Infections and Mutations

J. Díaz

LSI-UPC

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Contact Graph

An infinite or finite graph G where each node represents a healthy or infected agent. Usually healing depends of each agent and infection propagates through the edges.

Epidemics Model

- SIR (susceptible-infected-removed): a node become infected at rate $\lambda\times$ function of infected neighbors, stay infected for an exp. distributed with ratio 1, number of steps and after can not be infected again.
  M.Newman:*Spread of of epidemic disease on networks* (2002)
- SIS (susceptible-infected-susceptible): when one node heals it becomes susceptible to be re-infected.

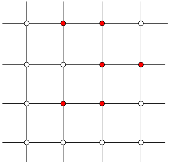
Early work on infinite graphs

The setting was the contact process (SIS):
Given an infinite G, define the continuous time Markov process on $\{0, 1\}^{V(G)}$,
where 0 denotes the vertex is healthy and 1 denotes the vertex is infected.
If A is the set of infected vertices,

- if $v \in A$, $A \to A \setminus \{v\}$ with rate $= 1$
- if $v \notin A$, $A \to A \cup \{v\}$ with rate $= \lambda \times \#$ infected neighbors.

Define an epidemic if the process keep for ever a large component that is infected.

Harris-74: Uses the infinite grid $\mathbb{Z}^k$.
If $k \geq 2 \Rightarrow$ a.a.s. epidemic for $\lambda \geq \frac{1}{2k-1}$
If $k = 1 \Rightarrow$ a.a.s. epidemic for $\lambda \geq 1.18$



Early work on SIS for infinite graphs

Liggett-99: infinite k-regular graphs

There exists epidemic thresholds $\lambda_1 < \lambda_2$ s.t. if the infection rate is λ, then

- if $\lambda > \lambda_2 \Rightarrow$ with prob. > 0 there is epidemic,
- if $\lambda < \lambda_1 \Rightarrow$ a.a.s. extinction.

Let λ_c be the threshold for epidemic.
For the grid $\mathbb{Z}^k$, if $k \geq 2$ $\lambda_1 = \lambda_2 = \lambda_c \sim \frac{1}{2k}$ (for large k).
If $k = 1 \Rightarrow 1.53 \leq \lambda_c \leq 2$. Experimentally $\lambda_c = 1.65$,

Open problem: Find analytically the value of λ_c for the case $k = 1$.

Some further work on SIS model

Over 150 papers on the contact model and some variations for different finite and infinite contact graphs

For finite graphs G ($|V(G)| = n$), an infection becomes an epidemic if the time it takes to die out is supper-polynomial in n

- Preferent Attachment model Berger,Borg,Chayes,Saberi (2005)
- Configuration model Ganesh,Massoulie, Towsley (2005)
- Bollobas-Chung Small World graphs Durrett, Jung (2007)
- Change healing/infection rate dynamically Borg,Chayes,Ganesh,Saberi (2010)

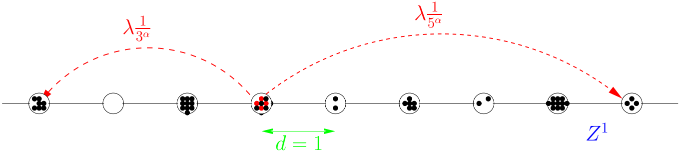
Nice survey: Durrett: *Some features of the spread of epidemics and information on a random graph*,PNAS, 2010

A new graph model for SIS infections

D., Pérez, Wormald

- The infinite $\mathbb{Z}^1$. Consider each integer as a bin.
- Distributed infinite points (agents) inside of of the bins according to a randomized or adversarial procedure.
- An infected point can infect another agent in a bin a distance d with rate $\lambda d^{-\alpha}$, where λ, α are given constants > 0. Points in same bin are consider to be at $d = 1$
- Every infected point can heal with rate 1.

Given any distribution of points in the integers, start with an infected point and find thresholds of λ and α for extinction and epidemic



Some results: Epidemic

Theorem

1.- For any distribution of points on the integers bins in $\mathbb{Z}^1$, there is a.a.s. an *epidemic* for the described process, for the following conditions on α, λ:

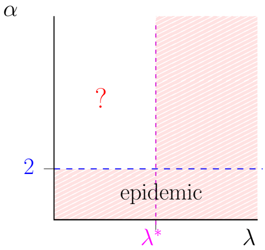
1.a- for $0 \leq \alpha \leq 2$ and any $\lambda \geq 0$,

1.b- for a sufficiently large λ and any $\alpha \geq 0$.

Theorem

The above result generalizes to k-dimensional graphs:

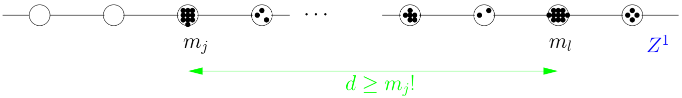
spread points in the integer vertices of $\mathbb{Z}^k$, one infected point heals with rate 1, and a healthy point infects with rate λ/d^α.



Some results: Extinction

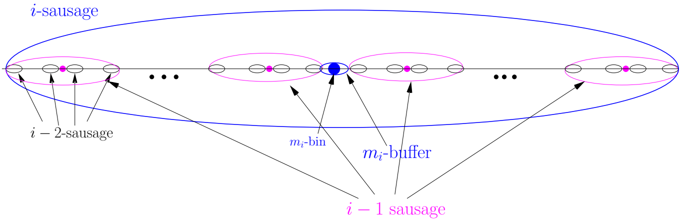
Define $\{m_j\}_0^n$ as $m_0 = 10$ and $m_j = m_0 \cdot (1.1)^j$.
For each bin $v \in \mathbb{Z}^1$ say the bin is of type m_j if $m_{j-1} < |v| \le m_j$.

2.- Let $\alpha \ge 7$ be a given constant. Then, for any deterministic or probabilistic distribution of points into the integer bins in $\mathbb{Z}^1$ such that any two bins $u, v \in \mathbb{Z}$ of types m_j and m_l ($m_j < m_l$) it must be that $d(u, v) \ge m_j!$. Then,
Theorem $\exists \lambda_c$ such that a.a.s. there will be extinction, for $\lambda < \lambda^*$.



Idea of the proof

Construct a worst case structures under the defined constrains: the *sausages*.

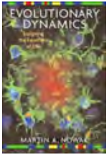


Prove that for $\alpha > 2$ there is a sufficiently small value of λ s.t. if we start from an infected point, a.a.s. the infection dies out.

Evolutionary graph theory

Moran (1958), Lieberman, Hauert, Nowak (2005)

Evolutionary graph theory gives a way to study how the topology of the interactions between the population affects the evolution.



Variation of the finite contact graph, SIS model, where the vertices represent single agents and edges the interactions among agents. Infection and healing are produced by other agents.

Two types of vertices: *mutants* and the *non-mutants*.

The fitness r of an agent denotes its reproductive rate, and will determine how often an offspring will take over adjacent vertices.

Evolutionary graph theory

Moran (1958), Lieberman, Hauert, Nowak (2005)

The dynamics of selection is studied by the Moran process:

• Select randomly an individual with probability proportional to its fitness,
• clone an offspring,
• replace a random selected neighbor by the new clone.

What are the probability a single mutant takes over the entire population?

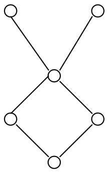
How long does it take?

Moran's process

Lieberman, Hauert, Nowak (2005)
Given a graph G, with n vertices, where
mutants have fitness r
non-mutants have fitness 1

At the beginning of the process all vertices are non-mutant.
Select randomly one vertex to mutate.

- Iterate:
  At any time $t > 0$, assume we have k mutant and $(n-k)$ non-mutant vertices
- choose u with probability $\frac{r}{kr+(n-1)}$ if u is mutant and $\frac{1}{kr+(n-1)}$ othrwise, and create a clone
- choose uniformly at random a $v \in \mathcal{N}(u)$, and replace v with the clone of u



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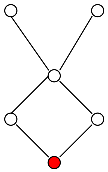

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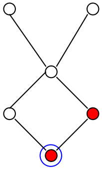

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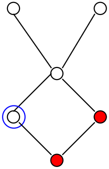

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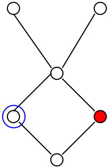
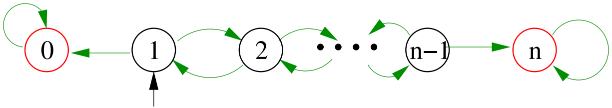


Moran Process

This random process defines discrete, transient Markov chain, on states $\{0, 1, \ldots, n-1, n\}$ with two absorbing states: n fixation (all mutants) and 0 extinction (no mutants).



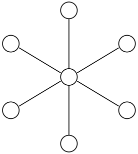
Let G be a contact graph and initially choose randomly a $v \in V(G)$ to mutate.
The fixation probability $f_G(r)$ of G is the probability that a mutant with takes over the whole graph G.
The extinction probability of G is $1 - f_G(r)$.

Given G and r, compute $f_G(r)$.

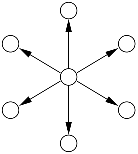
Moran Process

Theorem[Lieberman et al., 2005]
Given a symmetric or regular contact graphs G, with $|V(G)| = n$, if $r \geq 1$ then $\rho = f_G(r) = \frac{1-1/r}{1-1/r^n} \sim 1 - \frac{1}{r}$.

G is said to be an amplifier if $f_G(r) > \rho$.



G is said to be an suppressor if $f_G(r) < \rho$.



Absorption time for undirected graphs

D., Goldberg, Mertzios, Richerby, Serna, Spirakis (2012)

Given a G and r, let τ be a r.v. counting the absorption time.

Theorem

- if $r < 1$ $\mathbf{E}[\tau] \leq \frac{1}{1-r} n^3$
- if $r > 1$ $\mathbf{E}[\tau] \leq \frac{r}{r-1} n^4$
- if $r = 1$ $\mathbf{E}[\tau] \leq n^6$

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Corollary

There is an FPRAS for computing the fixation probability, for any $r \geq 1$.

There is an FPRAS for computing the extintion probability, for any $r \geq 0$.

Simulate N times the Moran process for the above upper bounds (where $N = O(\frac{n^2}{\varepsilon^2})$ if $r \geq 1$ and $N = O(\frac{(r+n)^2}{\varepsilon^2})$ if $r < 1$).

Absorption time for undirected graphs

D., Goldberg, Richerby, Serna (2013)

Given a digraph G and r, let τ be a r.v. counting the absorption time of the Moran process.

Theorem

If G is strongly connected and Δ-regular with $V(G) = n$,

$$\mathbf{E}[\tau] \leq n^2 \Delta,$$

and the upper bound us sharp.

Notice this bound is smaller than the general one for the undirected case.

On the other hand, and differently from the undirected case, there is an infinite family of strongly connected graphs with exponential absorption time.

Shakarian, Ross, Johnson: A review of evolutionary graph theory with applications to game theory (2012)

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Thank you for your attention