Random directed graphs are robustly Hamiltonian

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Joint work with
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Benny Sudakov

Hamilton cycles in undirected graphs

Definition:

A Hamilton cycle of a graph G is a spanning connected 2-regular subgraph of G.

Theorem (Dirac 1952):

Let G be a graph on $n \geq 3$ vertices. If $\delta(G) \geq n/2$, then G is Hamiltonian. Easily seen to be best possible.

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Local resilience

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Let G be a graph and let $\mathcal{P}$ be a monotone increasing graph property. The local resilience of G with respect to $\mathcal{P}$ is

$$r_\ell(G,\mathcal{P}) := \min_{\substack{H\subseteq G \\ G\setminus H\notin\mathcal{P}}} \max_{u\in V(G)} \left\{ \frac{\deg_H(u)}{\deg_G(u)} \right\}.$$

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Theorem (Erdős and Rényi 1959):

If $p \leq (1-\varepsilon)\log n/n$, then a.a.s. $r_\ell(G(n,p),\mathcal{H}) = 0$.

Theorem (Komlós and Szemerédi 1983 and Bollobás 1984):

If $p \geq (1+\varepsilon)\log n/n$, then a.a.s. $r_\ell(G(n,p),\mathcal{H}) > 0$.

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Let D be a strongly connected digraph on n vertices. If $\delta^+(D) + \delta^-(D) \geq n$, then D is Hamiltonian.

Corollary:

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Proof of the upper bound

Sketch

- Split $[n]$ into two sets of equal size V_1 and V_2.
- Expose all arcs of $D(n, p)$ with one endpoint in V_1 and the other in V_2.
- Delete all arcs directed from V_1 to V_2.
- a.a.s. we deleted at most $(1/2 + \alpha)deg^+_D(u)$ and at most $(1/2 + \alpha)deg^-_D(u)$ from every $u \in [n]$.

Remark:

This proof works for $p \geq c_1 \log n/n$ and $\alpha \geq c_2\sqrt{\frac{\log n}{np}}$.

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The lower bound - proof sketch for constant p

Main stages of the proof:

Let $D = ([n], E)$ be obtained from $D(n, p)$ by deleting at most $(1/2 - \alpha)deg_D^+(u)$ and at most $(1/2 - \alpha)deg_D^-(u)$ from every $u \in [n]$.

Stage 1: Build an almost spanning cycle C_1.

Stage 2: Absorb every remaining vertex into C_1.

Stage 2:

For every $u \in [n] \setminus C_1$, replace an arc $(x, y) \in E(C_1)$ with two arcs $(x, u), (u, y) \in E$.

Clearly we have to ensure that such arcs exist.

Remark:

Having so many triangles requires $p = \Omega(n^{-1/2})$.

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Main substages of Stage 1:

Stage 1.0: Preparation.

Stage 1.1: Build a short directed path P_1 which includes all problematic vertices.

Stage 1.2: Extend P_1 into an almost spanning path P_2 such that every $u \in [n] \setminus P_2$ has many arcs $(x, y) \in E(P_2)$ for which $(x, u), (u, y) \in E$.

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- Apply the Directed Regularity Lemma to D; let $\{V_0, V_1, \ldots, V_k\}$ be the resulting ε-regular partition.
- Let $R = R(D, \delta)$ be the resulting directed regularity graph: for every $1 \leq i \neq j \leq k$ the arc (v_i, v_j) is in $E(R)$ iff (V_i, V_j) is ε-regular with directed density $\geq \delta$.
- Prove that $deg_R^+(v_i) \geq (1+\beta)k/2$ and $deg_R^-(v_i) \geq (1+\beta)k/2$ for all but at most βk vertices of R.
- Deduce by the corollary of Ghouila-Houri's Theorem that R admits a directed cycle of length $r \geq (1-\beta)k$.
- Hence $V_1, \ldots, V_r, V_1$ is a "directed cycle" of clusters in D.

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- Hence $V_1, \ldots, V_r, V_1$ is a "directed cycle" of clusters in D.

Proof sketch for constant p cont'd

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- Apply the Directed Regularity Lemma to D; let $\{V_0, V_1, \ldots, V_k\}$ be the resulting ε-regular partition.
- Let $R = R(D, \delta)$ be the resulting directed regularity graph: for every $1 \leq i \neq j \leq k$ the arc (v_i, v_j) is in $E(R)$ iff (V_i, V_j) is ε-regular with directed density $\geq \delta$.
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- A vertex $v \in [n]$ is problematic if $|deg^+_{D(n,p)}(v, V_i) - \ell p| \geq \varepsilon \ell p$ or $|deg^-_{D(n,p)}(v, V_i) - \ell p| \geq \varepsilon \ell p$ for some $1 \leq i \leq r$.
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- Let v be a problematic vertex; there exist $1 \leq j_1, j_2 \leq r$ such that v has at least $\ell p/3$ in-neighbours in V_{j_1} and at least $\ell p/3$ out-neighbours in V_{j_2}.
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Stage 1.2: Extending P_1 to an almost spanning P_2

As long as P_2 is not long enough, traverse the cycle of clusters, at each step choosing a typical vertex uniformly at random.

Lemma:

At the end of Stage 1.2, for every $v \in [n] \setminus V(P_2)$ there are many arcs $(x, y) \in E(P_2)$ such that $(x, v), (v, y) \in E(D)$.

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Proof of the lemma:

- Since all problematic vertices are in P_2, for every $v \in [n] \setminus V(P_2)$ there are many pairwise far pairs (V_i, V_{i+1}) such that v has many in-neighbours in V_i and many out-neighbours in V_{i+1}.
- Every time we add an arc $(x, y) \in V_i \times V_{i+1}$ to P_2, the probability that $x \in N_D^-(v, V_i)$ and $y \in N_D^+(v, V_{i+1})$ is more or less $\frac{|N_D^-(v, V_i \setminus P_2)| \cdot |N_D^+(v, V_{i+1} \setminus P_2)|}{|V_i \setminus P_2| \cdot |V_{i+1} \setminus P_2|}$.
- For far away pairs these attempts are independent.
- The number of successful attempts dominates a binomial distribution.
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Stage 1.3: Closing P_2 to a cycle C_1

- Traverse the cycle of clusters, using only typical vertices, until reaching $x \in V_{r-2}$.
- Since v_1 is good, it has many in-neighbours in V_r.
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Additional difficulties for $p = o(1)$

Stage 1.0: Apply the Sparse Diregularity Lemma.

Stage 1.1: The use of the Sparse Diregularity Lemma gives rise to many problematic vertices of different types; many more than the size of a neighbourhood of a typical vertex.

Stage 1.2: Essentially the same but more involved technically.

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Summary and future work

- We have proved that a.a.s. $1/2 - \alpha \leq r_\ell(\mathcal{D}(n,p),\mathcal{H}) \leq 1/2 + \alpha$ whenever $p \gg \log n/\sqrt{n}$.
- The upper bound holds for $p = \Omega(\log n/n)$ and $\alpha = O(\sqrt{\frac{\log n}{np}})$.
- For constant p, the lower bound holds with $\alpha = O(\sqrt{\frac{\log n}{np}})$ as well.
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Summary and future work

- We have proved that a.a.s. $1/2 - \alpha \leq r_\ell(\mathcal{D}(n,p),\mathcal{H}) \leq 1/2 + \alpha$ whenever $p \gg \log n/\sqrt{n}$.
- The upper bound holds for $p = \Omega(\log n/n)$ and $\alpha = O(\sqrt{\frac{\log n}{np}})$.
- For constant p, the lower bound holds with $\alpha = O(\sqrt{\frac{\log n}{np}})$ as well.
- What is the local resilience of sparser digraphs? Is it $1/2 + o(1)$ for every $p \gg \log n/n$?

Thank you for your attention

Questions?