Energy and wealth redistribution models

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based on

1. Carinci, G.; Giardinà, C.; Giberti, C.; Redig, F. Duality for stochastic models of transport. J. Stat. Phys. 152 (2013), no. 4, 657-697.
2. Cirillo, P.; Redig, F.; Ruszel, W. Duality and stationary distributions of wealth distribution models. J. Phys. A 47, 085203, (2014)
3. Redig, F., Ruszel. W., Covariances in non-equilibrium models of transport. in preparation.

Wright Fisher diffusion as a transport model

- Brownian energy process with parameter $2k$:

$$L_{12}^{BEP(2k)} = x_1 x_2 (\partial_{x_1} - \partial_{x_2})^2 - 2k(x_1 - x_2)(\partial_{x_1} - \partial_{x_2})$$

  conserves $x_1 + x_2$ (total energy, population size).

- Invariant product measure

$$\mu(dx_1 dx_2) = \prod_{i=1}^{2} \frac{x_i^{2k-1} e^{-x_i/\theta}}{\theta^{2k}\Gamma(2k)}$$

  i.e., product of $\Gamma(2k, \theta)$ distributions.

- With this basic model in two variables, one can build a transport model

$$L = \sum_{i,j \in S} p(i,j) L_{i,j} + \text{boundary generators}$$

- E.g. linear chain with boundary reservoirs on left and right ends

$$\sum_{i=1}^{N} L_{i,i+1} + \mathscr{L}_0 + \mathscr{L}_N$$

Thermalization

- Is a procedure to go from the diffusion processes $BEP(2k)$ to energy redistribution models of the type KMP (Kipnis, Marchioro, Presutti, 1982) , where energy is redistributed at Poisson event times.
- If X_1, X_2 has distribution

$$\mu(dx_1 dx_2) = \prod_{i=1}^{2} \frac{x_i^{2k-1} e^{-x_i/\theta}}{\theta^{2k}\Gamma(2k)}$$

then, given $X_1 + X_2 = E$, $X_1 = E\epsilon$ where ϵ is $\nu_{2k} := Beta(2k, 2k)$ distributed

$$\nu_{2k}(dx) = \frac{x^{2k-1}(1-x)^{2k-1}dx}{\int_0^1 x^{2k-1}(1-x)^{2k-1}dx}$$

- Therefore, by ergodicity of the diffusion with generator L_{12}

$$\lim_{t\to\infty} e^{tL_{12}} f(x_1, x_2) = \int f(\epsilon(x_1 + x_2), (1-\epsilon)(x_1 + x_2))\nu_{2k}(d\epsilon)$$

- The energy redistribution model $KMP(2k)$ is then defined by its generator

$$L_{12}^{KMP(2k)} f(x_1, x_2) = \lim_{t\to\infty} (\mathbb{E}_{x_1,x_2}^{BEP(2k)} f(x_1(t), x_2(t))) - f(x_1, x_2)$$
$$= \int_0^1 f(\epsilon(x_1 + x_2), (1-\epsilon)(x_1 + x_2))\nu_{2k}(d\epsilon) - f(x_1, x_2)$$

- I.e., on Poisson event times, x_1, x_2 is replaced by a sample of its unique stationary distribution under the process with generator L_{12} starting from (x_1, x_2).

Recovering KMP

- Put $2k = 1$, then $L_{12}^{BEP(2k)}$ has products of exponentials as invariant measure. Therefore, given $X + Y = E$, $X = \epsilon E$ with ϵ uniformly distributed. Therefore, the thermalized model has generator

$$L^{KMP(1)} f(x_1, x_2) = \int_0^1 f(\epsilon(x_1+x_2), (1-\epsilon)(x_1+x_2)) d\epsilon - f(x_1, x_2)$$

which gives exactly the original model of Kipnis Marchioro Presutti.

Duality $BEP(2k)$ with $SIP(2k)$

[Giardinà, Kurchan, R., Vafayi, JSP, 2009]
Define, for $\xi_1, \xi_2 \in \mathbb{N}$ $x_1, x_2 \geq 0$

$$\mathscr{D}(\xi_1, \xi_2; x_1, x_2) = \prod_{i=1}^{2} d(\xi_i, x_i)$$

with

$$d(n, x) = \frac{x^n \Gamma(2k)}{\Gamma(2k+n)}$$

a polynomial naturally associated to the $\Gamma(2k, \theta)$ distribution via

$$\mathbb{E}_{\Gamma(2k,\theta)}(d(n, X)) = \theta^n$$

- Define the process $SIP(2k)$ (symmetric inclusion process with parameter $2k$) via its generator

$$\begin{aligned} L^{SIP(2k)} f(\xi_1, \xi_2) &= \xi_1(2k + \xi_2)(f(\xi^{1,2}) - f(\xi)) \\ &+ \xi_2(2k + \xi_1)(f(\xi^{2,1}) - f(\xi)) \end{aligned}$$

- Then we have the duality relation

$$\mathbb{E}^{BEP(2k)} \mathscr{D}(\xi_1, \xi_2; x_1(t), x_2(t)) = \mathbb{E}^{SIP(2k)} \mathscr{D}(\xi_1(t), \xi_2(t); x_1, x_2)$$

- As a consequence between the thermalized models

$$\mathbb{E}^{KMP(2k)} \mathscr{D}(\xi_1, \xi_2; x_1(t), x_2(t)) = \mathbb{E}^{TSIP(2k)} \mathscr{D}(\xi_1(t), \xi_2(t); x_1, x_2)$$

where the $TSIP(2k)$ is the thermalized $SIP(2k)$

- $TSIP(2k)$ has generator

$$L^{TSIP(2k)} f(\xi_1, \xi_2) = \sum_{r=0}^{\eta_1+\eta_2} \nu_{2k}(r|\eta_1+\eta_2)(f(r, \eta_1+\eta_2-r) - f(\eta_1, \eta_2))$$

with

$$\nu_{2k}(r|\eta_1 + \eta_2) = \frac{\binom{2k+r-1}{r}\binom{2k+\eta_1+\eta_2-r-1}{\eta_1+\eta_2-r}}{\binom{4k+\eta_2+\eta_1-1}{\eta_1+\eta_2}}$$

i.e., on Poisson event times, η_1, η_2 is redistributed according to a negative hypergeometric distribution.

- Special case $2k = 1$,

$$\nu_1(r|\eta_1 + \eta_2) = \frac{1}{\eta_1 + \eta_2 + 1}$$

i.e., starting from (η_1, η_2), at Poisson event times we have discrete uniform redistribution among the (r, m) with $(r + m) = \eta_1 + \eta_2$. This is exactly the dual of the original KMP process.

Self-duality of the $SIP(2k)$

- Besides being the dual of $BEP(2k)$, the $SIP(2k)$ is also self-dual with self-duality function

$$\mathscr{D}(\xi_1, \xi_2; \eta_1, \eta_2) = \prod_{i=1}^{2} \frac{\eta_i!}{(\eta_i - \xi_i)!} \frac{\Gamma(2k)}{\Gamma(2k + \xi_i)}$$

  i.e.,

$$\mathbb{E}_{\eta}^{SIP(2k)} \mathscr{D}(\xi_1, \xi_2; \eta_1(t), \eta_2(t)) = \mathbb{E}_{\xi}^{SIP(2k)} \mathscr{D}(\xi_1(t), \xi_2(t); \eta_1, \eta_2)$$

- As a consequence, the thermalized discrete process, $TSIP(2k)$ is also self-dual with the same duality functions. This self-duality function comes from a symmetry, i.e., a non-trivial operator commuting with the generator

Generalized energy redistribution model

Is process with generator

$$\begin{aligned} & L_{12} f(x_1, x_2) \\ = & \int_0^1 f(\epsilon(x_1 + x_2), (1-\epsilon)(x_1 + x_2))\nu(\epsilon) d\epsilon - f(x_1, x_2) \end{aligned}$$

with ν the redistribution density, of which we assume

1. Full support, C^1: $\nu(\epsilon) > 0$.
2. Symmetric around $\epsilon = 1/2$: $\nu(\epsilon) = \nu(1-\epsilon)$.

We call this model: energy redistribution model with redistribution measure ν.

Boundary reservoir setting

With this model we can build again a non-equilibrium chain with generator

$$\mathscr{L}_0 + \mathscr{L}_N + \sum_{i=1}^{N} L_{i,i+1}$$

with boundary generators modelling reservoirs, exchanging with a left, right "ghost" site $0, N+1$, with distribution μ_0, μ_{N+1}

$$\mathscr{L}_0 = \gamma_0 \left(\int f((1-\epsilon)(x_0 + x_1), x_2, \ldots, x_N) \nu(d\epsilon) \mu_0(dx_0) - f(x) \right),$$

$$\mathscr{L}_N = \gamma_{N+1} \left(\int f(x_1, \ldots, x_{N-1}, \epsilon(x_N + x_{N+1})) \nu(d\epsilon) \mu_{N+1}(dx_{N+1}) - f(x) \right)$$

In this talk $\gamma_0 = \gamma_{N+1} = 1$. For appropriate choices of $\mu_0 \neq \mu_{N+1}$, this process has a a unique stationary measure which is called the non-equilibrium stationary state $\pi^{\mu_0, \mu_{N+1}, \nu}$.

Often coinciding facts

1. Product invariant measures in equilibrium.
2. Existence of a discrete dual process.
3. Factorization of the duality functions.
4. Linear energy profile in non-equilibrium stationary state.
5. Multilinear covariances in non-equilibrium stationary state

Connections, implications ??

Product measures for energy redistribution model

- The energy redistribution model has product invariant measures if and only if ν is a $Beta(2k, 2k)$ distribution, in which case it is exactly the $KMP(2k)$ model.
- *Remark: a different proof of a more general result is given by Grigo, Khanin and Ssazs in the context of billiard models.*

A sketch of the (simple) proof.

- If a product measure with marginals μ exists then the redistribution measure ν and the marginal μ are related via

$$\nu(a) = \frac{\mu(as)\mu((1-a)s)}{\int_0^1 \mu(as)\mu((1-a)s)da}$$

  Put $\psi(s) = \int_0^1 \mu(as)\mu((1-a)s)da$.

- Using symmetry and full support property of ν we can write $\nu(a) = K(a)K(1-a)$ we obtain

$$\frac{\mu(as)\mu((1-a)s)}{K(a)K(1-a)} = \psi(s)$$

- Putting $J(a,s) = \mu(as)/K(a)$ we find

$$\frac{\partial_a J(a,s)}{J(a,s)} = \frac{\partial_a J(1-a,s)}{J(1-a,s)}$$

which leads to

$$s\frac{\mu'(as)}{\mu(as)} - \frac{K'(a)}{K(a)} = s\frac{\mu'((1-a)s}{\mu((1-a))s} + \frac{K'(1-a)}{K(1-a)}$$

- Putting $a = 0$ now gives

$$\frac{\mu'(s)}{\mu(s)} = C_1 + \frac{C_2}{s}$$

which leads to

$$\mu(s) = s^c e^{-c's}$$

- In the general case we have the following characterization of all invariant measures: all are of the form "distribution of $(\epsilon S, (1-\epsilon)S)$" where ϵ is distributed according to ν and where S is non-negative random variable.
- Only in the case that ϵ is $Beta(2k, 2k)$ distributed, one can choose S such that the pair $(\epsilon S, (1-\epsilon)S)$ is i.i.d. Namely, choose S independent $\Gamma(4k, \theta)$.

Discrete duals for energy redistribution model

- The energy redistribution model always has a discrete dual process.
- The duality functions factorize, i.e. $\mathscr{D}(\xi_1, \xi_2; x_1, x_2) = d(\xi_1, x_1) d(\xi_2, x_2)$ if and only if the redistribution measure is $Beta(2k, 2k)$.

Discrete dual

- Fill in $f_{n,m}(x_1, x_2) = x_1^n x_2^m$ in the generator. We find

$$\begin{aligned} Lf_{nm} &= \sum_{k=0}^{n+m} \binom{n+m}{k} \nu_{n,m} f_{k,n+m-k} - f_{n,m} \\ &= \sum_{r,s:r+s=n+m} a_{nm;rs} f_{rs} \end{aligned}$$

  with $\nu_{nm} = \int_0^1 \epsilon^n (1-\epsilon)^m \nu(d\epsilon)$.
- We see that $a_{nm;rs} \geq 0$ for $(n,m) \neq (r,s)$ and $a_{nm;nm} \leq 0$. As a consequence, by a general result of Barbour, Ethier, Griffiths: if for an invariant measure μ one defines $c_{nm} = \int x^n y^m \mu(dxdy)$

$$D(n, m; x_1, x_2) = \frac{x_1^n x_2^m}{c_{nm}}$$

  then the discrete dual process has transition rates

$$q_{nm;rs} = a_{nm;rs} \frac{c_{rs}}{c_{nm}}$$

Coming back to a product measure case:

$$\nu(d\epsilon) = d\epsilon$$

$$\nu_{n,m} = \frac{n!m!}{(n+m+1)!}$$

$$\mu(dxdy) = e^{-x}e^{-y}$$

$$c_{nm} = n!m!$$

$$q_{nm;rs} = \delta_{n+m,r+s}\frac{1}{n+m+1}$$

which gives us once more the dual KMP process.

Profiles and covariances

Consider now the chain with left and right reservoirs we put

$$T_L = \int x d\mu_0, T_R = \int x d\mu_{N+1}$$

$$\alpha = \int \epsilon(1-\epsilon)\nu(d\epsilon) < 1/2$$

$$L^2 = \int x^2 d\mu_0, R^2 = \int x^2 d\mu_{N+1}$$

We then define

- The energy profile:

$$\mu_i = \int x_i \; \pi^{\mu_0, \mu_{N+1}, \nu}(dx)$$

- The energy covariance

$$C_{ij} = \int x_i x_j \; \pi^{\mu_0, \mu_{N+1}, \nu}(dx) - \mu_i \mu_j$$

- The energy profile is *always linear*

$$\mu_i = T_L \left(1 - \frac{i}{N+1}\right) + T_R \frac{i}{N+1}, i = 1, \ldots, N$$

  This is related to the fact that μ_i solves a discrete Laplace equation.

- The discrete dual process with a single particle is always a symmetric random walk, i.e.,

$$\mathbb{E}_x x_i(t) = \widehat{\mathbb{E}}_i x_{\mathscr{X}(t)}$$

  where $\mathscr{X}(t)$ is continuous-time nearest neighbor symmetric random walk

- The covariance is multilinear, i.e., of the form $a + bi + cj + dij$ if and only if

$$L^2 = \int x^2 d\mu_0 = \frac{1-2\alpha}{2\alpha} T_L^2 + \frac{\alpha(1-2\alpha)(T_L - T_R)^2}{(N+1)(1+2\alpha(N-1))2\alpha}$$

  and similarly for R^2
- Example: if $\nu(d\epsilon) = d\epsilon$, i.e., KMP model, then $\alpha = \int_0^1 \epsilon(1-\epsilon)d\epsilon = 1/6$ so we must have

$$L^2 = \int x^2 d\mu_0 = 2T_L^2 + O(1/N^2)$$

  which corresponds (up to the correction term) to an exponential distribution for μ_L which is exactly the stationary distribution of the KMP model. This corresponds to a case computed before by Bertini et al. (2007).

Wealth distribution

- Define

$$T_{\lambda,\epsilon}(x_1, x_2) = (\lambda x_1 + (1-\lambda)\epsilon(x_1+x_2), \lambda x_2 + (1-\lambda)(1-\epsilon)(x_1+x_2))$$

- Here $0 \leq \lambda \leq 1$ is called the propensity, and ϵ has distribution $\nu(d\epsilon)$ with the same conditions as the energy distribution model. For $\lambda = 0$, we recover the energy distribution model, for $\lambda = 1$ nothing happens. We will consider now $\lambda \in (0, 1)$.
- The generator of the wealth distribution model with propensity λ and redistribution measure ν is then defined as

$$L^{wealth} f(x_1, x_2) = \int f(T_{\lambda,\epsilon}(x_1, x_2))\nu(d\epsilon) - f(x_1, x_2)$$

Invariant measures

The invariant measures are of the form "distribution of $(\epsilon^\lambda S, (1-\epsilon^\lambda)S)$" where S is a non-negative random variable, and

$$\epsilon^\lambda = (1-\lambda)\sum_{n=0}^{\infty} \lambda^n \epsilon_n$$

where ϵ_n are i.i.d. copies of the redistribution measure. In particular, there is no product invariant measure.

To see this, we look at at $r = x_1/(x_1 + x_2)$, this is a Markov process with generator

$$Af(r) = \int (f(\lambda r + (1-\lambda)\epsilon) - f(r))\nu(d\epsilon)$$

from that we can easily deduce that the fixed point of

$$r_{n+1} = \lambda r_i + (1-\lambda)\epsilon_i$$

is invariant. This gives the distribution of ϵ^λ for the stationary distribution of this process.

Additional results for the wealth distribution model

1. Linearity: For a generator of the form

$$\sum_{ij} L_{ij}^{wealth} p(ij)$$

expected wealth spreads as a random walk, i.e.,

$$\mathbb{E}_x x_i(t) = \sum_j p_{(1-\lambda)t}(i,j) x_j(0)$$

where p_t is the continuous-time, rate-one random walk associated to $p(i,j)$.

2. The wealth distribution model has a discrete dual, but the duality functions do not factorize.
3. The wealth distribution model on a linear chain coupled to reservoirs has a linear profile. When a more complicated relation between $T_L, L,\ T_R, R$ involving both α, λ is satisfied, then also the wealth distribution model also has multilinear covariances.

Some words on asymmetric models

Starting again with the generator of $SIP(2k)$

$$\begin{aligned} L^{SIP(2k)} f(\xi_1, \xi_2) &= \xi_1(2k + \xi_2)(f(\xi^{1,2}) - f(\xi)) \\ &+ \xi_2(2k + \xi_1)(f(\xi^{2,1}) - f(\xi)) \end{aligned}$$

one might think that a good choice for the $ASIP(2k)$ is

$$\begin{aligned} L^{ASIP(2k)} f(\xi_1, \xi_2) &= p\xi_1(2k + \xi_2)(f(\xi^{1,2}) - f(\xi)) \\ &+ q\xi_2(2k + \xi_1)(f(\xi^{2,1}) - f(\xi)) \end{aligned}$$

with $p > q$. This has indeed the same invariant measures, but has no self-duality properties.

A better choice is based on a general procedure that constructively yields processes with non-trivial symmetries. Symmetries are related to self-duality functions as follows. If D is a self-duality function, i.e., if

$$L_1 D = L_2 D$$

where L_1 (resp. L_2) is L acting on the first (resp. second) variable and S is commuting with L, then also

$$L_1(S_1 D) = S_1 L_1 D = S_1 L_2 D = L_2(S_1 D)$$

so $S_1 D$ is also a self-duality function. I.e., the more commuting operators (symmetries), the more self-duality functions.

How to construct generators with many symmetries

[Koelink, Groenevelt]

- Start from a representation of a Lie algebra $\mathscr{A}$.
- Start from the Casimir element C, i.e., a non-trivial element commuting with all the elements of $\mathscr{A}$.
- Apply a co-product to it, i.e., a algebra homomorphism from

$$\Delta : \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$$

  in order to produce an operator acting on two variables. This operator will then automatically commute with all elements of the form $\Delta(X)$, $X \in \mathscr{A}$ (because $[\Delta(X), \Delta(C)] = \Delta([X, C]) = \Delta(0) = 0$).

This procedure applied to $SU_q(1,1)$ yields the following new asymmetric inclusion process $ASIP(q,2k)$

$$\begin{aligned} & L_{12}^{ASIP(q,2k)} f(\eta_1,\eta_2) \\ = \quad & q^{\eta_1-\eta_2+(2k-1)}[\eta_1]_q[2k+\eta_2]_q(f(\eta_1-1,\eta_2+1)-f(\eta_1,\eta_2)) \\ + \quad & q^{\eta_1-\eta_2-(2k-1)}[\eta_2]_q[2k+\eta_1]_q(f(\eta_1+1,\eta_2-1)-f(\eta_1,\eta_2)) \end{aligned}$$

with $0 < q < 1$ and with $[k]_q = (q^k - q^{-k})/(q - q^{-1})$ the k-th q number. This process is self-dual (with somewhat complicated non-local self-duality function) and reduces for $q = 1$ to the $SIP(2k)$.

How the general procedure leads to the $SIP(2k)$

- Starting from $\mathscr{A} = SU(1,1)$, a well known discrete left representation is given by

$$\begin{aligned} K^+ f(n) &= \left(\frac{m}{2} + n\right) f(n+1) \,, \\ K^- f(n) &= n f(n-1) \,, \\ K^0 f(n) &= \left(\frac{m}{4} + n\right) f(n) \,. \end{aligned} \tag{1}$$

These satisfy the "dual" $SU(1,1)$ commutation relations

$$[\mathbf{K^o}, \mathbf{K}^\pm] = -(\pm)\mathbf{K}^\pm \,, \qquad [\mathbf{K}^-, \mathbf{K}^+] = -\mathbf{2K^o} \,, \tag{2}$$

- The Casimir element is given by

$$C = K^+K^- - K^o(K^o - 1)$$

- The coproduct is given by

$$X \to X \otimes I + I \otimes X = X_1 + X_2$$

for $X \in \{K^\pm, K^0\}$

- Applying the coproduct to the Casimir element gives (up to an additive constant) exactly the generator of the $SIP(2k)$.

$$L^{SIP(2k)} = \frac{1}{2}\left(\mathbf{K_1^+K_2^-} + \mathbf{K_1^-K_2^+} - \mathbf{2K_1^oK_2^o} + \frac{\mathbf{k^2}}{\mathbf{2}}\right), \quad (3)$$

- One then knows automatically that the generator commutes with all elements of the form

$$\Delta(X), X \in \mathscr{A}$$

in particular with $K_1^\pm + K_2^\pm$, $K_1^0 + K_2^0$.

- The self-duality function is produced exactly from one of these commuting elements $K_1^+ + K_2^+$.

Deformation of $SU(1,1)$

[Carinci, Giardinà, R. , Sasamoto]

- $SU_q(1,1)$ is the q-deformation of $SU(1,1)$ generated by K^+, K^-, K^0 with commutation relations

$$[K^+, K^-] = -[2K^0]_q, \ [K^0, K^\pm] = \pm K^\pm$$

with

$$[2K^0]_q = \frac{q^{2K^0} - q^{-2K^0}}{q - q^{-1}}$$

- The Casimir element is given by

$$[K^0]_q[(K^0 - 1)]_q - K^+K^-$$

- The coproduct is given by

$$\Delta(K^{\pm}) = K_1^{\pm} \otimes q^{-K_2^0} + q^{K_1^0} \otimes K_2^{\pm}$$

$$\Delta(K^0) = K_1^0 \otimes I_2 + I_1 \otimes K_2^0$$

- The coproduct applied to the Casimir gives, after a suitable ground-state transformation exactly the generator of the $ASIP(q, 2k)$

The process with generator

$$\sum_{i=1}^{N} L_{i,i+1}^{ASIP(q,2k)}$$

is then self-dual with self-duality function

$$\mathscr{D}(\xi,\eta) = \prod_{i=1}^{N} \frac{\binom{\eta_i}{\xi_i}}{\binom{\xi_i+2k-1}{\xi_i}} q^{(\eta_i-\xi_i)(2\sum_{m=1}^{i-1}\xi_m+\xi_i)-4ki\xi_i} 1_{\xi_i\leq\eta_i}$$

In particular, when there is only one dual particle $\xi = \delta_i$ then

$$\mathscr{D}(\xi, \eta) = \frac{q^{-4ki+1}}{q^{2k} - q^{-2k}}(q^{2N_i(\eta)} - q^{2N_{i+1}(\eta)}$$

and so the duality relation applied to this gives that

$$\mathbb{E}_{\eta}^{ASIP(q,2k)} \mathscr{D}(\xi, \eta_t) = \widehat{\mathbb{E}}_i \frac{q^{-4kX_t+1}}{q^{2k} - q^{-2k}}(q^{2N_{X_t}(\eta)} - q^{2N_{X_{t+1}}(\eta)}$$

with X_t a single continuous-time random walk making jumps to the right with rate $q^{2k}2k$ and to the left with rate $q^{-2k}2k$.

Thanks for your attention !