High Order Phase Transition in Random Hypergraphs

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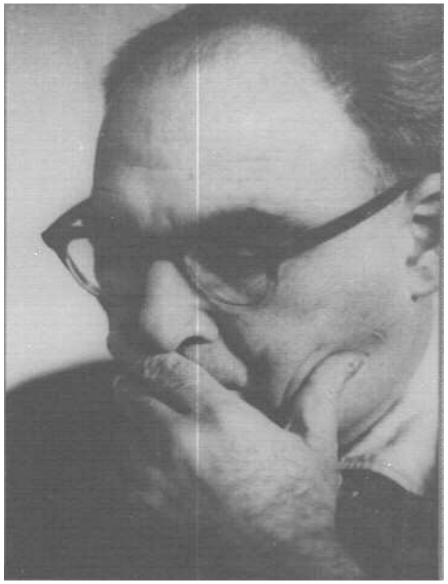
Phase transition of random graphs

Paul Erdős



Alfréd Rényi

Paul Erdős and A. Rényi, On the evolution of random graphs *Magyar Tud. Akad. Mat. Kut. Int. Kozl.* **5** (1960) 17-61.

Phase transition of random graphs

ON THE EVOLUTION OF RANDOM GRAPHS

by

P. ERDÖS and A. RÉNYI
Institute of Mathematics
Hungarian Academy of Sciences, Hungary

1. Definition of a random graph

Let $E_{n,N}$ denote the set of all graphs having n given labelled vertices $V_1, V_2, \cdots, V_n$ and N edges. The graphs considered are supposed to be not oriented, without parallel edges and without slings (such graphs are sometimes called linear graphs). Thus a graph belonging to the set $E_{n,N}$ is obtained by choosing N out of the possible $\binom{n}{2}$ edges between the points $V_1, V_2, \cdots, V_n$, and therefore the number of elements of $E_{n,N}$ is equal to $\binom{\binom{n}{2}}{N}$. A random graph $\Gamma_{n,N}$ can be defined as an element of $E_{n,N}$ chosen at random, so that each of the elements of $E_{n,N}$ have the same probability to be chosen, namely $1/\binom{\binom{n}{2}}{N}$. There is however an other slightly

Classical random graphs models

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Rich literature on the phase transition of these two models: **Erdős, Rényi, Bollobás, Spencer, Janson, Knuth, Łuczak, Ruciński, Pittel, Sudakov, Krivelevich, Wormald, ...**

Evolution of $G(n, p)$

Erdős-Rényi 1960s:

- $p \sim c/n$ for $0 < c < 1$: The largest connected component of $G_{n,p}$ is a tree and has about $\frac{1}{\alpha}(\log n - \frac{5}{2}\log\log n)$ vertices, where $\alpha = c - 1 - \log c$.

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- $p \sim 1/n + \epsilon/n^{4/3}$, the critical window, double jump.
- $p \sim c/n$ for $c > 1$: Except for one "giant" component, all the other components are relatively small. The giant component has approximately $f(c)n$ vertices, where

$$f(c) = 1 - \frac{1}{c}\sum_{k=1}^{\infty}\frac{k^{k-1}}{k!}(ce^{-c})^k.$$

General question

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This is a hard problem in general. The answer depends on the model.

Chung-Lu model

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- $w_1, w_2, \ldots, w_n$: vertices weight/expected degrees
- $p_{ij} = w_i w_j / \sum_{i=1}^{n} w_i$: the probability of the pair (i, j) being created as an edge, independently.

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Chung and Lu (2004)
If the average degree is strictly greater than 1, then almost surely the giant component in a graph G in $G(w_1, \ldots, w_n)$ has volume $(\lambda_0 + O(\sqrt{n}\frac{\log^{3.5} n}{\mathrm{vol}(G)}))\mathrm{vol}(G)$, where λ_0 is the unique positive root of the following equation:

$$\sum_{i=1}^{n} w_i e^{-w_i \lambda} = (1 - \lambda) \sum_{i=1}^{n} w_i.$$

Graph percolation

Percolation problem: Given a host graph G, find the the threshold $p_c(G)$ so that the random subgraph G_p has a giant component if $p > p_c$ and no giant component if $p < p_c$.

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- **Chung, Lu, Horn [2009]:** $p_c \approx \frac{1}{\mu}$, for sparse graphs under some spectral conditions.

Hypergraphs

$H = (V, E)$ is an r-uniform hypergraph (r-graph, for short).

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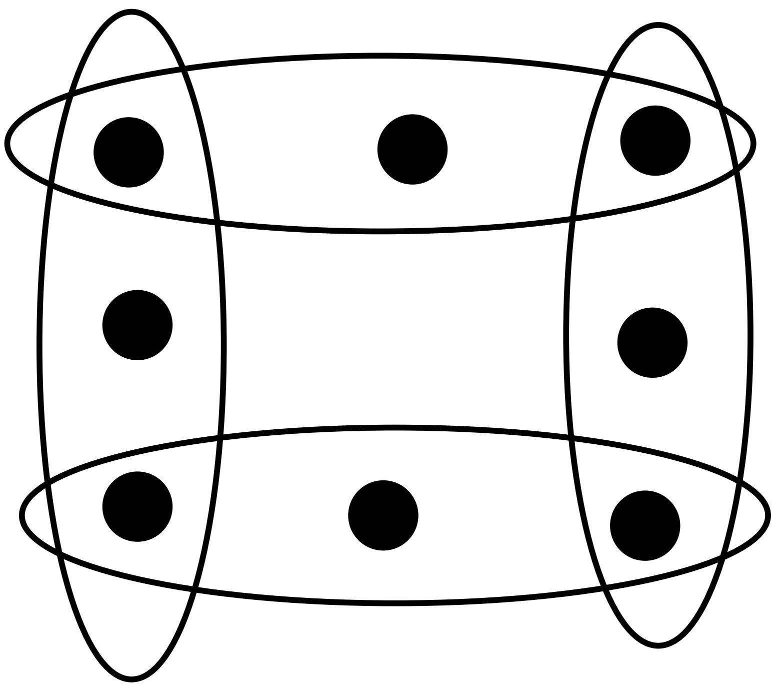
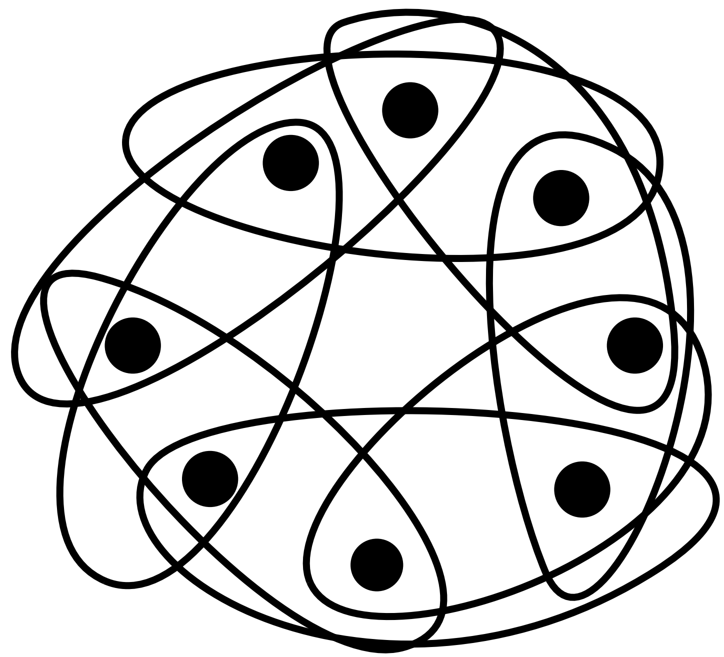


A 3-uniform loose cycle



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Connections in 3-graphs

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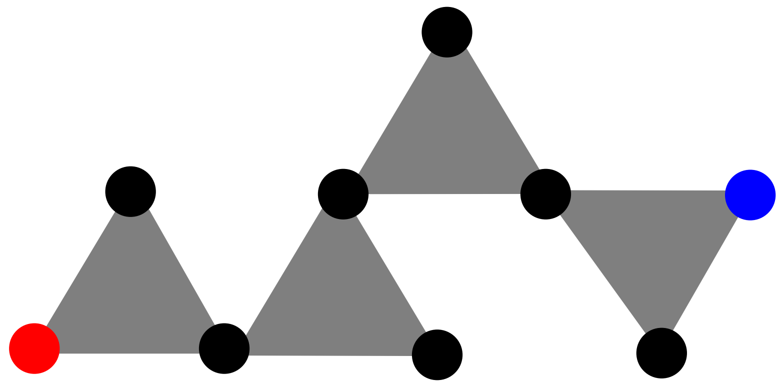

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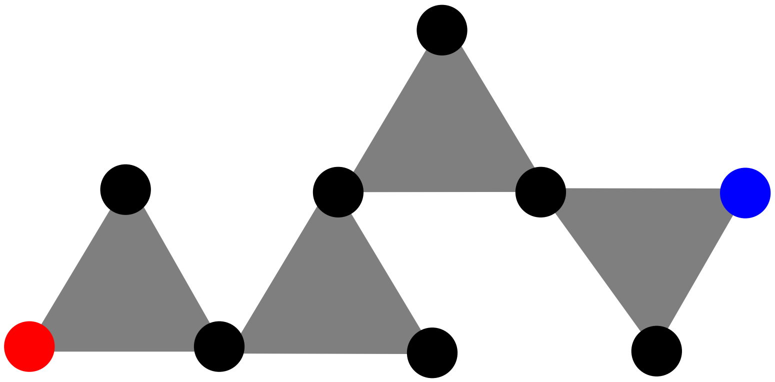
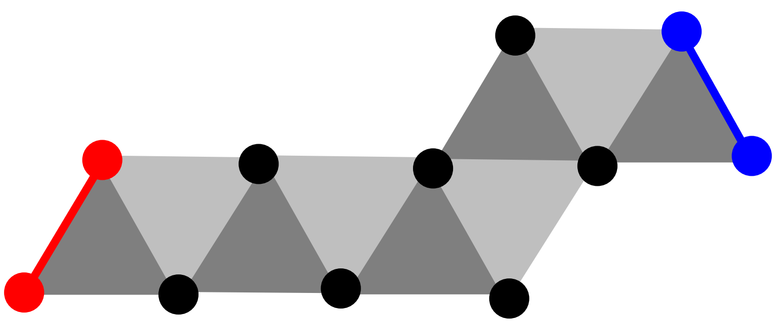


- Pair to Pair



High order components of H

Given an r-graph $H = (V, E)$, for $1 \leq s \leq r - 1$, define an auxiliary graph G^s:

- vertex set $\binom{V}{s}$, an s-set is called a stop.
- a pair (S, T) forms an edge in G^s if $S \cup T \subseteq F \in E$.

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Main question: how does the distribution of s-th order components evolve as the number of edges in random hypergraph increases?

Random hypergraphs

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Two models are more or less "equivalent" via $N = \binom{n}{r}p$. Here we use the binomial model.

Phase transition of s-components

$s = 1$ (vertex-to-vertex connection)

- Schmidt-Pruzan and Shamir [1985]
- Karoński and Łuczak [2002]
- Coja-Oghlan, Moore, and Sanwalani [2007]
- Kang, Behrisch and Coja-Oghlan[2010]

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$s \geq 2$

- This talk.
- Also independently by **Cooley-Kang-Person [2013+]**.

Our Results

Theorem I [Lu-Peng 2013+] For any $1 \leq s \leq r-1$, set $p = \frac{c}{\binom{n}{r-s}}$ and $m = \binom{r}{s} - 1$. Then the following statements hold for the s-th order components in $H_r(n,p)$.

- If $mc < 1 - \epsilon$, then almost surely all s-th-order connected components have size $O(\ln n)$.
- If $mc > 1 + \epsilon$, then almost surely there is a unique giant s-th-order connected component of size $(f(c) + o(1))\binom{n}{s}$, where $f(c)$ is the unique positive root of $1 - x = e^{c[(1-x)^m - 1]}$ and

$$f(c) = 1 - \sum_{k=0}^{\infty} \frac{(mk+1)^{k-1} c^k}{k! e^{(km+1)c}}.$$

Our Results

Theorem II [Lu-Peng 2013+] For any $1 \leq s \leq r-1$, set $p = \frac{c}{\binom{n}{r-s}}$ and $m = \binom{r}{s} - 1$. For any $k \geq 0$, almost surely the number of s-th-order components in $H_r(n,p)$ of exactly k-edges is $(a_k + o(1))\binom{n}{s}$, where

$$a_k = \frac{(mk+1)^{k-1}c^k}{k!e^{(km+1)c}}.$$

The majority of the small components are (r,s)-trees.

(r, s)-trees

For any $1 \leq s \leq r-1$, an (r, s)-tree T_k is an s-th-order component of k edges with maximum number of vertices.

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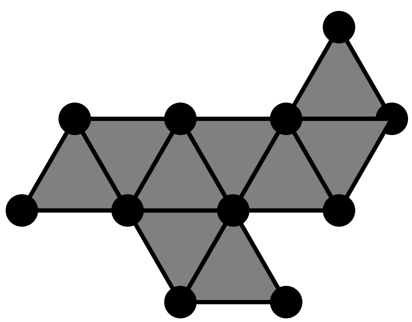
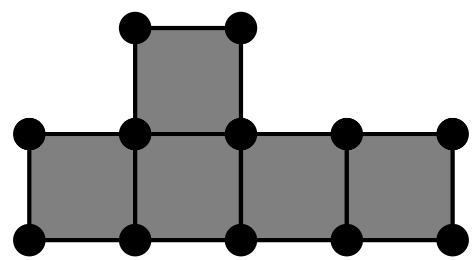
- T_0 is a single stop S_0.
- T_{k+1} can be obtained by first adding a set E_{k+1} of $r-s$ new vertices to T_k, selecting a stop $S \subset F \in E(T_k)$, then adding an new edge $S \cup E_{k+1}$.



a $(3, 2)$-tree



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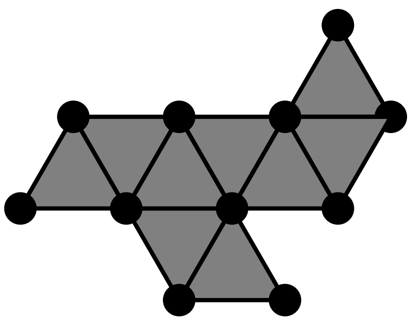

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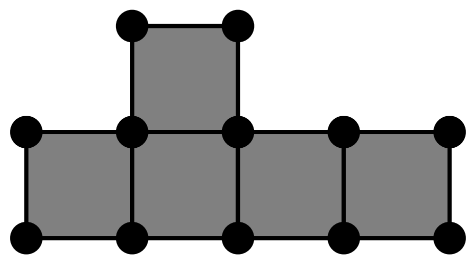

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Canonical partition: $S_0 \cup E_1 \cup E_2 \cup \cdots \cup E_k$

(Here $E_1, \ldots, E_k$ are indistinguishable.)

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Theorem [Lu-Peng 2013+]: For any $1 \leq s \leq r-1$, the number of (r, s)-trees with a fixed canonical partition $S_0 \cup E_1 \cup E_2 \cup \cdots \cup E_k$ is

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Remark: This theorem generalizes Cayley's formula. We also found the generalized Prüfer codes for (r, s)-trees.

Method

We couple the branching process with a special Galton-Watson process called "m-fold Poisson process" $T^{po}_{m,c}$:

- Start with one live node (the root).
- For each round, select a live node u if there is one.
- Roll a Poisson dice to produce an non-negative integer k with probability $e^{-c}c^k/k!$.
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Easy fact:

- If $mc < 1$, then the process terminates after finite steps.
- If $mc > 1$, then with probability $1 - x$, the process will survive forever. Here x is the solution of

$$x = e^{c(x^m - 1)}.$$

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DFS helps. but we use different analysis.

Sketch

Our approaches

- We also use DFS.
- We book-keep dead stops, live stops, available edges, and potentially bad t-sets for each live stops.
- If the number of potentially bad t-sets and used edges exceeding $\epsilon\binom{n}{r-s}$ at any live stop, we restart the process.

With all efforts, we are able to find a path of length $\Omega(\frac{n^s}{log^{C_s}n})$. Then we use sprinkling to show the giant component exists and is unique.

Variations

There are different ways to define two s-sets are adjacent. For example, we define $G^{(s)}$:

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The connected components in $G^{(s)}$ are the same as in G^s unless $r = 2s$. It is interesting that the threshold happens when the average degree in $G^{(s)}$ much smaller than 1.
For $r = 2s$, the components in $G^{(s)}$ is similar to $G(n, p)$ and has an easier proof.

Related topics

For $1 \leq s \leq \frac{r}{2}$, we define the Laplacian eigenvalues of $G^{(s)}$ as the s-th-order Laplacian eigenvalues of the hypergraph H.

- These eigenvalues can effectively control the mixing rate of high-ordered random walks, the generalized distances/diameters, and the edge expansions.
- The Laplacian eigenvalues of $H_r(n,p)$ follows the Semi-circle Law.
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