Decomposition of Wasserstein Geodesics

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WASSERSTEIN GEODESICS

Basic Definitions : - (X, d) complete and separable metric space

- $P_2(X) = \{ \mu \in \mathcal{P}(X) : \int d^2(o,x)\, \mu(dx) < \infty \}$

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L^2-Wasserstein transportation distance: for $\mu_0, \mu_1 \in \mathcal{P}_2(X)$

$$W_2^2(\mu_0,\mu_1) := \inf \left\{ \int d^2(x,y)\, \pi(dx\,dy) : \pi \in \mathcal{P}(X \times X),\ P_1 \# \pi = \mu_0,\ P_2 \# \pi = \mu_1 \right\}.$$

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$\mathrm{Geo}(X)$ space of geodesic on (X,d), for $t \in [0;1]$ $e_t : \mathrm{Geo}(X) \longrightarrow X$, $e_t(\gamma) = \gamma_t$,

$(\mu_t)_{t\in[0;1]} \in \mathrm{Geo}(\mathcal{P}_2(X)) \rightsquigarrow \exists\, \nu \in \mathcal{P}(\mathrm{Geo}(X)) : (e_t)\# \nu = \mu_t$, for all $t \in [0;1]$.

$\rightsquigarrow$ Geodesics of (X,d) and of $(\mathcal{P}_2(X), W_2)$ are strongly linked.

Optimal Geodesics

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$$T_t(x) := \exp_x(-t \cdot \nabla \varphi(x)), \qquad \mu_t := T_{t\,\#}\mu_0 .$$

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For $t\in[0,1)$, $\mu_t \ll m$ with $\varrho_t m = \mu_t$. By change of variable formula

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Fact: If $\mathrm{Ric}_g \geq K$ and $\dim M \leq N$, denote with $J_t(x) = \det(dT_t(x))$ then

$$\frac{d^2}{dt^2} J_t^{1/N} \leq -\frac{K\theta^2}{N} \cdot J_t^{1/N}, \quad \text{with } \theta(x) = d(x, T(x))$$

$$\Rightarrow J_t^{1/N}(x) \geq \sigma_{K,N}^{(1-t)}(\theta) J_0(x)^{1/N} + \sigma_{K,N}^{(1-t)}(\theta) \cdot J_1(x)^{1/N}, \quad \text{where } \sigma_{K,N}^{(t)}(\theta) = \frac{\sin\left(t \cdot \theta \sqrt{\frac{K}{N}}\right)}{\sin\left(\theta \cdot \sqrt{\frac{K}{N}}\right)}$$

OPTIMAL GEODESICS

Improved Inequality: Curvature only in N-1 directions , $J_t(x) = A_t(x) \cdot L_t(x)$

$$\frac{d^2}{dt^2} A_t^{\frac{1}{N-1}} \leq -\frac{k\theta^2}{N-1} A_t^{\frac{1}{N-1}}, \qquad L_t(x) \geq (1-t)L_0(x) + t\,L_1(x).$$

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Denote $\tau_{K,N}^{(t)}(\theta) = t^{\frac{1}{N}} \cdot \sigma_{K,N-1}^{(t)}(\theta)^{\frac{N-1}{N}}$, then using Hölder inequality

$$J_t(x)^{\frac{1}{N}} \geq \tau_{k,N}^{(1-t)}(d(x,T(x))) \cdot J_0(x)^{\frac{1}{N}} + \tau_{k,N}^{(t)}(d(x,T(x))) \cdot J_1(x)^{\frac{1}{N}} \qquad (*)$$

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Key Fact: From change of variable $J_t(x) = \rho_0(x) \cdot \rho_t(T_t(x))^{-1}$, therefore formulation in terms of densities of geodesics in $(P_2(M), W_2)$.

Consequence: Formulation of "$Ric \geq k$ & $\dim \leq N$" with no smooth structures.

CURVATURE-DIMENSION CONDITION

(X, d, m) is non-branching if: for $\gamma^1, \gamma^2 \in \mathrm{Geo}(X)$, s.t. $\gamma^1_0 = \gamma^2_0$, $\gamma^1_1 = \gamma^2_1$
if $\exists\, s \in (0,1)$ s.t. $\gamma^1_s = \gamma^2_s$, then $\gamma^1 = \gamma^2$.

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<u>Definition</u> (X, d, m) non-branching satisfies $CD(K,N)$, $K, N \in \mathbb{R}$, $N \geq 2$, iff $\forall \mu_0, \mu_1 \in \mathcal{P}_2(X)$, $\mu_0, \mu_1 \ll m$, there exists $\nu \in \mathcal{P}(\mathrm{Geo}(X))$,

- $(e_t \# \nu)_{t \in [0,1]} = (\mu_t)_{t \in [0,1]} \in \mathrm{Geo}(\mathcal{P}_2(X))$, $\mu_t = e_t\, m$;
- $e_t(\gamma_t)^{-\frac{1}{N}} \geq e_0(\gamma_0)^{-\frac{1}{N}} \tau^{(1-t)}_{K,N}(d(\gamma_0, \gamma_1)) + e_1(\gamma_1)^{-\frac{1}{N}} \cdot \tau^{(t)}_{K,N}(d(\gamma_0, \gamma_1))$, $t \in [0,1]$, ν-a.e. γ.

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- $(e_{t\#}\nu)_{t\in[0,1]}=(\mu_t)_{t\in[0,1]}\in Geo(\mathcal{P}_2(X))$, $\mu_t=\varrho_t\, m$;
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If $(X,d,m)\in CD(K,N)$, many geometric and analytical properties can be proved to hold.

DECOMPOSITION OF GEODESICS

Objective: given $(\mu_t) \in \mathrm{Geo}(\mathcal{P}_2(X))$, $\mu_t = \rho_t \cdot m$, decompose ρ_t

$$\rho_t = l_t \cdot h_t, \quad l_t^{-1} \text{ concave}, \quad \frac{d^2}{dt^2} h_t^{-\frac{1}{N-1}} \le -\frac{K\theta^2}{N-1} h_t^{-\frac{1}{N-1}}.$$

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First Application / Motivation: Globalization property for $CD(K,N)$.

Let $\{U_i\}_{i\in\mathbb{N}}$ be an open covering of X so that $CD(K,N)$ is satisfied on U_i, that is: $\forall \mu_0, \mu_1 \in \mathcal{P}_2(U_i, d, m)$ $\exists$ $\mu_t = \varrho_t m \in \mathcal{P}_2(X, d, m)$ geodesic st

$$\varrho_t(\gamma_t)^{-\frac{1}{N}} \geq \varrho_0(\gamma_0)^{-\frac{1}{N}} \tau_{K,N}^{(1-t)}(d(\gamma_0,\gamma_1)) + \varrho_1(\gamma_1)^{-\frac{1}{N}} \tau_{K,N}^{(t)}(d(\gamma_0,\gamma_1)),$$

and $\varrho_t m$ not necessarily supported in U_i. $\overset{?}{\Longrightarrow}$ (X,d,m) verifyies $CD(K,N)$

GLOBALIZATION PROBLEM

Why the answer is not clear:

$CD(k,N)$ can be restated as $t \mapsto e_t(\gamma_t)^{-\frac{1}{N}}$ is concave but the combination between $e_0(\gamma_0)^{-\frac{1}{N}}$ and $e_1(\gamma_1)^{-\frac{1}{N}}$ is done with $\tau^{(t)}_{k,N}(\cdot) = \sigma^{(t)}_{k,N-1}(\ )^{\frac{N-1}{N}} \cdot t^{\frac{1}{N}}$
where

$$\frac{d^2}{dt^2}\,\sigma^{(t)}_{k,N-1}(\theta) = -\frac{k}{N-1}\cdot\theta^2\cdot\sigma^{(t)}_{k,N-1}(\theta)$$

$\Longrightarrow$ $t \mapsto \sigma^{(t)}_{k,N-1}(\theta)$ has good "concavity" property. But $t \mapsto \tau^{(t)}_{k,N}(\theta)$ not!

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$\Rightarrow$ $CD^*(k,N)$.

Globalization Problem

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$\Rightarrow$ $CD^*(K,N)$. Less information: $CD^*(K,N) \Rightarrow CD(K^*,N)$ for $K^* = K(N-1)/N$.

Bacher-Sturm '10: CD^* enjoys globalization property (non-branching).

GLOBALIZATION PROBLEM

3. Separate dynamic ruled by $\sigma^{(t)}_{K,N-1}$ from the one ruled by t.
repeat in opposite direction the argument on Riem. mfld where

$$J_t(x) = A_t(x) \cdot L_t(x), \quad \frac{d^2}{dt^2} L_t \geq 0, \quad \frac{d^2}{dt^2} A_t^{\frac{1}{N-1}} \leq -\frac{K}{N-1}\theta^2 \cdot A_t^{\frac{1}{N-1}} \quad (\text{or } CD^*(K,N-1))$$

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Main problem of decomposition: find the sets of codimension one.

Heuristic. The distortion of the set evolving like L_t is not affected by curvature it has to evolve in the same direction of the optimal transport.

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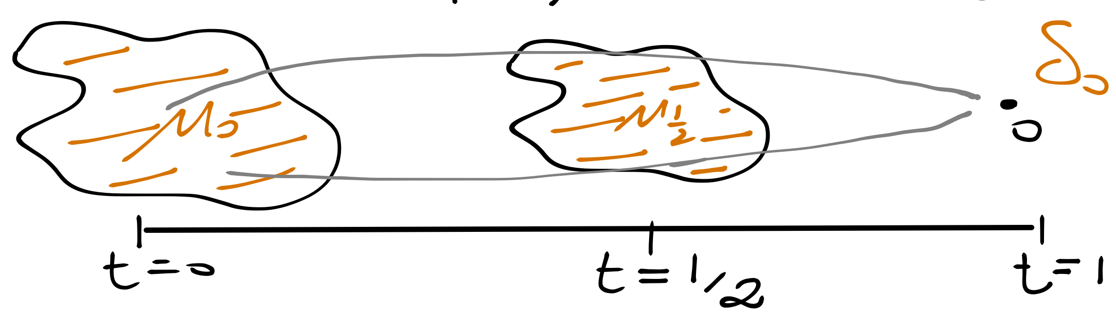
Study Case. Fix $\mu_0 \in \mathcal{P}_2(X,d,m)$ and $\mu_1 = \delta_o$, with $o \in \operatorname{supp}[m]$.
Then $\nu \in \mathcal{P}(\mathrm{Geo}(X))$ so that $e_{0\#}\nu = \mu_0$, $e_{1\#}\nu = \delta_o$, $e_{t\#}\nu = \mu_t$ geodesic.



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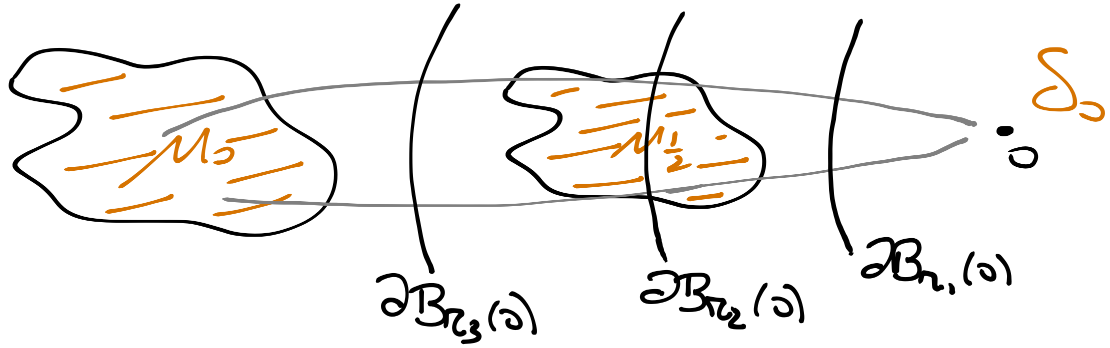

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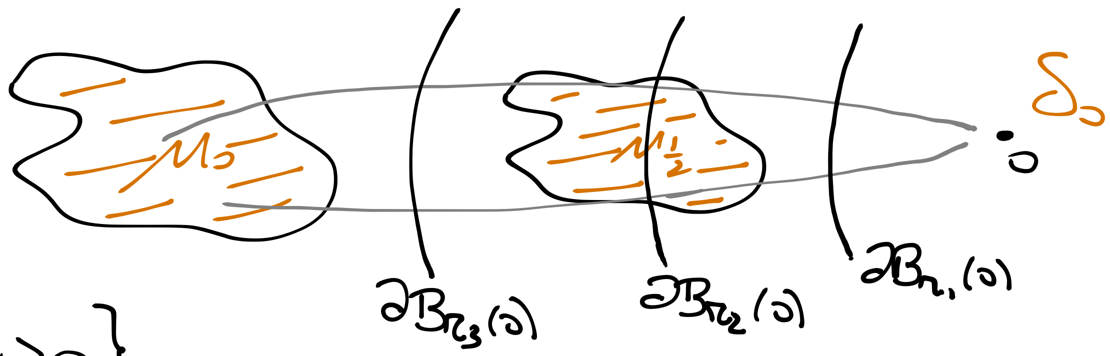

$$\underline{\{ \partial B_r(0) : r > 0 \}}.$$

$$\mu = \int_{(0,\infty)} \mu_r \, dr , \quad \mu_r(\partial B_r(0)) = \|\mu_r\| , \quad v = \int_{(0,\infty)} v_r \, dr .$$

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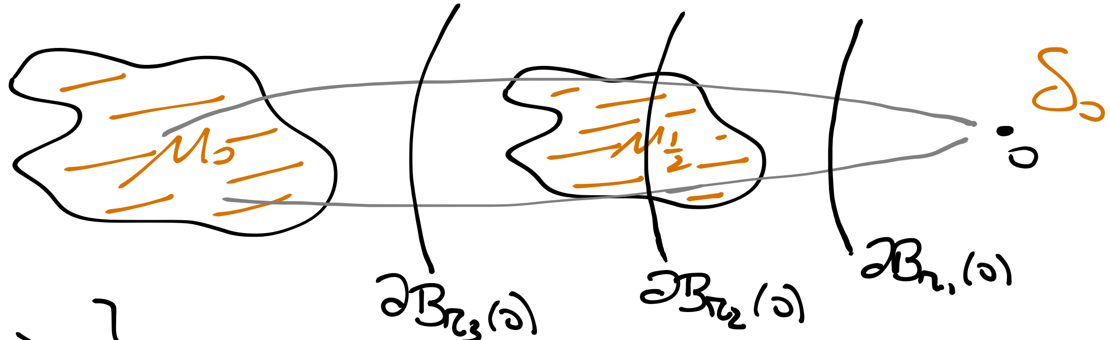
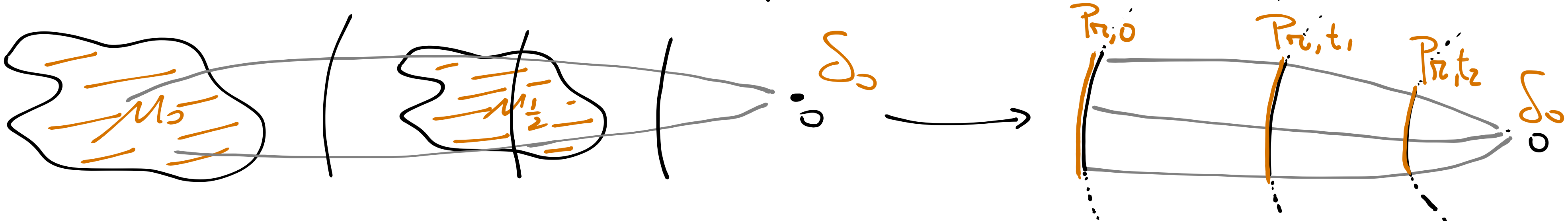


$$\{\partial B_r(o) : r > 0\}$$

$$\mu = \int_{(0;\infty)} \mu_r \, dr, \quad \mu_r(\partial B_r(o)) = \|\mu_r\|, \quad \nu = \int_{(0,\infty)} \nu_r \, dr.$$

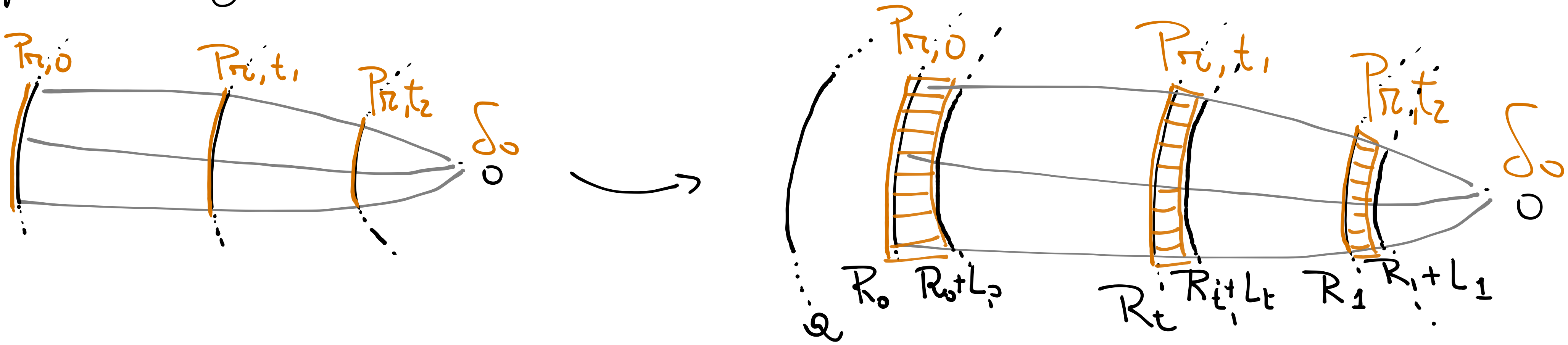
So $[0;1] \ni t \mapsto e_{t\#}(\nu_r) := P_{r,t} \in \mathrm{Geo}(\mathcal{P}_2(X))$ of "codimension one".



Consider its density w.r.t. μ_r: $\boxed{P_{r,t} = h_{r,t} \, \mu_{r(1-t)}}$.

THICKENING

Thickening of $\{P_{R,t}\}_{t\in[0,1]}$: produce W_2-geodesic with "N-dim" support approximating the evolution in codimension 1.



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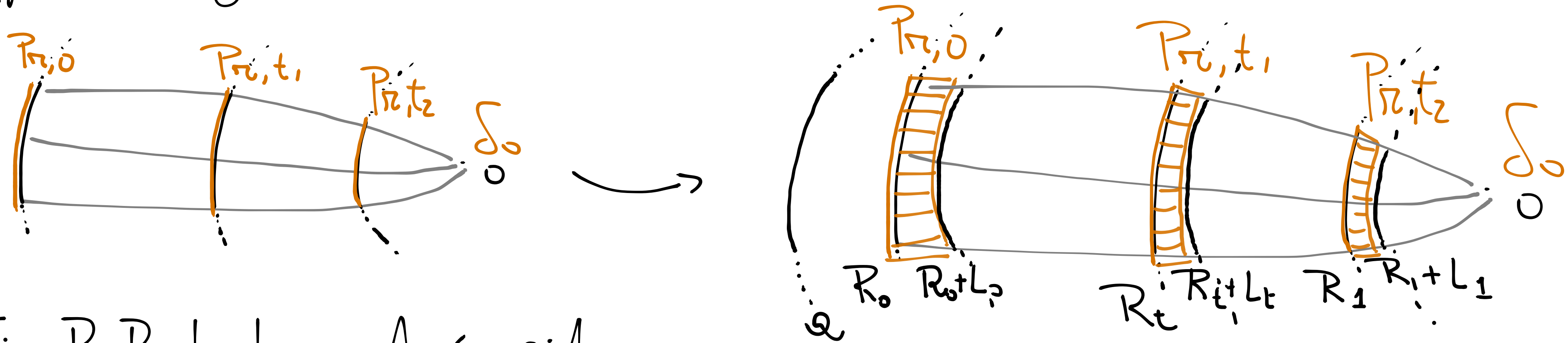

Fix R_0, R_1, L_0, L_1 and consider:

$$[0,1]\ni t\longmapsto \eta_t := \frac{1}{L_t}\cdot\int_{[R_t,R_t+L_t]}\left(e_{\left(1-\frac{r}{a}\right)}\right)_{\#} \nu_a\,\mathcal{L}^1(dr),$$

where $R_t=(1-t)R_0+tR_1$, $L_t=(1-t)L_0+tL_1$. Then η_t is a W_2-geodesic.

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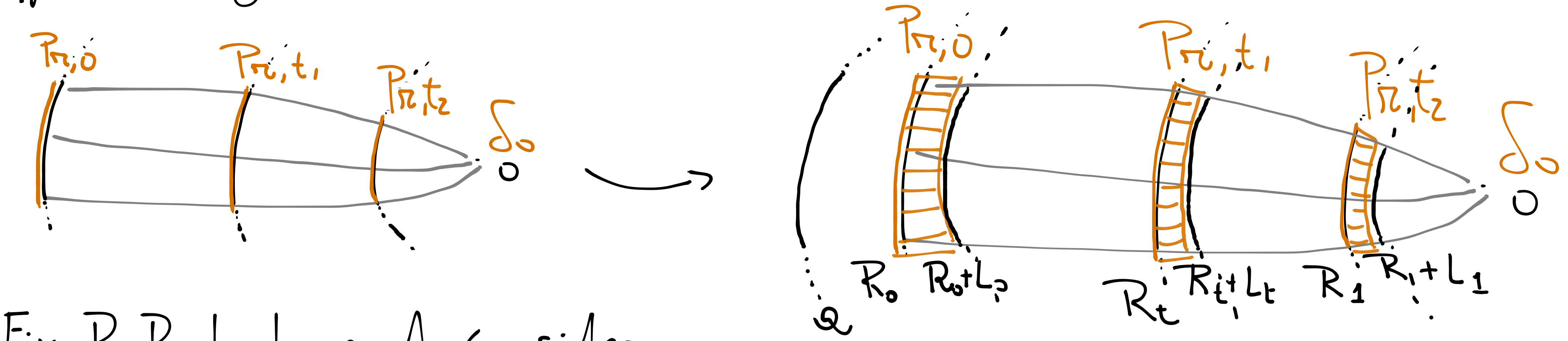

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where $R_t = (1-t)R_0 + tR_1$, $L_t = (1-t)L_0 + tL_1$. Then ν_t is a W_2-geodesic.

Density of ν_t: $\left\| \frac{1}{L_t} h_{a,1-\frac{r}{a}} \right\| \rightsquigarrow \|$ Optimize entropy inequality in $h_{a,r}$ varying L_t $\|$

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We prove that $\{p_{r,t}\}$ verifies $CD^*(k,N-1)$.

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$$(X,d,m) \in CD_{loc}(k,N) \text{ and non-branching,}$$

Consider $\nu \in Geo(\mathcal{P}_2(X))$ so that $e_1\#(\nu) = \delta_o$, $e_t\#(\nu) = e_t \cdot m$, $t \in [0,1)$.
then $e_t = l_t \cdot h_t$, with $h_t \in CD^*(k,N-1)$, l_t^{-1} linear.

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then $e_t = l_t \cdot h_t$, with $h_t \in CD^*(K,N-1)$, l_t^{-1} linear.

In particular we obtain the following:

Theorem (C.-Sturm '11) Let (X,d,m) verify $CD_{loc}(K,N)$ and $\{\mu_t\}_{[0,1]}$ be a geodesic in $\mathcal{P}_2(X,d,m)$ with $\mu_1 = \delta_o$. Then μ_t verifies $CD(K,N)$.
In particular, (X,d,m) verifies $MCP(K,N)$.

Now we consider decomposition for general $\nu \in Geo(\mathcal{P}_2(X))$.

DECOMPOSITION OF GEODESICS

Same idea for a general ν with $e_0\#(\nu), e_1\#(\nu) \ll m$.

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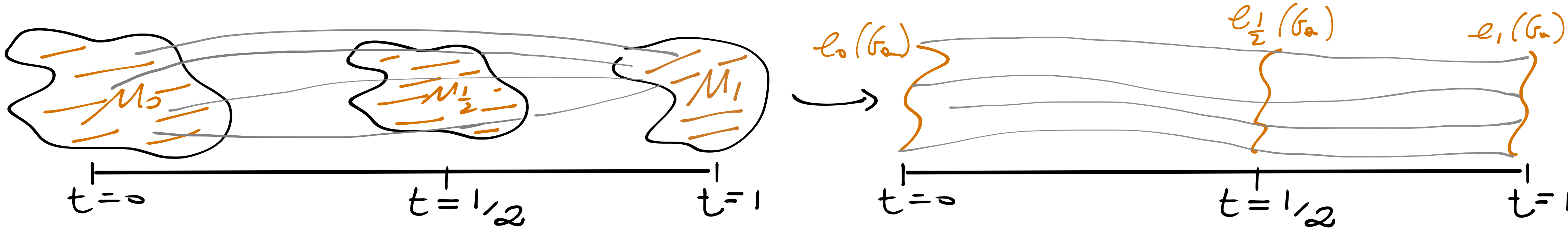
‖At time $t=0$ the orthogonal sets are $\{\varphi = a\}_{a\in\mathbb{R}}$.‖

‖For $t\in(0,1)$ the sets of codim 1 are $\{\gamma_t : \gamma \in \mathrm{spt}(\nu),\ \varphi(\gamma_0) = a\}_{a\in\mathbb{R}}$.‖

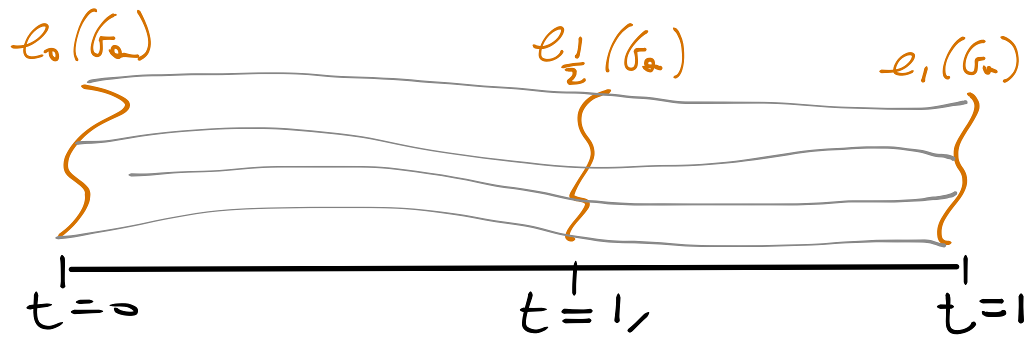
If $G = \mathrm{spt}(\nu)$, $G_a = \{\gamma \in G : \varphi(\gamma_0) = a\}$, then



EVOLUTION IN CODIMENSION 1

Since the family $\{G_a\}_{a\in\mathbb{R}}$ is a partition of G, Disintegration Thm gives:

$$\nu = \int_{\mathbb{R}} \nu_a \, \mathcal{L}^1(da), \qquad \nu_a(G_a) = \|\nu_a\|, \quad t \mapsto e_{t\#}(\nu_a) \text{ geodesic.}$$



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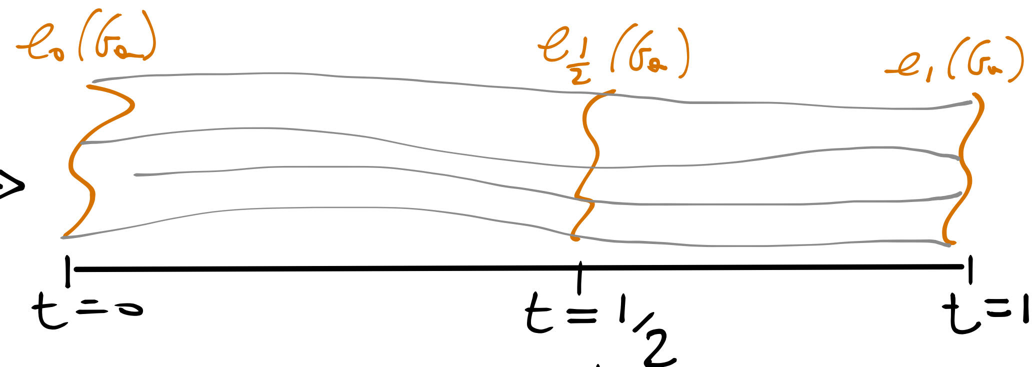

Lemma: The set $(e_0,e_1)(G_a) = \{(\gamma_0,\gamma_1) : \gamma\in G_a\}$ is d-cyclically monotone.

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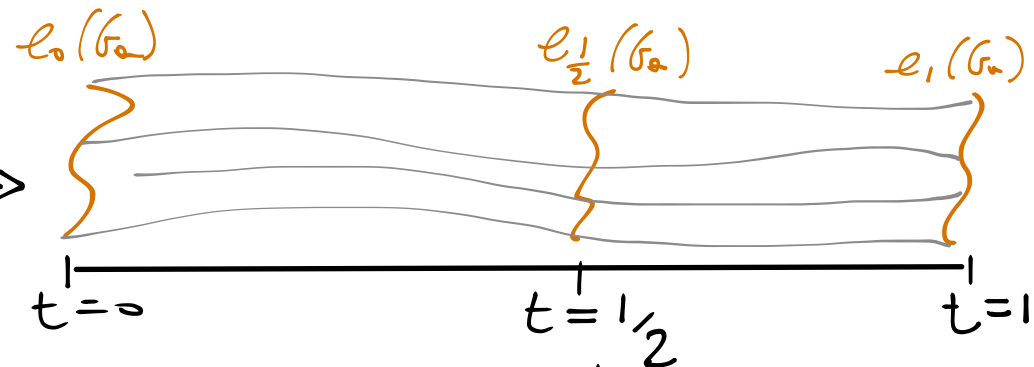

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$$\rightsquigarrow \quad m\llcorner_{e_{[0;1]}(G_a)} = \int m_{a,t} \, \mathcal{L}^1(dt), \qquad m_{a,t}(e_t(G_a)) = \|m_{a,t}\|.$$

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Then it holds: $e_{t\#}(\nu_a) = h_{a,t} \cdot m_{a,t}$. Use "thickening" (as for $\mu_1 = \delta_0$).

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So let $\phi : X \to \mathbb{R}$ 1-Lipschitz, $\Gamma = \{(x,y) : \phi(x) - \phi(y) = d(x,y)\}$

Lemma: Let $\Delta \subset \Gamma$ be any set s.t.

$$(x_0,y_0),(x_1,y_1) \in \Delta \implies (\phi(y_1) - \phi(y_0)) \cdot (\phi(x_1) - \phi(x_0)) \geq 0.$$

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$\rightsquigarrow$ Use curvature information also for d-monotone sets.

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Theorem (C. '13) $(X, d, m) \in CD_{loc}(k, N)$, non-branching, $(\mu_t) \in Geo(\mathcal{P}_2(X))$ with $\mu_t = \varrho_t \cdot m$ so that (A) holds. Then

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- $$\frac{1}{\lambda_t(\gamma_t)} = \lim_{s \to 0^+} \frac{\Phi_t(\gamma_t) - \Phi_t(\gamma_{t+s})}{s}.$$

If (X, d, m) is infinitesimally strictly convex: $1/\lambda_t = D\Phi_t(\nabla \varphi_t)$.

APPLICATIONS OF DECOMPOSITION

Globalization problem.

Consider $\nu \in \mathcal{P}(\mathrm{Geo}(X))$, $(e_t \# \nu) \in \mathrm{Geo}(\mathcal{P}_2(X))$ and there exists $f : \mathbb{R} \to (0; \infty)$:

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Theorem (C. '13) Let $(X,d,m) \in CD_{loc}(K,N)$ be non-branching. Let $\mu_t = e_{t\#}\nu$, be in $\mathrm{Geo}(\mathcal{P}_2(X))$ s.t.

$$L(\gamma) = f(\varphi(\gamma_0)),$$

for some $f : \mathbb{R} \to \mathbb{R}$. Then $t \mapsto \lambda_{a,t}$ is linear. Therefore $t \mapsto \mu_t \in CD(K,N)$.

APPLICATIONS OF DECOMPOSITION

Monge Problem in $RCD^*(K,N)$

Let $(X,d,m) \in RCD^*(K,N)$, consider $\mu_0, \mu_1 \in \mathcal{P}(X)$, with $\mu_0 \ll m$.
Study

$$\min \left\{ \int_X d(x,T(x))\,\mu_0(dx) : T_\sharp(\mu_0) = \mu_1 \right\} \qquad (MP)$$

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$$\Downarrow$$

Use essential non-branching of $Geo(\mathcal{P}_2(X))$
for $(X,d,m) \in RCD^*(K,N)$

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<u>Proposition</u> Neglecting a set of m-measure zero, the transport set of (MP) is formed by a family of disjoint geodesics.

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Disintegration theorems can be used. Moreover RCD^* implies regularity of disintegration.

Applications of Decomposition

Monge Problem in $RCD^*(K,N)$

Proposition Neglecting a set of m- measure zero, the transport set of (MP) is formed by a family of disjoint geodesics.

Disintegration theorems can be used. Moreover RCD^* implies regularity of disintegration.

Theorem (C. '14) $(X,d,m) \in RCD^*(K,N)$. Let $\mu_0, \mu_1 \in \mathcal{P}_1(X)$ with $\mu_0 \ll m$.
There exists $T: X \to X$, $T_\# \mu_0 = \mu_1$ s.t.

$$\int_X d(x,T(x))\,\mu_0(dx) = \min\left\{ \int_{X\times X} d(x,y)\,\pi(dxdy) : \pi \in \Pi(\mu_0,\mu_1) \right\}.$$

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