Maker-Breaker Games on Random Geometric Graphs

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joint work with
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2. Structure of the RGG
   - Dissection of $[0,1]^2$ into Tiny Cells
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3. Maker-Breaker Games
   - Connectivity Game
   - Hamilton Game
   - Perfect Matching Game
4. Conclusion
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Maker-Breaker Game on a graph $G = (V, E)$

- Two player, complete information game
- Collection of winning subsets $\mathcal{F} \subset 2^{E(G)}$
- Breaker and Maker alternately claim edges of G
- Maker wins if he claims some subset in $\mathcal{F}$.
  Otherwise Breaker wins.
- Typically,

$$\mathcal{F} = \{F \subset E \mid G[F] \text{ has property } \mathcal{P}\}$$

where $\mathcal{P}$ is an **increasing graph property** (e.g. has spanning tree, Hamilton cycle, or perfect matching)

Maker-Breaker games on graphs and hypergraphs

Two Classic Results

- P. Erdős and J. Selfridge, *On a Combinatorial Game*, 1973.
- V. Chvátal and P. Erdős, *Biased Positional Games*, 1978.

The Book on Combinatorial Games

- J. Beck, *Combinatorial Games: Tic-Tac-Toe Theory*, 2008.

A Recent Break-Through

- M. Krivelevich, *The Critical Bias for the Hamiltonicity Game is* $(1 + o(1))n/\ln n$, 2011.

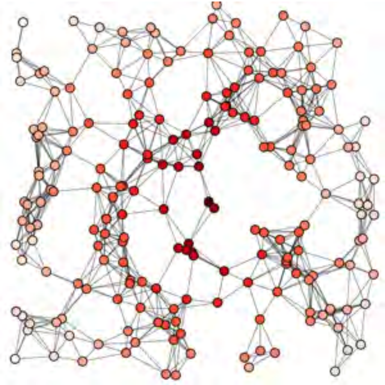
Maker-Breaker games on Random Graphs

Some additional recent results

- M. Stojaković and T. Szabó, *Positional Games on Random Graphs*, 2005
- D. Hefetz, M. Krivelevich, M. Stojaković, and T. Szabó, *A Sharp Threshold for the Hamilton Cycle Maker-Breaker game*, 2009.
- S. Ben-Shimon, M. Krivelevich, and B. Sudakov, *Local Resilience and Hamiltonicity Maker-Breaker Games in Random Regular Graphs*, 2011.
- S. Ben-Shimon, A. Ferber, D. Hefetz, and M. Krivelevich, *Hitting Time Results for Maker-Breaker Games*, 2011.

Random Geometric Graph

Random Geometric Graph $G(n, r_n)$



- Pick random points $x_1, \ldots, x_n \in [0, 1]^2$
- Connectivity radius r_n
- $x_i x_j \in E \iff \|x_i - x_j\| \leq r_n$
- Study expected behavior as $n \to \infty$
- $\mathcal{A}_n$ holds *whp* means $\Pr(\mathcal{A}_n) = 1 - o(1)$

Connectivity of RGG

Theorem (cf. Penrose, *Random Geometric Graphs*, 2003)

Let $x \in \mathbb{R}$ be a constant. If

$$r_n^2 = \frac{\ln n + \omega(1)}{\pi n}$$

then

$$\lim_{n\to\infty} \mathbb{P}[G(n, r_n) \text{ connected }] = 1.$$

Key idea:

$$E(\deg(v)) = n \cdot \text{area}(B(v, r_n)) \approx \ln n$$

and this is enough to guarantee connectivity.

Hitting Radius of an Increasing Property

The *hitting radius* of increasing graph property $\mathcal{P}$ is

$$\rho_n(\mathcal{P}) = \inf\{r \geq 0 : G(n,r) \text{ satisfies } \mathcal{P}\}$$

Example:

The hitting radius for connectivity is

$$\rho_n(G \text{ is connected}) = \sqrt{\frac{\ln n}{\pi n}}.$$

Hitting Radii for RGG Minimum Degree

Theorem (cf. Penrose, *Random Geometric Graphs*, 2003)

Let $x \in \mathbb{R}$ be a constant.

- Hitting radius for minimum degree 2 is

$$\rho_n(\delta(G) \geq 2) = \sqrt{\frac{\ln n + \ln \ln n}{\pi n}}$$

- Hitting radius for minimum degree 4 is

$$\rho_n(\delta(G) \geq 4) = \sqrt{\frac{\ln n + 5 \ln \ln n}{\pi n}}$$

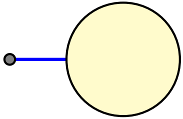
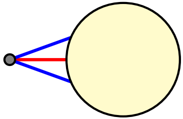
Maker-Breaker Hitting Radii for RGG

Theorem (BDFMS 13+)

The hitting radius for the random geometric graph $G(n, r)$ to be Maker's win corresponds to a simple minimum degree condition as follows:

- *Connectivity game* $\iff \delta(G(n, r)) \geq 2$
- *Hamilton Cycle game* $\iff \delta(G(n, r)) \geq 4$
- *Perfect Matching game* $\iff \delta(G(n, r)) = 2$ *and minimum edge degree* ≥ 3.

Why are these minimum degree conditions necessary?

when $\delta(G)$ is...	then Breaker wins...	
1	Connectivity game	
3	Hamilton Cycle game	

because Breaker goes first!

Maker-Breaker Hitting Radii for RGG

Game	Minimum Degree Condition	Hitting Radius (essentially)
Connectivity Game	$\delta(G) \geq 2$	$r = \sqrt{\frac{\ln n + \ln \ln n}{\pi n}}$
Perfect Matching Game	$\delta(G) \geq 2$, and if $x_i x_j \in E(G)$ then $\lvert N(\{x_i, x_j\})\rvert \geq 3$	$r = \sqrt{\frac{\ln n + \ln \ln n}{\pi n}}$
Hamilton Cycle Game	$\delta(G) \geq 4$	$r = \sqrt{\frac{\ln n + 5 \ln \ln n}{\pi n}}$

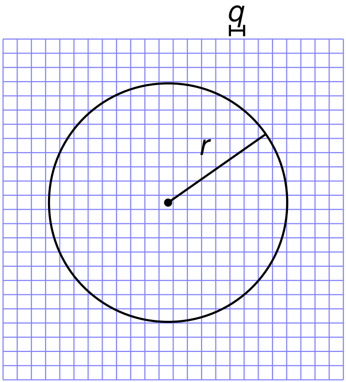
The Problem: vertices with very low degree

If we had $\delta(G) = \omega(1)$, then our games would be easy Maker-win. Must deal with vertices of constant degree.

- We **dissect** the square $[0, 1]$ into **very small cells** (squares).
- **The good news:** most points have lots of neighbors in nearby dense cells.
- **The not-so-bad news:** the rest are in clusters of well-separated sparse cells.

The dense cells provide the backbone of our strategy. We use them to handle the sprinkling of sparse cells.

Dissection $\mathcal{D}$ of unit square $[0,1]^2$ into cells



Given

$$r^2 = \frac{\ln n + \Theta(\ln \ln n)}{\pi n}.$$

Let $\eta > 0$ be a small constant. Choose $q = q(n)$ such that

$$q = \eta r$$

This ensures that you need

$$\approx \frac{1}{\eta^2} < \infty$$

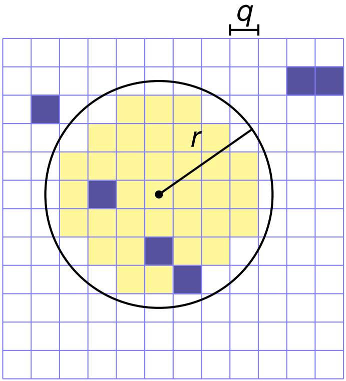
$q \times q$ squares to cover $B(v, r)$.

The Structure of Γ

- Fix a large constant $T > 0$.
- A cell c is **good** if $|V \cap c| \geq T$. Otherwise, c is **bad**.

Define graph Γ using good cells of dissection $\mathcal{D}$.

- $V(\Gamma) =$ all good cells
- $E(\Gamma) = \{cc' : \text{dist}(c, c') \leq r\}$



Gives rise to connected components $\Gamma_{\max}$ and other smaller components $\Gamma_2, \Gamma_3, \ldots$

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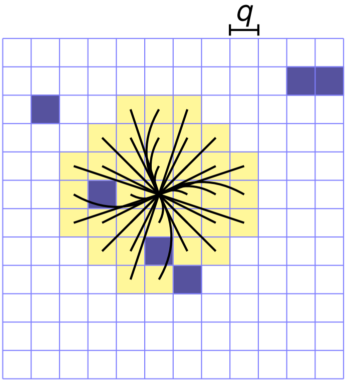
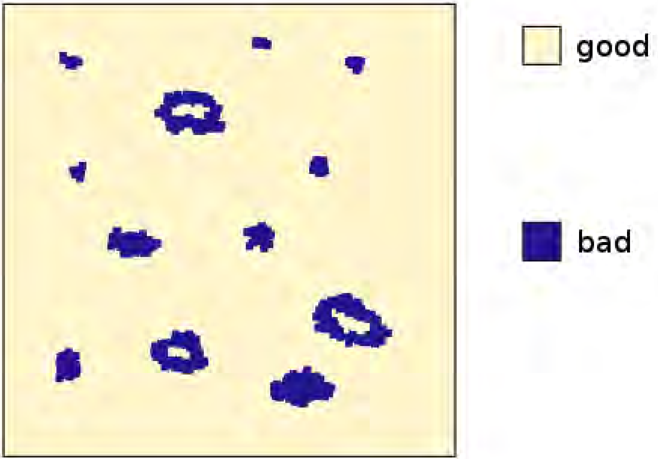


Gives rise to connected components $\Gamma_{\max}$ and other smaller components $\Gamma_2, \Gamma_3, \ldots$

Cells of Γ are Good or Bad



Vertices of G are Safe, Risky or Dangerous

Components are $\Gamma_{\max}$ and the smaller $\Gamma_2, \Gamma_3, \ldots$

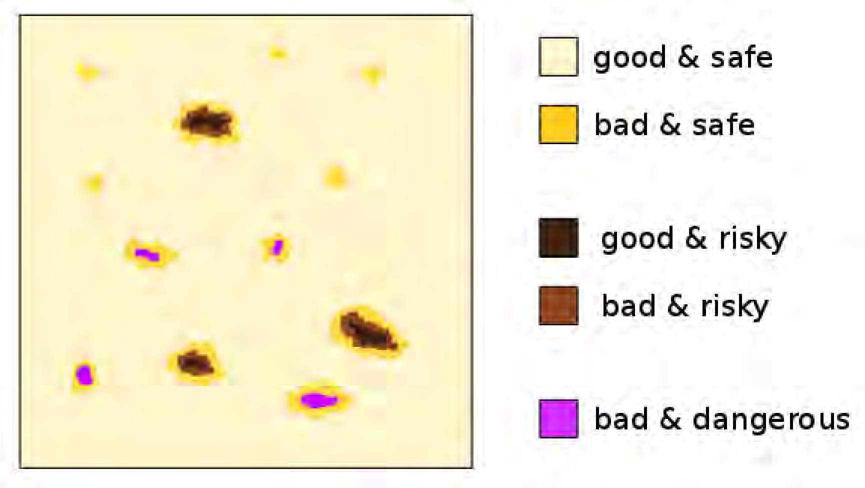
Categorize each $v \in V$ as follows:

- v is **safe**: has $\geq T$ neighbors in a good cell c of $\Gamma_{\max}$
- v is **risky**: has $\geq T$ neighbors in a good cell c of Γ_i, for $i \geq 2$
- v is **dangerous**: otherwise

Vertices in good cells are safe or risky.

Vertices in bad cells can be safe, risky or dangerous.

The Giant and the Obstructions



The Giant and the Obstructions

Partition G into the unique **Giant** and a collection of two types of **Obstructions.**

The Giant

- $\Gamma^+_{\mathrm{max}} = \Gamma_{\mathrm{max}}$ and its nearby safe points

The Obstructions

- $\Gamma^+_i = \Gamma_i$ and its nearby risky points
- **Dangerous Cluster**: a maximal clique of dangerous points

Global Structure of RGG $G(n, r)$

The Dissection Lemma

The largest component $\Gamma^+_{\max}$ is giant.

$$\Gamma_{\max} \text{ contains } \geq 0.99 \cdot |\mathcal{D}| \text{ cells } whp.$$

The Obstructions are small and very far apart.

Whp, for obstructions $\mathcal{O}_i \neq \mathcal{O}_j$

- $\mathrm{diam}(\mathcal{O}_i) < r/100$
- $\mathrm{dist}(\mathcal{O}_i, \mathcal{O}_j) > r \cdot 10^{10}$

Obstructions = small components and dangerous clusters

$\Gamma_{\max}$ contains $\geq 0.99 \cdot |\mathcal{D}|$ cells

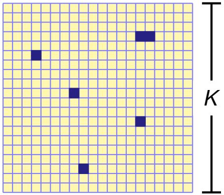
Lemma (The Giant)

$\Gamma_{\max}$ contains $\geq 0.99 \cdot |\mathcal{D}|$ cells *whp*.

Recall: cell c has side length $q = \eta r$

Set $K > \frac{1}{\eta^2} > 0$.

Pick any $\mathcal{B} = K \times K$ block of cells.



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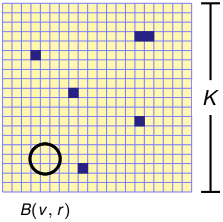
$$\text{Area}(\mathcal{B}) = \frac{K^2}{\eta^2} B(v, r)$$



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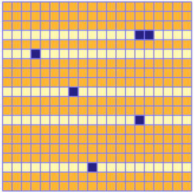
Recall: cell c has side length $q = \eta r$

Set $K > \dfrac{1}{\eta^2} > 0$.

Pick any $\mathcal{B} = K \times K$ block of cells.

- 0.99% rows/columns have no bad cells, because $E(|V \cap c|) = \Theta(\log n) \gg T$.



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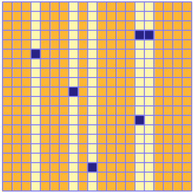

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Recall: cell c has side length $q = \eta r$

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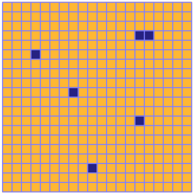
Pick any $\mathcal{B} = K \times K$ block of cells.

- 0.99% rows/columns have no bad cells, because $E(|V \cap c|) = \Theta(\log n) \gg T$.
- Creates largest component in $\mathcal{B}$



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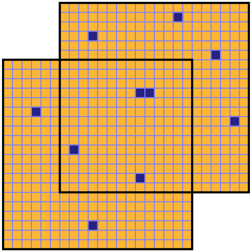
Recall: cell c has side length $q = \eta r$

Set $K > \dfrac{1}{\eta^2} > 0$.

Pick any $\mathcal{B} = K \times K$ block of cells.

- 0.99% rows/columns have no bad cells, because $E(|V \cap c|) = \Theta(\log n) \gg T$.
- Creates largest component in $\mathcal{B}$
- Take overlapping blocks to get $\Gamma_{\max}$ $\square$

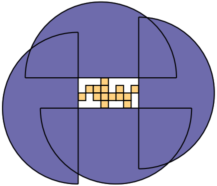


$\mathrm{diam}(\Gamma_i^+) < r/100$ when $i \geq 2$

Lemma

Whp, $\mathrm{diam}(\Gamma_i^+) < r/100$ for $i \geq 2$.

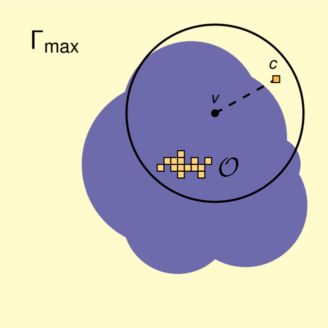
- No good cells in surrounding half-disks of radius r
- If $\mathrm{diam}(\Gamma_i^+) \geq r/100$ then there are too many bad cells in a small area □



Similar proofs that other obstructions are small & that pairs of obstructions are well-separated

Crucial & Important Vertices for an Obstruction

Assign vertices to help with obstruction $\mathcal{O}$



Point $v \in V$ is **crucial** for $\mathcal{O}$ if

- v is **safe**, and
- $\mathcal{O} \subset B(v; r)$, and

Recall: v safe $\Rightarrow \exists c \in \Gamma_{max}$ with $|B(v; r) \cap c \cap V| \geq T$

The T vertices in c are **important** for v and for $\mathcal{O}$.

Obstructions have Crucial Vertices

The Obstruction Lemma

Consider $G(n, r)$ where

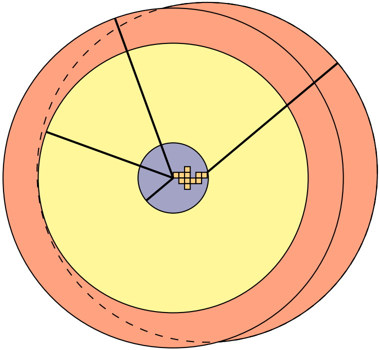
$$\pi r^2 = \ln n + (2k-3)\ln\ln n + O(1),$$

with $k \geq 2$ fixed. *Whp* the following holds for all obstructions $\mathcal{O}$. Let $|\mathcal{O}| = s$

- If $2 \leq s \leq T$ then $\mathcal{O}$ has $\geq k + s - 2$ crucial vertices;
- If $s \geq T$, then $\mathcal{O}$ has $\geq k$ crucial vertices.

Note: Obstructions far apart $\Rightarrow$ crucial vertices for $\mathcal{O}_i \neq \mathcal{O}_j$ are distinct.

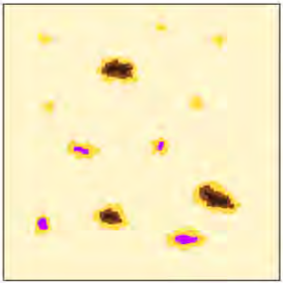
Obstructions have Crucial Vertices



- $|\mathcal{O}| \leq T$
- Must be a finite number of vertices in outer ring
- Forces existence of vertices in middle ring
  - These vertices adjacent to $\mathcal{O}$
  - Not part of $\mathcal{O} \Rightarrow$ safe or risky
  - Must be adjacent to good cells in Γ_{max}

Summary: Structure of RGGs

- There is a giant component Γ_{max} of dense cells
- Obstructions are small and far from one another
- Obstructions have enough crucial vertices to help connect them to Γ_{max}



Terminology Reminders

Minimum Degree $\delta(G) = \min_{v \in V} \deg(v)$

With High Probability (*whp*)
Event $A = A_n$ holds *whp* if $\lim_{n \to \infty} \Pr(A_n) = 1$.

Hitting Radius
The *hitting radius* of increasing graph property $\mathcal{P}$ is

$$\rho_n(\mathcal{P}) = \inf\{r \geq 0 : G(n, r) \text{ satisfies } \mathcal{P}\}$$

- If $r < \rho_n$ then $G(n, r)$ DOES NOT have property $\mathcal{P}$ *whp*.
- If $r \geq \rho_n$ then $G(n, r)$ DOES have property $\mathcal{P}$ *whp*.

Hitting Radius for the Connectivity Game

Theorem (BDFMS 2013+)

Whp, the RGG process $G(n,r)$ satisfies

$$\rho_n(\text{Maker wins connectivity game}) = \rho_n(\delta(G(n,r)) \geq 2).$$

In particular, if

$$\pi n r^2 = \ln n + \ln\ln n + x_n$$

then

$$\lim_{n\to\infty} \mathbb{P}(\text{Maker wins}) = \begin{cases} 1 & \text{if } x_n \to +\infty, \\ e^{-(e^{-x}+\sqrt{\pi e^{-x}})} & \text{if } x_n \to x \in \mathbb{R}, \\ 0 & \text{if } x_n \to -\infty. \end{cases}$$

Hitting Radius for the Connectivity Game

Breaker wins when $\delta(G) \leq 1$

- Breaker makes an isolated vertex on the very first move

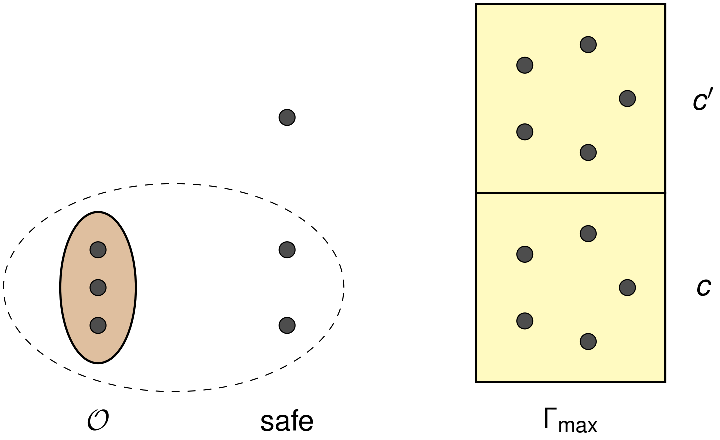
When $\delta(G) \geq 2$

- We use the Shannon Switching Game result

Theorem (A. Lehman, 1964)

Tthe connectivity game is Maker-win if and only if G admits two disjoint spanning trees.

Two Disjoint Spanning Trees in $G(n, r)$



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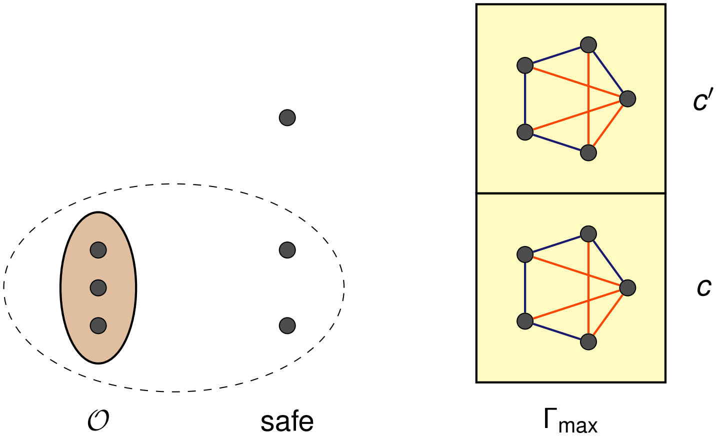

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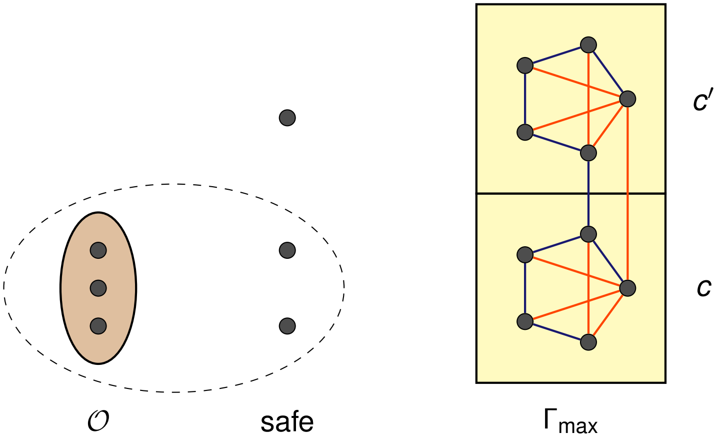

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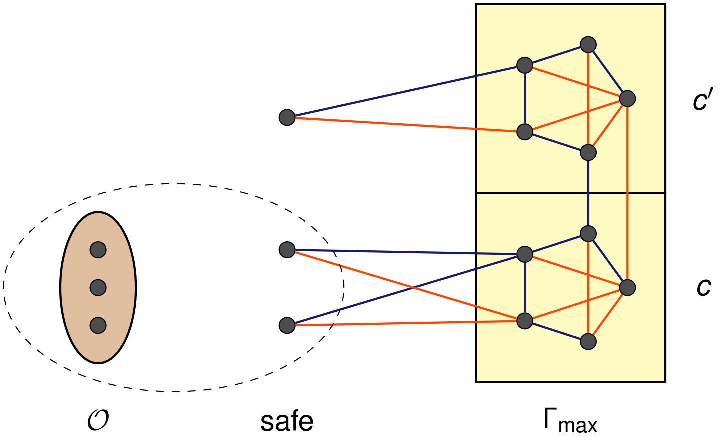

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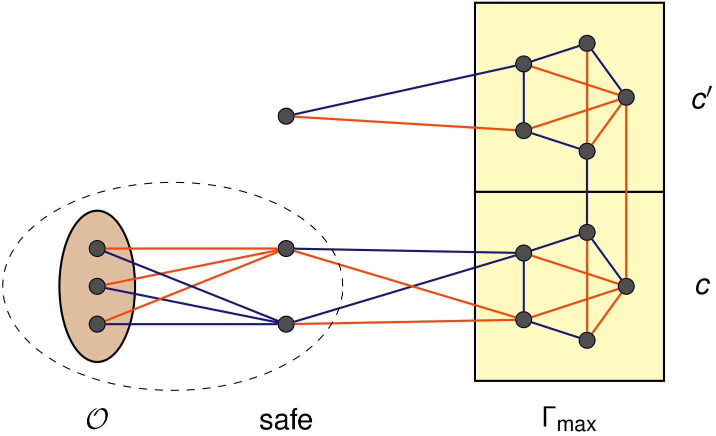


Hitting Radius for the Hamilton Game

Theorem (BDFMS 2013+)

Whp, the RGG process $G(n, r)$ satisfies

$$\rho_n(\text{Maker wins Hamilton game}) = \rho_n(\delta(G(n, r)) \geq 4).$$

In particular, if

$$\pi n r^2 = \ln n + 5 \ln \ln n - 2 \ln 6 + x_n$$

then

$$\lim_{n \to \infty} \mathbb{P}(\text{Maker wins}) = \begin{cases} 1 & \text{if } x_n \to +\infty, \\ e^{-e^{-x}} & \text{if } x_n \to x \in \mathbb{R}, \\ 0 & \text{if } x_n \to -\infty. \end{cases}$$

Maker's Hamilton Strategy Overview

Before the Game Begins:

Pick a spanning tree $\mathcal{T}$ of Γ_{max} with maximum degree ≤ 5

- Such a tree $\mathcal{T}$ exists because Γ_{max} is a geometric graph

Every good cell $c \in \Gamma_{\text{max}}$

- At most $T = O(1)$ vertices are **marked.** They will be used to (a) connect with vertices in bad cells, and (b) create matchings between cells adjacent in $\mathcal{T}$.
- The remaining vertices in c are **unmarked.** These will become the bulk of the Hamilton cycle. We make a soup of flexible **blob cycles.**

Maker's Hamilton Strategy Overview

During the Game, Maker plays lots of mini-games:

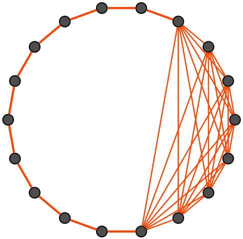
1. Create a path through each obstruction and each safe cluster, ending in marked vertices in the same cell
2. Claim two edges between cells adjacent in $\mathcal{T}$
3. Create soup of flexible **blob cycles** in the unmarked vertices
4. Claim half the edges from each marked to vertex to the set of unmarked vertices.

After the Game, Maker stitches together the Hamilton Cycle

Blob Cycles

Let $k \geq s$. An ***s*-blob cycle** on k vertices is the union of

- A k-cycle on $v_1, \ldots, v_k$
- A complete graph on $v_1, \ldots, v_s$



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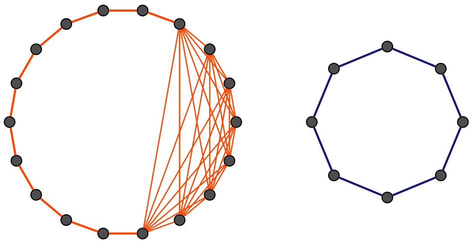

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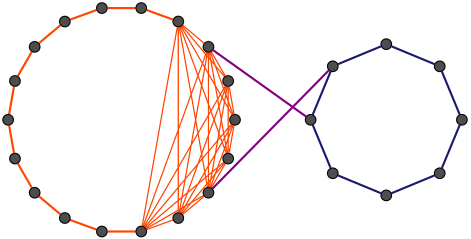

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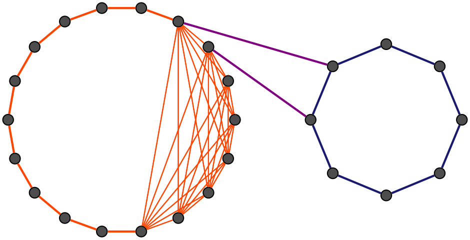

Blob Cycles

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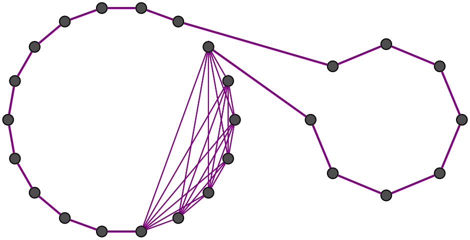
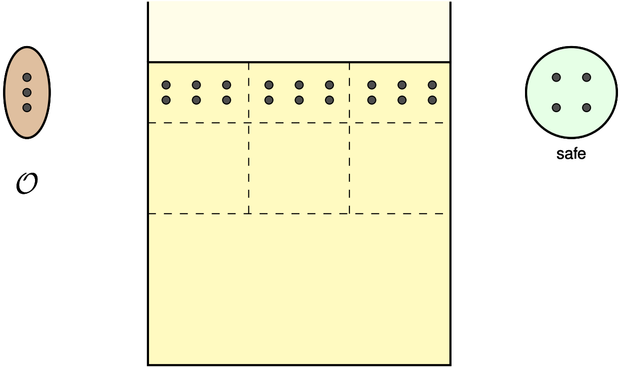


Maker's Hamilton Strategy for each Good Cell c

Mark $T = O(1)$ vertices for connecting to nearby cells, obstructions and safe vertices. Make blob cycle soup in the rest.



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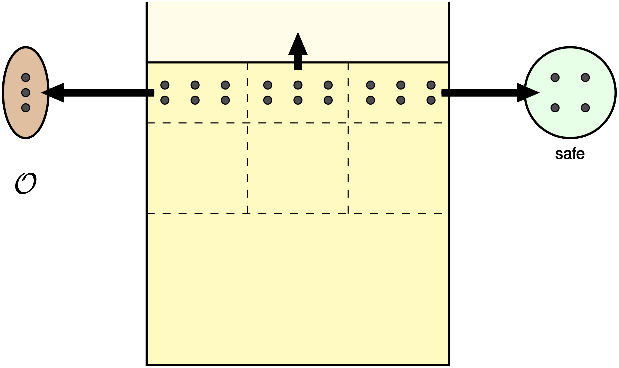
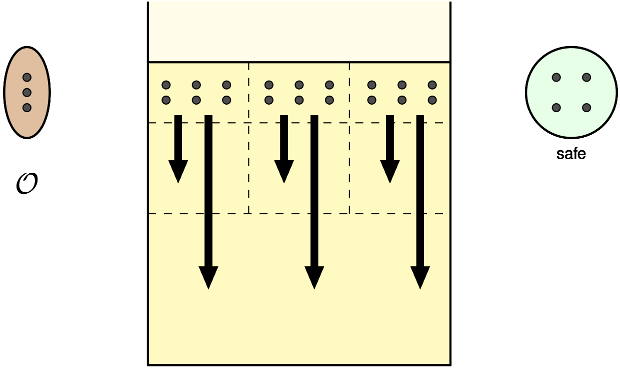
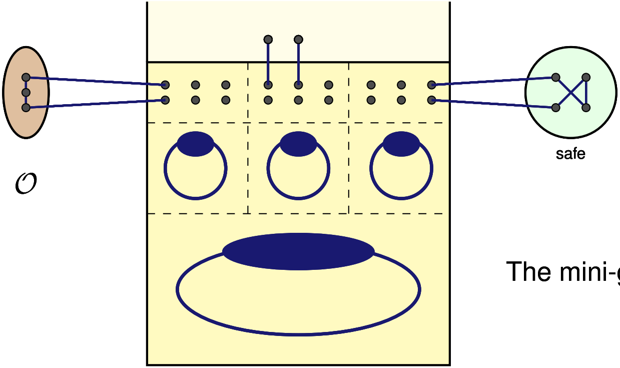


Maker's Hamilton Strategy for each Good Cell c

Claim half the edges from each vertex to lower level

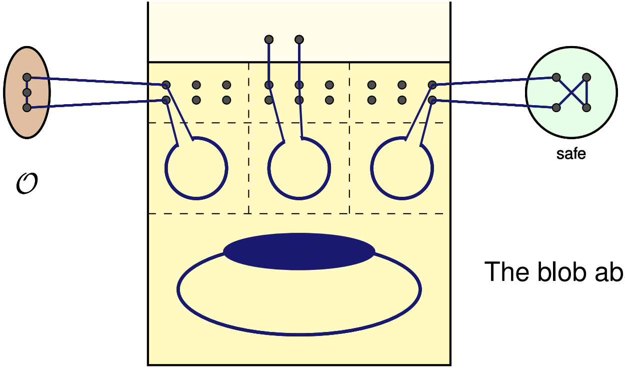


Maker's Hamilton Strategy for each Good Cell c



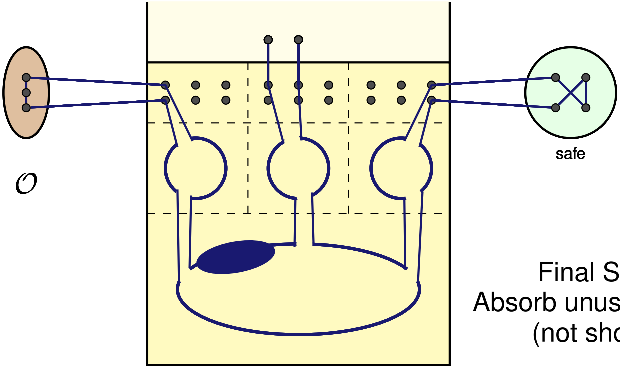
The mini-games

Maker's Hamilton Strategy for each Good Cell c



The blob absorption

Maker's Hamilton Strategy for each Good Cell c



Final Step:
Absorb unused points
(not shown)

Hitting Radius for Perfect Matching Game

Theorem (BDFMS 2013+)

Whp, the random geometric graph process satisfies, for n even:

$$\rho_n(\text{Maker wins p. m. game}) = \rho_n(\delta(G) \geq 2 \text{ and } \delta_e \geq 3)$$

where $\delta_e(G) = \min_{uv \in E(G)} |N(\{u, v\})|$. In particular, if

$$\pi n r^2 = \ln n + \ln \ln n + x_n$$

then

$$\lim_{\substack{n\to\infty,\\ n \text{ even}}} \mathbb{P}(\text{Maker wins}) = \begin{cases} 1 & \text{if } x_n \to +\infty, \\ e^{-((1+\pi^2/8)e^{-x} + \sqrt{\pi}(1+\pi)e^{-x/2})} & \text{if } x_n \to x \in \mathbb{R}, \\ 0 & \text{if } x_n \to -\infty. \end{cases}$$

Summary

Game	**Minimum Degree Condition**	**Hitting Radius (essentially)**
Connectivity Game	$\delta(G) \geq 2$	$r^2 = \frac{\ln n + \ln\ln n}{\pi n}$
Perfect Matching Game	$\delta(G) \geq 2$, and if $x_i x_j \in E(G)$ then $\|N(\{x_i, x_j\})\| \geq 3$	$r^2 = \frac{\ln n + \ln\ln n}{\pi n}$
Hamilton Cycle Game	$\delta(G) \geq 4$	$r^2 = \frac{\ln n + 5\ln\ln n}{\pi n}$

Future Directions

Biased Games

- What happens when Breaker claims b edges on every turn, while Maker only claims 1?
- Our results should extend to constant b, but what about when $b = b(n) = \omega(1)$?

Higher Dimensions

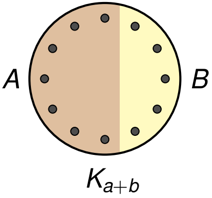
- What is the critical radius for each of these games for a 3D (and higher) random geometric graph?

Thank you!

Mini-game: the (a, b) Path Game

The (a, b) Path Game:

- Played on K_{a+b} partitioned into sets A, B of sizes a, b.
- Maker Goal: create a path between any two B-vertices that contains all A-vertices.



Lemma

The (a, b) Path Game is Maker-win when

- $b \geq 6$, or;
- $a = 3$ and $b \geq 5$, or;
- $a \in \{1, 2\}$ and $b \geq 4$.

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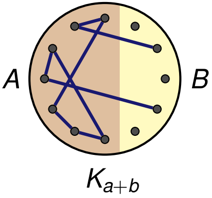

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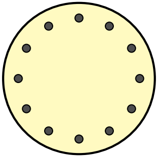
Mini-game: Blob Cycle Game

s-Blob Cycle Game

- Played on K_m
- Maker tries to make an *s*-blob on *m* vertices



Lemma

For $s \geq 4$, there is a constant $N = N(s)$ such that the *s*-Blob Game is Maker-win on K_m for $m \geq N(s)$.

Fun fact: the proof uses Krivelevich's result on the critical bias of the Hamilton cycle game on K_n.

Mini-game: Blob Cycle Game

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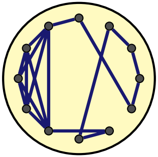

K_m

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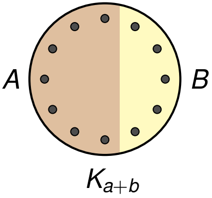
Mini-game: the (a, b) Matching Game

The (a, b) Matching Game

- Played on K_{a+b} partitioned into sets A, B of sizes a, b.
- Maker Goal: create a matching that saturates A



Lemma

The (a, b) matching game is Maker-win when

- $b \geq 4$, or;
- $a \in \{2, 3\}$ and $b \geq 3$, or;
- $a = 1$ and $b \geq 2$.

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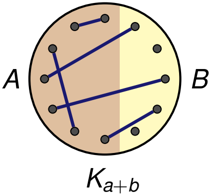

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