A few random open problems, some of them for random graphs

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Outline

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Definition

- Place one *agent* on each vertex.
- Every pair of agents sharing an edge is declared to be *acquainted*, and remains so throughout the process.
- In each round, we chose some matching M (any matching, perhaps it is a single edge). For each edge of M, we swap the agents occupying its endpoints.
- The *acquaintance time*, $AC(G)$, is the minimum number of rounds required for all agents to become acquainted with one another.

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Clearly,

$$AC(G) \geq \frac{\binom{|V(G)|}{2}}{|E(G)|} - 1$$

How large can this be for n-vertex connected graph?

Theorem (Benjamini, Shinkar, Tsur, 2014+)

$$AC(G) = O\left(\frac{n^2}{\log n / \log\log n}\right)$$

Theorem (Kinnersley Mitsche, Pralat, 2013)

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Random graphs $G(n, p)$

Conjecture (Benjamini, Shinkar, Tsur, 2014+)

For $p = p(n) \geq (\log n + \omega)/n$ we have a.a.s.

$$AC(G) = O\left(\frac{polylog(n)}{p}\right)$$

They conjectured this despite the fact that no non-trivial upper bound was known.

Trivial upper bound: $O(n)$, provided that $\omega - \log\log n \to \infty$ (since $AC(G) = O(n)$ for any graph with Hamiltonian path).

Trivial lower bound: $\Omega(n^2/m) = \Omega(1/p)$.

Random graphs $G(n, p)$

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$$AC(G) = O\left(\frac{\log n}{p}\right)$$

Is it tight?

$\overline{AC}(G)$: each agent has a helicopter and can, on each round, move to any vertex she wants. Clearly,

$$AC(G) \geq \overline{AC}(G)$$

$\overline{AC}(G)$ also represents the minimum number of copies of a graph G needed to cover all edges of a complete graph of the same order.

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Theorem (Kinnersley Mitsche, Pralat, 2013)

Let $\varepsilon > 0$. For $p = p(n) \geq n^{-1/2+\varepsilon}$ and $p \leq 1 - \varepsilon$ we have a.a.s.

$$\overline{AC}(G) = \Theta(AC(G)) = \Theta\left(\frac{\log n}{p}\right)$$

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The behaviours of $AC(G)$ and $\overline{AC}(G)$ for sparse random graphs remain undetermined!

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Check what is going on below the threshold for Hamiltonian path and when $p \to 1$?

Random geometric graphs $G(n, r)$

Theorem (Muller, Pralat, 2014+)

If $\pi n r^2 - \log n \to \infty$, *then a.a.s.*

$$AC(G) = \Theta(r^{-2}).$$

(Recall that if $\pi n r^2 - \log n \to -\infty$, then G is a.a.s. disconnected and so the acquaintance time is not defined.)

Percolated random geometric graphs $G(n, r, p)$

Theorem (Muller, Pralat, 2014+)

Let $\varepsilon > 0$. If $pnr^2 \geq n^{1/2+\varepsilon}$, then a.a.s.

$$AC(G) = \Theta(r^{-2}p^{-1}\log n).$$

(In fact, an upper bound works whenever $pnr^2 \geq K \log n$ for some large constant K.)

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The behaviour for sparser graphs remains undetermined.

Outline

Definition

- Initially, each vertex has weight 1.
- In each step, we are allowed to move weight from a vertex u to a neighbouring vertex v, provided that before the move the weight on v is at least the weight on u.
- The *total acquisition number* $a_t(G)$ is the minimum possible size of the final set of vertices with positive weight.

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Also, $a_t(G) \geq a_t(H)$ if G is a subgraph of H (on the same vertex set).

So $a_t(G(n, p_1)) \leq a_t(G(n, p_2))$, provided $p_1 > p_2$.

Random graphs $G(n, p)$

Question (LeSaulnier, Prince, Wenger, West, Worah, 2013):
Find the smallest $d = p(n-1)$ such that a.a.s. $a_t(G(n, p)) = 1$.

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- $d = C \log^2 n / \log\log n$ (Bal, Bennett, Dudek, Pralat, ≈ 40 days ago).

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Note that $\frac{1}{\log 2} \approx 1.44$ so it is above the connectivity threshold.

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Sharp! For $d = \frac{1-\varepsilon}{\log 2} \log n$, a.a.s. $a_t(G(n,p)) > n^{\varepsilon + o(1)}$ (Bal, Bennett, Dudek, Pralat, $\approx$ 25 days ago).

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Investigate $f(c) = \log_n a_t(G(n, c \log n/n))$.

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Investigate what is going on when $d = \frac{1+o(1)}{\log 2} \log n$.

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What about other models, for example, random geometric graphs?

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- The robber starts at 'level' 0, vertex $(0, 0, \dots, 0)$.
- Cops start at 'level' n, vertex $(n, n, \dots, n)$.

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Greedy strategy for the robber gives:

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\begin{aligned}
c(n) &\geq 2^m = 2^{n/2}, \text{ provided } n = 2m \\
c(n) &\geq 2^{-m}\binom{2m+1}{m+1} = \Theta(2^{n/2}/\sqrt{n}), \text{ provided } n = 2m+1
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Where is the right value?

Outline

Classic game

- $G(n,p)$: dense graphs (Bonato, Pralat, Wang, 2009)
- $G(n,p)$: very dense graphs (Pralat, 2010)
- $G(n,p)$: zig-zag (Luczak, Pralat, 2010)
- $G(n,p)$: sparse graphs (Bollobas, Kun, Leader, 2013)
- $G(n,p)$: Meyniel's conjecture (Pralat, Wormald, 2014+)
- random d-regular graphs: Meyniel's conjecture (Pralat, Wormald, 2014++)
- $G(n,r)$: (Beveridge, Dudek, Frieze, Muller, 2012) and (Alon, Pralat, 2014+)
- $G(n,r,p)$: (Alon, Pralat, 2014+)

Other variants for $G(n, p)$

- Fast robber (Alon, Mehrabian, 2014+)
- Playing on edges (Dudek, Gordinowicz, Pralat, 2014+)
- Cops have limited fuel or time (Fomin, Golovach, Pralat, 2012)
- Cops can shoot from distance (Bonato, Chiniforooshan, Pralat, 2010)
- ...
- Lazy Cops (Bal, Bonato, Kinnersley, and Pralat, 2014+)

Hypercube

$$c_L(Q_n) = O\left(\frac{2^n \log n}{n^{3/2}}\right)$$

(Offner and Ojakian, 2014+) $c_L(Q_n) = \Omega\left(2^{\sqrt{n}/20}\right)$

(Bal, Bonato, Kinnersley, and Pralat, 2014+) For any $\varepsilon > 0$, $c_L(Q_n) = \Omega\left(\frac{2^n}{n^{7/2+\varepsilon}}\right)$

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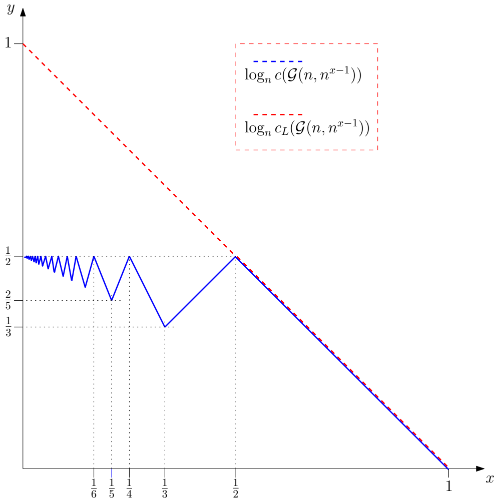
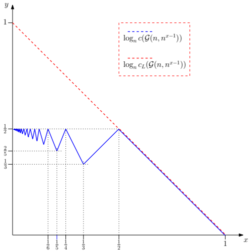
$G(n, p)$—Big picture



$G(n, p)$—More details



The ratio between upper and lower bounds:

a) $x = 1 : 1 + o(1)$.

b) $\frac{1}{2} < x < 1 : \Theta(1)$.

c) $\frac{1}{j+1} < x < \frac{1}{j}$ for some $j \geq 2 : \Theta(\log n)$.

d) $x = \frac{1}{j}$ for some $j \geq 2 : \Theta(\log^3 n)$.

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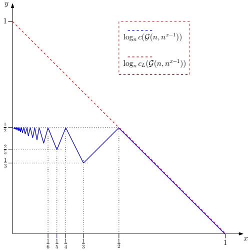

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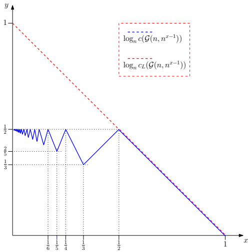

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Improve the ratio?