Embedding trees into graphs

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Extremal graph theory is one of the important, deep and fast developing areas in Discrete Mathematics. In my lecture I will consider two famous Turán type extremal graph conjectures: I will speak of the exact solution of the Erdős-T. Sós and Komlós-Sós conjectures.

The first part is a joint work with M. Ajtai, J. Komlós, and E. Szemerédi, the second part with Maya Stein, Jan Hladký, János Komlós, Diana Piguet, and Endre Szemerédi.

The main result is that both conjectures hold for large trees. I will sketch the proof of the Erdős-Sós and the Loebl-Komlós-Sós conjectures, asserting that if either the average degree of a graph G_n or the median degree of a graph is large, then every k-vertex tree T_k is embeddable into this graph.

Conjecture 1 (Erdős-T. Sós conjecture)**.** *If T_k is a fixed tree of k vertices, then every graph G_n of n vertices and*

$$e(G_n) > \frac{1}{2}(k-2) \tag{1}$$

edges contains T_k.

Our main result is that

Theorem 1. *There exists an integer k_0 for which, if $k > k_0$, then Conjecture 1 holds.*

In the first part of the proof a weakened Erdős-T. Sós conjecture is proved, according to which for every $\eta > 0$ there exists an integer $n_0(\eta)$ such that if $n, k > n_0(\eta)$ and a graph G on n vertices contains no T_k then

$$e(G_n) \leq \frac{1}{2}(k-2)n + \eta n. \tag{2}$$

That proof, combined with some stability methods shows that in most cases either we know that $T_k \subseteq G_n$ even under the weaker condition (2) or we can prove that the structure of G_n is very near to the conjectured extremal graphs which is the union of small complete blocks or some complete bipartite graphs. Then, for $k > k_0$, applying some elementary arguments, we can embed T_k into G_n using only (1).

For technical reasons it is worth first to prove the "Dense Case", where one assumes that for some fixed (large) constant Ω, $n < \Omega k$. This enables us to apply the Szemerédi Regularity Lemma to our graph G_n.

Then comes the question, how can one handle the sparse case: this is the more involved part of the proof.

In the second part I will describe a strongly related topic with very similar results, having a long history:

Conjecture 2 (Loebl for $k = n$, Komlós-Sós for arbitrary $n \geq k$)**.** *Every graph of median degree at least $k - 1$ contains all trees with $k - 1$ edges as subgraphs.*

Theorem 2. *There exist a k_0 such that if $k > k_0$, T_k is an arbitrary k-vertex tree, $n \geq k$, G_n is an n-vertex graph with at least $n/2$ vertices of degree at least $(k - 1)$, then G_n contains T_k.*

The proofs are rather long and will be published in several papers. The lecture will be self-contained, however, will describe only the history, motivation, and the main lines of the proof and the background.