L^2–perturbation of Markov processes and applications

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The unperturbed Markov process

Ω Polish space

Markov process on Ω with càdlàg paths $(\eta_t)_{t\geq 0}$

$\mathbb{P}_\nu$ law when ν is initial distribution ($\mathbb{E}_\nu$ expectation)

$\mathbb{P}_\eta$, $\mathbb{E}_\eta$ if $\nu = \delta_\eta$

Assumption: $\exists$ **ergodic stationary distribution** μ

$S(t)f(\eta) := \mathbb{E}_\eta\big[f(\eta_t)\big]$.

$(S(t))_{t\geq 0}$ strongly continuous contractive semigroup on $L^2(\mu)$.

Assumption:$(S(t))_{t\geq 0}$ **satisfies Poincaré inequality**

Poincaré inequality

- **Poincaré inequality**: $\exists \gamma > 0$ such that

$$\|S(t)f - \mu(f)\| \leq e^{-\gamma t}\|f - \mu(f)\| \qquad \forall t \geq 0\,,\ f \in L^2(\mu)\,.$$

- $L : \mathcal{D}(L) \subset L^2(\mu) \to L^2(\mu)$ generator of $(S(t))_{t\geq 0}$
  Poincaré inequality: $\exists \gamma > 0$ such that

$$\gamma\|f\|^2 \leq \mu\big(f(-Lf)\big) \qquad \forall f \in \mathcal{D}(L) \text{ with } \mu(f) = 0\,.$$

- Reversible process (L self–adjoint):
  Poincaré inequality $\Leftrightarrow$ Positive spectral gap

Perturbed semigroup

- $\hat{L}_\varepsilon : L^2(\mu) \to L^2(\mu)$ with $\varepsilon := \|\hat{L}_\varepsilon\|$
- Set $L_\varepsilon := L + \hat{L}_\varepsilon$, $\mathcal{D}(L_\varepsilon) := \mathcal{D}(L)$.
- L_ε is the generator of a strongly continuous semigroup $(S_\varepsilon(t))_{t\geq 0}$ on $L^2(\mu)$.

Dyson–Phillips expansion

Defining iteratively

$$S_\varepsilon^{(0)}(t) := S(t)\,, \qquad S_\varepsilon^{(n+1)}(t) := \int_0^t S(t-s)\hat{L}_\varepsilon S_\varepsilon^{(n)}(s)ds\,,$$

$$\boxed{S_\varepsilon(t) = \sum_{n=0}^{\infty} S_\varepsilon^{(n)}(t)\,, \qquad t \geq 0}$$

norm k–th rest is $O((\varepsilon/\gamma)^k)$ for $\varepsilon < \gamma$

Perturbed Markov process

- Consider another Markov process on Ω with càdlàg paths.
- $\mathbb{P}_\nu^{(\varepsilon)}$ its law when initial distribution ν [$\mathbb{E}_\nu^{(\varepsilon)}$ expectation]
- **Assumption**: $S_\varepsilon(t)f(\eta) := \mathbb{E}_\eta^{(\varepsilon)}\big(f(\eta_t)\big)$ μ–a.s. for any f continuous, bounded
- T. M. Liggett, *Interacting particle systems*.

Invariant distribution μ_ε

PP: perturbed process

Theorem (First part)

Let $\varepsilon < \gamma$.

- *Exists a unique invariant distribution μ_ε for PP absolutely continuous w.r.t. μ.*
- $\frac{d\mu_\varepsilon}{d\mu} \in L^2(\mu)$ *and μ_ε is time–ergodic for PP.*
- *Consider PP with initial distribution $\nu \ll \mu$. Then $\nu_t \to \mu_\varepsilon$ weakly as $t \to \infty$.*

Theorem (2nd part)

- *(Dyson–Phillips expansion)* *For any* $f \in L^2(\mu)$

$$\mu_\varepsilon(f) = \mu(f) + \sum_{n=0}^{\infty} \int_0^{\infty} \mu\left(\hat{L}_\varepsilon S_\varepsilon^{(n)}(s) f\right) ds\,.$$

- *If for all* $t > 0$ *and* $B \subset \Omega$

$$\left\{\eta \in B^c \,:\, \mathbb{P}_\eta^{(\varepsilon)}(\eta_t \in B) = 0\,,\; \mathbb{P}_\eta(\eta_t \in B) > 0\right\}$$

has zero μ*–probability, then* μ *and* μ_ε *are mutually absolutely continuous.*

Theorem (3rd part)

- *(Semigroup convergence)*

$$\|S_\varepsilon(t)f - \mu_\varepsilon(f)\| \leq \frac{\gamma}{\gamma-\varepsilon} e^{-(\gamma-\varepsilon)t} \|f - \mu(f)\| \ , \ \forall f \in L^2(\mu)$$

$$\|S_\varepsilon(t)f - \mu_\varepsilon(f)\|_\varepsilon \leq \frac{\gamma}{\gamma-\varepsilon} e^{-\frac{\gamma-\varepsilon}{2}t} \|f - \mu(f)\|_\infty \, , \forall f \in L^\infty(\mu)$$

$\|\cdot\|$ norm in $L^2(\mu)$, $\|\cdot\|_\varepsilon$ norm in $L^2(\mu_\varepsilon)$, $\|\cdot\|_\infty$ norm in $L^\infty(\mu)$

Remarks

- Our analysis is based on Dyson–Phillips expansion
- Part of above theorem already obtained in

  T. Komorowski, S. Olla, *On Mobility and Einstein Relation for Tracers in Time-Mixing Random Environments*, Journal of Statistical Physics (118) 3/4, (2005).

  by different methods. Perturbation $\hat{L}_\varepsilon$ can be unbounded, it must satisfy sector condition.

Remarks

- In the applications, μ_ε and μ mutually a.c. In general, if a property is true μ_ε–a.s., then it is true apart a set of μ–probability $O(\varepsilon^2/(\gamma-\varepsilon)^2)$

Law of large number for additive functionals

μ_ε is time–ergodic for PP.

Corollary

Let $\varepsilon < \gamma$ and let $f : \Omega \to \mathbb{R}$ be a measurable function, nonnegative or in $L^1(\mu_\varepsilon)$ (e.g. bounded or in $L^2(\mu)$). Then

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(\eta_s) = \mu_\varepsilon(f)\,, \qquad \mathbb{P}_\eta^{(\varepsilon)} - \text{a.s.}$$

for μ_ε any starting configuration η

Quenched invariance principle for additive functional

Theorem

Let $\varepsilon < \gamma$ and let $f \in C_b(\Omega)$ be continuous. Given $n \in \mathbb{N}$, define

$$B_t^{(n)}(f) := \int_0^{nt} \frac{f(\eta_s) - \mu_\varepsilon(f)}{\sqrt{n}} ds\,, \qquad t \geq 0\,.$$

Then $\exists$ $\sigma^2 \geq 0$ such that, for μ_ε–any η, under $\mathbb{P}_\eta^{(\varepsilon)}$ the process $(B_t^{(n)})_{t\geq 0}$ weakly converges to a Brownian motion with diffusion coefficient σ^2.

- Martingale methods (Kipnis & Varadhan)
- Quenched improvements (Maxwell & Woodroofe, Rassoul–Agha & Seppalainen, ...)

Random walk driven by a dynamical random environment

- S Polish space. $\Omega = S^{\mathbb{Z}^d}$
- Translation τ_x on Ω for $x \in \mathbb{Z}^d$: $\tau_x \eta(y) := \eta(y - x)$
- **Dynamical environment** $(\xi_t)_{t \geq 0}$: Markov process on Ω with càdlàg paths and generator L_{env}
- **Random walk** $(X_t)_{t \geq 0}$ **on** $\mathbb{Z}^d$ **starting at** 0: $r(y, \tau_x \xi)$ is probability rate of jump $x \curvearrowright x + y$ with environment ξ

Environment viewed from the walker

$\eta_t := \tau_{X_t}\xi_t$

Markov generator:

$$
\begin{aligned}
L_{\mathrm{ew}} f(\eta) &:= L_{\mathrm{env}} f(\eta) + \sum_{y \in \mathbb{Z}^d} r(y,\eta)\big[f(\tau_y \eta) - f(\eta)\big] \\
&= L_{\mathrm{env}} f(\eta) + L_{\mathrm{jumps}} f(\eta)
\end{aligned}
$$

Environment viewed from the walker

Markov generator:

$$L_{\text{ew}} f(\eta) = L_{\text{env}} f(\eta) + L_{\text{jumps}} f(\eta)$$

Assumption:

- (i) the environment process with generator L_{env} has ergodic invariant distribution μ and satisfies Poincaré inequality in $L^2(\mu)$ with Poincaré constant γ
- (ii) $\mu(L_{\text{jumps}} f) = 0$

Perturbed random walk

- We modify the jump rates of the rw.
- $r_\varepsilon(y, \tau_x \xi)$: rate for jump $x \curvearrowright x + y$

$$r_\varepsilon(y, \cdot) := r(y, \cdot) + \hat{r}_\varepsilon(y, \cdot),$$

- We apply general theory with:
  $L := L_{\mathrm{ew}}$;
  $L_\varepsilon := L_{\mathrm{ew}}^{(\varepsilon)}$ (environment viewed from perturbed walker);
  $\hat{L}_\varepsilon := \sum_{y \in \mathbb{Z}^d} \hat{r}_\varepsilon(y, \eta) \big[f(\tau_y \eta) - f(\eta) \big]$

$$L_{\mathrm{ew}}^{(\varepsilon)} = L_{\mathrm{ew}} + \hat{L}_\varepsilon \, .$$

- Set $\phi(\varepsilon) := \sum_{y\in\mathbb{Z}^d} |\hat{r}_\varepsilon(y,\cdot)|_\infty$. Then $\|\hat{L}_\varepsilon\| < 2\phi(\varepsilon)$.
- We assume $\phi(\varepsilon) < \gamma$
- Switch on the machine...
- μ_ε unique invariant distribution for the *environment viewed from the perturbed random walk* absolutely continuous w.r.t. μ

Asymptotic velocity of perturbed random walk $(X_t^{(\varepsilon)})_{t\geq 0}$

Theorem

- *For* μ_ε*–almost any initial environment*

$$\lim_{t\to\infty} \frac{X_t^{(\varepsilon)}}{t} = v(\varepsilon) \qquad a.s.$$

- $v(\varepsilon)$ *admits the Dyson–Phillips expansion:*

$$v(\varepsilon) = \mu(j_\varepsilon) + \sum_{n=1}^{\infty} \int_0^\infty \mu(\hat{L}_\varepsilon S_\varepsilon^{(n)}(s) j_\varepsilon) ds$$

where $j_\varepsilon(\eta) := \sum_{y\in\mathbb{Z}^d} y r_\varepsilon(y,\eta)$

- n^{th}*–addendum* $=\mathcal{O}\left(\phi(\varepsilon)^n \|j_\varepsilon\|_\infty\right)$

Quenched invariance principle

Theorem

- $\exists$ *symmetric* $d \times d$ *matrix* $D_\varepsilon \geq 0$ *such that, for* μ_ε*–a.a. initial environment, the rescaled process*

$$\frac{X_{nt}^{(\varepsilon)} - v(\varepsilon)nt}{\sqrt{n}}, \qquad t \geq 0,$$

weakly converges as $n \to \infty$ *to a Brownian motion with covariance matrix* D_ε *.*

- *(under reversibility of environment viewed from the unperturbed rw) if* $\phi(\varepsilon)$ *is small enough, than* D_ε *is non–degenerate.*

Qualitative properties of μ_ε

We know: $\mu_\varepsilon \ll \mu$, $\frac{d\mu_\varepsilon}{d\mu} \in L^2(\mu)$

Fact

If $r(y,\eta) > 0 \implies r_\varepsilon(y,\eta) > 0$, *then* $\mu \ll \mu_\varepsilon$.

We have more general criterion

Finite speed of propagation for the environment process

$\exists\ \alpha(\cdot)$ vanishing at infinite and $C > 0$ such that:

$$\left| \mathrm{Cov}_{\mu}^{\mathrm{env}}[X, X'] \right| \leq \alpha(d(\Lambda, \Lambda')) \|X\|_\infty \|Y\|_\infty$$

- $d(\Lambda, \Lambda')$ Euclidean distance between $\Lambda, \Lambda' \subset \mathbb{Z}^d$
- $d(\Lambda, \Lambda') \geq Ct$
- X r.v. determined by process on Λ up to time t
- X' r.v. determined by process on Λ' up to time t

Approximation of μ_ε by μ at infinity

Theorem

Suppose

- *environment process has finite speed of propagation*
- *walker jump rates $r_\varepsilon(y,\eta)$ have finite range in y and finite support in η*

Then for any bounded local function $f:\Omega\to\mathbb{R}$, it holds

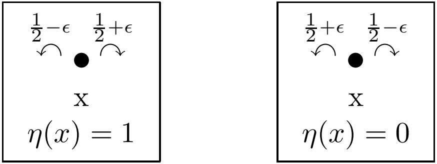
$$\lim_{|x|\to\infty}\mu_\varepsilon(\tau_x f)=\mu(f)\,.$$

- The proof provides bound on the error $|\mu_\varepsilon(\tau_x f) - \mu(f)|$.
- If α decay exponentially, then $|\mu_\varepsilon(\tau_x f) - \mu(f)| \leq e^{-c|x|}$

Random walk and 1d interacting particle system

- **Environment process**
  - 1d interacting particle system
  - State space $\{0,1\}^{\mathbb{Z}}$
  - μ reversible distribution, translation invariant
  - Poincaré inequality

Nearest–neighbor random walk $(X_t^{(\varepsilon)})_{t\geq 0}$



Asymptotic velocity $v(\varepsilon)$

Proposition

Take ε small. Then:

- *$v(\varepsilon)$ has Dyson–Phillips expansion.*
- *$v(\cdot)$ is antisymmetric: $v(\varepsilon) = -v(-\varepsilon)$*
- $$v(\varepsilon) = \begin{cases} 2\varepsilon\big[2\mu(\eta(0)) - 1\big] + O(\varepsilon^3) & \text{if } \mu(\eta(0)) \neq 1/2 \\ \varepsilon^3\kappa + O(\varepsilon^5) & \text{if } \mu(\eta(0)) = 1/2, \end{cases}$$

  with

  $$\kappa := -8\mu\left((2\eta(0) - 1)\Big\{\int_0^\infty \mathbb{E}_\eta[\eta_s(1) - \eta_s(-1)]ds\Big\}^2\right).$$

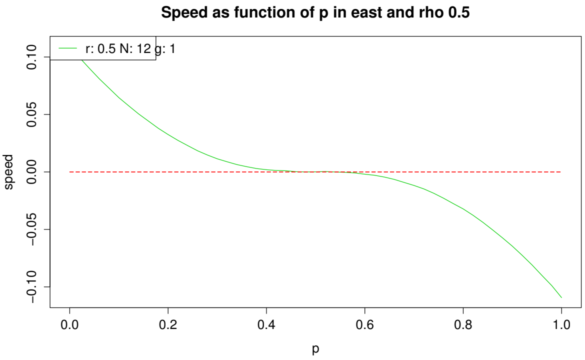
$v(\varepsilon)$ for $\mu(\eta(0)) = 1/2$

- Environment given by independent spins $\Rightarrow v(\varepsilon) \equiv 0$, $\kappa = 0$
- Environment given by East model $\Rightarrow$ simulations suggest $\kappa < 0$
- We have proved negative velocity for a random walk mimicking some mechanism of rw in East environment.

Antisymmetry $v(\varepsilon) = -v(-\varepsilon)$

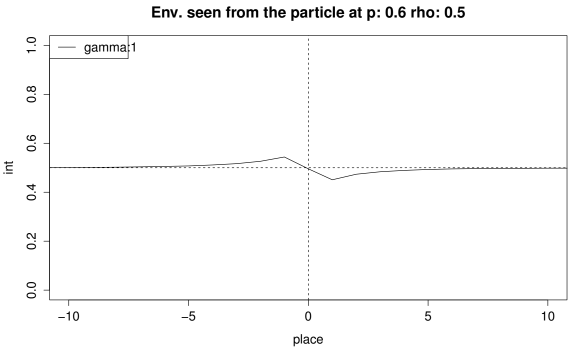
- the environment process is left invariant by space reflection $\Longrightarrow$ **easy**
- in general: algebraic derivation, **not related to dynamical internal symmetries** of environment process
- it holds for a larger class of random walks
- Replace $\mathbb{Z}$ by torus $\mathbb{T}_N := \mathbb{Z}/N\mathbb{Z}$. Instead of $X_t^{(\varepsilon)}/t$ study asymptotics of winding number per time unit. With V. Lecomte: **higher–level symmetry on LD rate function** (hence, generating function)

East environment $\rho = 1/2$: simulation of $v(\varepsilon)$ (P. Thomann)



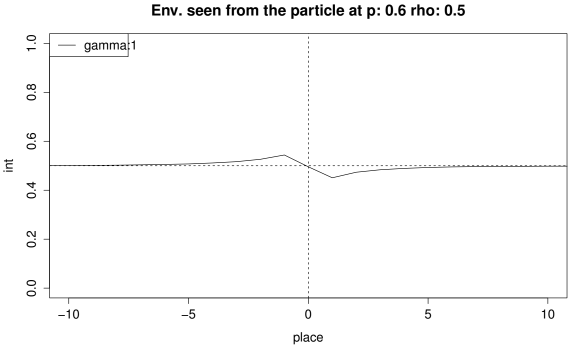
$\Rightarrow$ antisymmetry, negative mobility

East environment $\rho = 1/2$: simulation of $\mu_\varepsilon(\eta(x))$ (P. Thomann)



$\Rightarrow$ Asymptotic convergence to $\mu(\eta(x)) = 1/2$

East environment $\rho = 1/2$: simulation of $\mu_\varepsilon(\eta(x))$ (P. Thomann)



$\Rightarrow$ Asymptotic convergence to $\mu(\eta(x)) = 1/2$

East process

- Stochastic model for glassy systems, state space $\Omega = \{0,1\}^{\mathbb{Z}}$
- Constrained Glauber dynamics. Parameter $\rho \in (0,1)$
- At site x wait exponential time of mean 1.
  Then, only if $\eta(x+1) = 0$, update η_x:

$$\begin{cases} 1 & \text{with probability } 1-\rho\,, \\ 0 & \text{with probability } \rho\,. \end{cases}$$

- $(1-\rho)$–Bernoulli probability ν_ρ: reversible distribution
- it satisfies Poincaré inequality

East process and West process

- West process: same definition but now constraint "$\eta(x-1)=0$"
- Antisymmetry: for each ρ, $v_{\text{east}}(\varepsilon) = v_{\text{west}}(\varepsilon)$