigma > 0$, $\gamma \in (-1, 1)$

$$dX_t = -\beta[X_t - \gamma]dt + \sigma\sqrt{1 - X_t^2}dW_t$$

Eigenfunctions: $P_n^{(\beta(1-\gamma)\sigma^{-2}-1, \beta(1+\gamma)\sigma^{-2}-1)}(x)$ ($P_n^{(a,b)}$ Jacobi polynomial)

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$Y_t = \sin^{-1}(X_t)$ state space $(-\frac{\pi}{2}, \frac{\pi}{2})$, $\rho = \beta - \frac{1}{2}\sigma^2$, $\varphi = \beta\gamma/(\beta - \frac{1}{2}\sigma^2)$

$$dY_t = -\rho \frac{\sin(Y_t) - \varphi}{\cos(Y_t)}dt + \sigma d\tilde{W}_t$$

Eigenfunctions: $P_n^{(\rho(1-\varphi)\sigma^{-2}-\frac{1}{2}, \rho(1+\varphi)\sigma^{-2}-\frac{1}{2})}(\sin(x))$

Optimal martingale estimating functions

$$G_n(\theta) = \sum_{i=1}^n g(\Delta_i, X_{t_i}, X_{t_{i-1}}; \theta),$$

$$g(\Delta, y, x; \theta) = \sum_{j=1}^N a_j(x, \Delta; \theta) \left[\varphi_j(y; \theta) - e^{-\lambda_j(\theta)\Delta} \varphi_j(x; \theta) \right]$$

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Suppose

$$\varphi_j(x; \theta) = \psi_j(\kappa(x); \theta),$$

where κ is a real function independent of θ , and ψ_j is a polynomial of degree j :

$$\psi_j(y; \theta) = \sum_{k=0}^j p_{j,k}(\theta) y^k$$

Then the optimal weights $a_j^*(x, \Delta; \theta)$ can be found explicitly

Explicit optimal estimating functions

Optimal weight matrix ($A^* = \{a_1^*, \dots, a_N^*\}$):

$$A^*(x, \Delta; \theta) = B_h(x, \Delta; \theta)^T V_h(x, \Delta; \theta)^{-1}$$

$$B_h(x, \Delta; \theta)_{ij} = \sum_{k=0}^j \partial_{\theta_i} p_{j,k}(\theta) \int_{\ell}^r \kappa(y)^k p(\Delta, x, y; \theta) dy - \partial_{\theta_i} [e^{-\lambda_j(\theta)\Delta} \varphi_j(x; \theta)]$$

$$i = 1, \dots, p, \quad j = 1, \dots, N$$

$$\begin{aligned} V_h(\Delta, x; \theta)_{ij} &= \sum_{r=0}^i \sum_{s=0}^j p_{i,r}(\theta) p_{j,s}(\theta) \int_{\ell}^r \kappa(y)^{r+s} p(\Delta, x, y; \theta) dy \\ &\quad - e^{-[\lambda_i(\theta) + \lambda_j(\theta)]\Delta} \varphi_i(x; \theta) \varphi_j(x; \theta) \end{aligned}$$

$$i, j = 1, \dots, N$$

Explicit optimal estimating functions

Thus to find the optimal estimating function based on the first N eigenfunctions, we need to find the moments

$$\int_{\ell}^r \kappa(y)^i p(\Delta, x, y; \theta) dy \quad \text{for } 1 \leq i \leq 2N$$

If we integrate both sides of

$$\varphi_i(y; \theta) = \sum_{j=0}^i p_{i,j}(\theta) \kappa(y)^j$$

with respect to $p(\Delta, x, y; \theta)$ for $i = 1, \dots, 2N$, we obtain a system of linear equations

$$e^{-\lambda_i(\theta)\Delta} \varphi_i(x; \theta) = \sum_{j=0}^i p_{i,j}(\theta) \int_{\ell}^r \kappa(y)^j p(\Delta, x, y; \theta) dy, \quad i = 1, \dots, 2N$$

Asymptotics - low frequency

$$G_n(\theta) = \sum_{i=1}^n g(\Delta_i, X_{t_i}, X_{t_{i-1}}; \theta),$$

Assume that X is ergodic with invariant measure $\mu_\theta(x)$, that $t_i = \Delta i$, and weak regularity conditions.

Then a consistent estimator $\hat{\theta}_n$ that solves the estimating equation $G_n(\theta) = 0$ exists and is unique in any compact subset of Θ containing θ_0 with a probability that goes to one as $n \rightarrow \infty$. Moreover,

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{\mathcal{D}} N(0, S_{\theta_0}^{-1} V_{\theta_0} (S_{\theta_0}^T)^{-1})$$

under P_{θ_0} . Here

$$V_\theta = Q_{\theta_0}^\Delta (g(\Delta, \theta)g(\Delta, \theta)^T) \quad \text{and} \quad S_\theta = \left\{ Q_{\theta_0}^\Delta (\partial_{\theta_j} g_i(\Delta; \theta)) \right\},$$

where $Q_\theta^\Delta(x, y) = \mu_\theta(x)p(\Delta, x, y; \theta)$

More general jumps

$$\begin{aligned} dX_t &= -\beta(X_t - \alpha)dt + \sqrt{2\beta(aX_t^2 + bX_t + c)}dW_t \\ &\quad + \int_{\|y\| < c} g_\gamma(y, X_{t-})(\mu_\xi - \nu_\xi)(dt, dy) + \int_{\|y\| \geq c} g_\gamma(y, X_{t-})\mu_\xi(dt, dy) \end{aligned}$$

μ_γ is a Poisson random measure on $(0, \infty) \times \mathbb{R}^d$ with intensity measure

$$\nu_\xi(dt, dy) = F_\xi(dy)dt.$$

Generator:

$$\begin{aligned} L\varphi &= \beta(ax^2 + bx + c)\varphi'' - \beta(x - \alpha)\varphi' \\ &\quad + \int_{\|y\| < c} [\varphi(x + g_\gamma(y, x)) - \varphi(x) - g_\gamma(y, x)\varphi'(x)]F_\xi(dy) \\ &\quad + \int_{\|y\| \geq c} [\varphi(x + g_\gamma(y, x)) - \varphi(x)]F_\xi(dy) \end{aligned}$$

Polynomial estimating functions

$$\varphi_n(x) = \sum_{i=0}^n p_{n,i} x^i, \quad p_{n,n} = 1$$

$$\begin{aligned} & \varphi_n(x + g_\gamma(y, x)) - \varphi_n(x) - g_\gamma(y, x)\varphi'_n(x) \\ &= \sum_{i=0}^n p_{n,i} (x + g_\gamma(y, x))^i - \sum_{i=0}^n p_{n,i} x^i - \sum_{i=1}^n p_{n,i} i x^{i-1} g_\gamma(y, x) \\ &= \sum_{i=0}^n p_{n,i} \sum_{k=0}^i \binom{i}{k} x^{i-k} g_\gamma(y, x)^k - \sum_{i=0}^n p_{n,i} x^i - \sum_{i=1}^n p_{n,i} i x^{i-1} g_\gamma(y, x) \\ &= \sum_{i=2}^n p_{n,i} \sum_{k=2}^i \binom{i}{k} x^{i-k} g_\gamma(y, x)^k \end{aligned}$$

Polynomial estimating functions

Assume that

$$\int_{\mathbb{R}^d} g_\gamma(y, x)^k F_\xi(dy) = \sum_{j=0}^k \kappa_{j,k} x^j, \quad k = 2, \dots, K$$

and

$$\int_{\|y\| \geq c} g_\gamma(y, x) F_\xi(dy) = \kappa_{0,1} + \kappa_{1,1} x.$$

Then $L\varphi_n$ is a polynomial of the same order as φ_n , and we find the same formulae for the eigenfunctions as previously.