The threshold for the Maker-Breaker H-game

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Joint work with Rajko Nenadov and Angelika Steger.

Introduction

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- The **board** – a finite set X,
- the **winning sets** – $\mathcal{F} \subseteq 2^X$, a collection of subsets of X.
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- Played by two players - **Maker** and **Breaker**,
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A Maker-Breaker positional game **on the complete graph**:

- The **board** is the **edge set** of the complete graph K_n,
- the **winning sets** are usually representatives of a graph-theoretic structure.

Example

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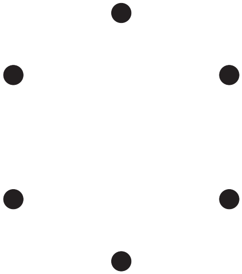

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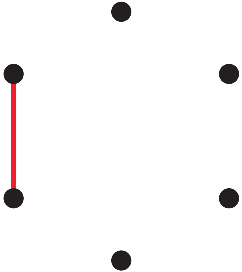

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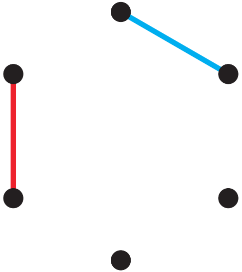

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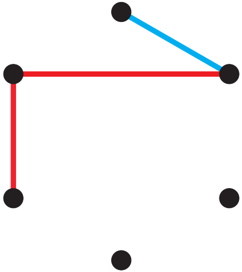

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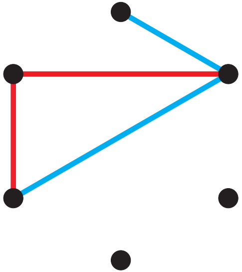

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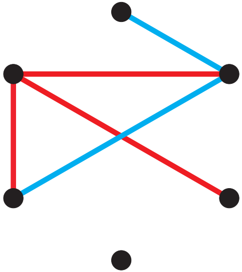

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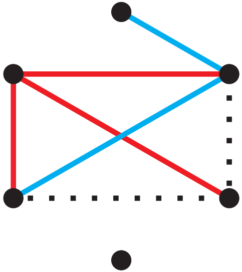

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- Connectivity game: $\mathcal{T}$ – set of all spanning trees;
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- Let Breaker claim more than one edge in each move – **biased game**,
- Randomly remove some of the edges of the base graph before the game starts – **random game**.

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Biased game $(1 : b)$ – Maker claims 1, and Breaker claims b edges per move.
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- H-game: $b_{\mathcal{G}_H} = \Theta\left(n^{\frac{1}{m_2(H)}}\right)$.
  [Bednarska-Łuczak, 2000]
  ...where $m_2(H) = \max_{H' \subseteq H, v(H') \geq 3} \frac{e(H')-1}{v(H')-2}$.

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Clique game: $b_{\mathcal{G}_H} \sim n^{\frac{1}{m_2(H)}}$.
But the threshold for appearance of H in $G(n,p)$ is $n^{-\frac{1}{m(H)}}$,
...where $m(G) = \max_{G' \subseteq G} \frac{e(G')}{v(G')}$.

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The threshold probability surely exists, as *"being Maker's win"* is an increasing graph property.

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  Again, we have $p_{\mathcal{G}_H} = n^{-\frac{\ell}{\ell-1}} < n^{-1} = b_{\mathcal{G}_H}^{-1}$.

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- Remaining cases: H with $m_2(H) = 2$, with max. 2-density determined only by triangle subgraphs.

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As it will turn out, the threshold can be placed almost arbitrarily between $n^{-5/9}$ and $n^{-1/2}$.

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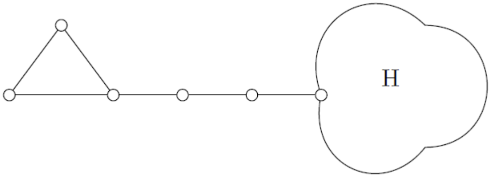
Graph H_P:



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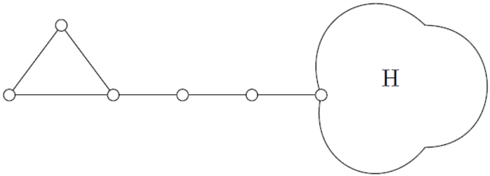

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If H is such that $9/5 < m_2(H) < 2$,
then $p_{\mathcal{G}_{H_P}} = n^{-\frac{1}{m_2(H)}}$.

Note: $m_2(H_P) = 2$.

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So, Maker can play as **Container-Complement-Breaker**! Winning sets are $\{E(K_n) \setminus C_i\}_i$, each of size $\geq \delta\binom{n}{2}$, and there is not too many of them → win e.g. by Erdős-Selfridge Theorem. □

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- Triangle game:
  Hitting time for Maker's win = hitting time for appearance of $K_5 - e$, a.a.s.
  [Müller-St. 2014+]

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  - Not known for the H-game, not even for the clique game, if $1 < b < \log n$.