Biased random walk among random conductances: Einstein relation and monotonicity of the speed

Nina Gantert

joint work with Jan Nagel and Xiaoqin Guo

Outline

1. The Random Conductance Model
2. Random walks on supercritical percolation clusters
3. Biased random walks among random conductances
4. Einstein-Relation
5. Strategy of the proof
6. A result about monotonicity

We define a random medium by giving random weights - we call them "conductances" - to the bonds of the lattice.

- Nearest-neighbour random walk in $\mathbb{Z}^d$
- Random conductances $\omega = \{\omega_{x,y}\}_{x \sim y}$ with law P such that
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  - (i) $\{\omega_{x,y}\}_{x \sim y}$ are iid
  - (ii) $\frac{1}{\kappa} \leq \omega_{x,y} \leq \kappa \quad \forall\, x \sim y$ (uniform ellipticity)

For fixed ω, the RW in the environment ω is the Markov chain given by

$$P^x_\omega(X_{n+1} = x + e | X_n = x) = \frac{\omega_{x,x+e}}{\sum_{|e'|=1} \omega_{x,x+e'}} \qquad \text{(quenched law)}$$

Average over environments: $\mathbb{P}^x = \int P^x_\omega(\cdot) P(d\omega)$ (averaged law)

Question

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Answer: Yes! $\frac{1}{\sqrt{n}} X_{\lfloor n \cdot \rfloor} \xrightarrow[n\to\infty]{d}$ Brownian motion with covariance $\Sigma = \sigma \cdot \mathrm{Id}$ (under P_ω). There are several papers by A. de Masi/P. A. Ferrari/S. Goldstein/W. D. Wick, D. Boivin, L. Fontes/P. Mathieu, S. M. Kozlov, V. Sidoravicius/A.-S. Sznitman,... leading to this result, and it has been extended to the case of bounded, strictly positive conductances.

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An active direction of research is the extension of this theorem to the case when the conductances form a stationary, ergodic random field. It is not true in general, but true under boundedness conditions on the conductances. In this case, the covariance matrix Σ is not diagonal in general.

Question

How does Σ depend on the law of the conductances?

Random walks on supercritical percolation clusters

Consider bond percolation with parameter p on the d-dimensional lattice: all bonds are *open* with probability p and *closed* with probability $1-p$, independently of each other. This corresponds to conductances with values either 1 or 0.

Take bond percolation on $\mathbb{Z}^d$, $d \geq 2$. Choose p close enough to 1 such that there is a (unique) infinite cluster.
Condition on the event that the origin is in the infinite cluster.
Start a random walk in the infinite cluster which can only walk on open bonds, and which goes with equal probabilities to all neighbours. (In particular, this random walk never leaves the infinite cluster.)

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Method of proof: decompose the walk in a martingale part and a "corrector". Show that the corrector can be neglected and apply the CLT for martingales.

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Question

How does Σ depend on p?

Einstein-Relation

The Einstein relation gives a different interpretation of the variance as the derivative of the speed of the random walk, when one has a drift in a "favourite" direction ℓ. This leads us to consider **biased random walks in random environments**.

Biased random walks among random conductances

- RW with bias of strength $\lambda > 0$:

$$P^x_{\omega,\lambda}(X_{n+1} = x + e | X_n = x) = \frac{\omega_{x,x+e}\, e^{\lambda \ell \cdot e}}{\sum_{|e'|=1} \omega_{x,x+e'}\, e^{\lambda \ell \cdot e'}}$$

- $\mathbb{P}^x_\lambda = \int P^x_{\omega,\lambda}(\cdot) P(d\omega)$

For the random conductance model, Lian Shen proved that for fixed λ, $\frac{1}{n}X_n \xrightarrow[n\to\infty]{} v(\lambda)$ $\mathbb{P}_\lambda^x$-a.s. where $v(\lambda)$ is deterministic and $v(\lambda)\cdot\ell > 0$.

For the RW among random conductances, the Einstein relation holds.

Theorem

Einstein-Relation (NG, Jan Nagel und Xiaoqin Guo, in progress)
Assume that the conductances are iid and uniformly elliptic. Then,

$$\lim_{\lambda \to 0} \frac{v(\lambda)}{\lambda} = \Sigma \ell \,.$$

Further, $v(\lambda)$ is differentiable for all λ and we can write its derivative as a covariance.

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It can be proved easily in the one-dimensional case and in the periodic case.

Why should it be true?

For simplicity, we give an argument for the continuous case.

Einstein relation for symmetric diffusions in random environment:

Consider a diffusion $X(t)$ in $\mathbb{R}^d$, with generator

$$L^\omega f(x) = \frac{1}{2} e^{2V^\omega(x)} \operatorname{div}(e^{-2V^\omega} a^\omega \nabla f)(x),$$

where V^ω is a real function and a^ω is symmetric matrix. V^ω and a^ω are realizations of a random environment V^ω and a^ω are realizations of a random environment. Then, with $\sigma^\omega = \sqrt{a^\omega}$ and $b^\omega = \frac{1}{2}\operatorname{div} a^\omega - a^\omega \nabla V^\omega$, X solves the stochastic differential equation

$$dX(t) = b^\omega(X(t))\, dt + \sigma^\omega(X(t)) dW_t$$

where W is a Brownian motion. Now, add a local drift in the equation satisfied by X: let $\ell \in \mathbb{R}^d$ be a vector, $\ell \neq 0$, and take the equation

$$dX^\lambda(t) = b^\omega(X^\lambda(t))dt + \sigma^\omega(X^\lambda(t))dW_t + a^\omega(X^\lambda(t))\lambda\ell dt.$$

A key ingredient is **Girsanov transform**. For any t, the law of $(X^\lambda(s))_{0\leq s\leq t}$ is absolutely continuous w. r. t. the law of $(X(s))_{0\leq s\leq t}$ and the Radon-Nikodym density is the exponential martingale

$$e^{\lambda B(t)-\frac{\lambda^2}{2}\langle B\rangle(t)}$$

where

$$B(t)=\int_0^t \ell^T\sigma^\omega(X(s))\cdot dW_s$$

and

$$\langle B\rangle(t)=\int_0^t \left|\ell^T\sigma^\omega(X(s))\right|^2 ds$$

In particular,

$$\mathbb{E}_0\left[X^\lambda(t)\right] = \mathbb{E}_0\left[X(t)e^{\lambda B(t) - \frac{\lambda^2}{2}\langle B\rangle(t)}\right]$$

Hence

$$\frac{d}{d\lambda}\,\mathbb{E}_0\left[X^\lambda(t)\right]\Big|_{\lambda=0} = \mathbb{E}_0\left[X(t)B(t)\right]$$

and

$$\lim_{t\to\infty}\frac{1}{t}\frac{d}{d\lambda}\,\mathbb{E}_0\left[X^\lambda(t)\right]\Big|_{\lambda=0} = \lim_{t\to\infty}\frac{1}{t}\mathbb{E}_0\left[X(t)B(t)\right]$$

Exchanging the order of the limits yields

$$\frac{d}{d\lambda}\left. v(\lambda)\right|_{\lambda=0} = \lim_{t\to\infty} \frac{1}{t}\mathbb{E}_0\left[X(t)B(t)\right]$$

A symmetry argument (using the reversibility) shows that

$$\mathbb{E}_0\left[X(t)B(t)\right] = \mathbb{E}_0\left[X(t)(\ell\cdot X(t))\right]$$

and we conclude that

$$\frac{d}{d\lambda}\left. v(\lambda)\right|_{\lambda=0} = \lim_{t\to\infty} \frac{1}{t}\mathbb{E}_0\left[X(t)(\ell\cdot X(t))\right] = \Sigma\ell$$

The argument can be made precise in the following sense: Joel Lebowitz and Hermann Rost showed, using the invariance principle and Girsanov transform:

Theorem

(Joel Lebowitz, Hermann Rost, 1994) Let $\alpha > 0$. Then

$$\lim_{\lambda \to 0\,, t \to +\infty\,, \lambda^2 t = \alpha} \mathbb{E}_0 \left[\frac{X^\lambda(t)}{\lambda t} \right] = \Sigma\, \ell\,.$$

Under assumptions on V^ω and a^ω (translation invariance, ergodicity,, uniform ellipticity, finite range dependence) the Einstein relation for symmetric diffusions in random environment was proved by NG, Pierre Mathieu and Andrey Piatnitski.

The Einstein relation for the random conductance model is a consequence of the following expansion. Let Q_λ denote the invariant measure for the process $\bar{\omega} = (\bar{\omega}_n)_{n\geq 0}$, the *environment seen from the particle*, where $\bar{\omega}_n = \theta_{X_n}\omega$ and the law of $(X_n)_n$ is given by P_{ω^λ}. Write $\mathcal{F}$ for the set of bounded functions $f : \Omega \to \mathbb{R}$ depending only on a finite set of conductances. We show the following first order expansion of Q_λ around $\lambda = 0$.

Theorem

There exists a functional Λ *on* $\mathcal{F}$*, such that*

$$\lim_{\lambda \to 0} \frac{Q_\lambda f - Q_0 f}{\lambda} = \Lambda f$$

for any $f \in \mathcal{F}$*.*

More precisely, Λ is given as follows. Let $d(\omega, x) = E_\omega^x[(X_1 - X_0)]$, consider the 2-dimensional process

$$\frac{1}{\sqrt{n}}\left(\sum_{k=1}^{n} f(\bar{\omega}_k), \sum_{k=1}^{n} d(\bar{\omega}_k, 0)\right),$$

By the Kipnis-Varadhan Theorem, this process converges in distribution under $\mathbb{P}$ to a 2-dimensional Brownian motion (N_f, N_d). Then $\Lambda(f) = -Cov(N_f, N_d)$.

To see how the Einstein relation follows, note that, defining $\mathbb{Q}_\lambda = Q_\lambda \otimes P_{\omega^\lambda}$, by the ergodic theorem we can write the velocity as

$$v(\lambda) = \lim_{n\to\infty} \frac{X_n}{n} = \mathbb{E}_{\mathbb{Q}_\lambda}[d(\omega^\lambda, 0)]$$

where the limit is almost surely under $\mathbb{Q}_\lambda$ and

$$d(\omega^\lambda, 0) = \sum_{|e|=1} (C_0^\lambda)^{-1} \omega_{0,e} e^{\lambda \ell \cdot e} e,$$

with $C_0^\lambda = \sum_{|e'|=1} \omega_{0,e'} e^{\lambda \ell \cdot e'}$. Write

$$d(\omega^\lambda, 0) = d(\omega, 0) + A(\omega) + o(\lambda)$$

Consequently, we can apply the expansion to (the components of) the drift $d(\cdot, 0) \in \mathcal{F}$:

$$\lim_{\lambda \to 0} \frac{v(\lambda)}{\lambda} = \lim_{\lambda \to 0} \frac{Q_\lambda d(\omega^\lambda, 0) - Q_0 d(\omega, 0)}{\lambda}$$

$$= \lim_{\lambda \to 0} \left(\frac{Q_\lambda d(\omega, 0) - Q_0 d(\omega, 0)}{\lambda} + Q_\lambda[A] \right)$$

$$= \Lambda d(\cdot, 0)(\omega) + Q_0[A].$$

To prove the expansion for Q_λ, we show

Theorem

Diffusivity part

For any $t \geq 1$ and $f \in \mathcal{F}$, we have

$$\lim_{\lambda \to 0} \frac{\frac{\lambda^2}{t}\mathbb{E}_{Q,\lambda} \sum_{k=0}^{t/\lambda^2} f(\bar{\omega}_k) - Q_0 f}{\lambda} = \Lambda f,$$

where $\mathbb{E}_{Q,\lambda}$ is the expectation with respect to $Q_0 \otimes P_{\omega^\lambda}$.

Theorem

Mobility part

There exists a constant C depending only on the dimension and the ellipticity constants, such that for any $t \geq 1$ and $f \in \mathcal{F}$

$$\frac{\frac{\lambda^2}{t}\mathbb{E}_{Q,\lambda} \sum_{k=0}^{t/\lambda^2} f(\bar{\omega}_k) - Q_\lambda f}{\lambda} \leq \frac{C}{\sqrt{t}}.$$

Proof of the Diffusivity part

- Consider first term $\frac{\lambda}{t}\sum_{k=0}^{t/\lambda^2}(f(\bar{\omega}_k) - Q_0 f)$ and assume without loss of generality $Q_0 f = 0$. From Kipnis-Varadhan, have a decomposition

$$\sum_{k=0}^{n} f(\bar{\omega}_k) = M_n^* + R_n,$$

where M_n^* is a martingale under $\mathbb{Q}_0$ and $\frac{R_n}{\sqrt{n}}$ converges in law to 0.

- With $d_\ell(\omega, x) = E_\omega^x[(X_1 - X_0)\cdot \ell]$ the expected displacement in direction ℓ, apply martingale CLT to get the joint convergence

$$\lambda\left(\sum_{k=0}^{t/\lambda^2} f(\bar{\omega}_k),\ \ell\cdot X_{t/\lambda^2} - \sum_{i=1}^{t/\lambda^2} d_\ell(\omega, X_{i-1})\right) \xrightarrow[\lambda\to 0]{} (N_t^*, N_t)$$

in distribution under $\mathbb{Q}_0$ to some 2-dimensional Brownian motion.

- Show convergence of density: under P_ω and in L^p:

$$\log \frac{dP_{\omega,\lambda}}{dP_\omega}(X_s)_{0\le s\le t} \xrightarrow[\lambda\to 0]{} N_t - \frac{1}{2}E[N_t^2]$$

where (N_t) is a Brownian motion.

- Prove boundedness statements to conclude convergence of the expectation

$$\mathbb{E}_{Q,\lambda}\left[\frac{\lambda}{t}\sum_{k=0}^{t/\lambda^2} f(\bar\omega_k)\right] = \mathbb{E}_{Q,0}\left[\frac{\lambda}{t}\sum_{k=0}^{t/\lambda^2} f(\bar\omega_k)\frac{dP_{\omega,\lambda}}{dP_\omega}(X_s)_{0\le s\le (t/\lambda^2)}\right]$$

Hence

$$\mathbb{E}_{Q,\lambda}\left[\frac{\lambda}{t}\sum_{k=0}^{t/\lambda^2} f(\bar\omega_k)\right] \xrightarrow[\lambda\to 0]{} \frac{1}{t}E\left[N_t^* e^{N_t-\frac{1}{2}E[N_t^2]}\right]$$

- Apply Girsanov's formula

$$E\left[N_t^* e^{N_t - \frac{1}{2}E[N_t^2]}\right] = E\left[[N^*, N]_t\right] = [N^*, N]_t,$$

- Remains to identify $[N^*, N]_t$.

Proof of the Mobility part

We define a suitable regeneration structure. Need a-priori estimates: For $h \in \mathbb{R}$ we define the hitting time

$$T_h = \inf\{n \geq 0 | (X_n - X_0) \cdot e_1 = h\}.$$

Lemma

There exist $L_0 \in \mathbb{N}, \lambda_0 > 0$ *such that*

$$P_{\omega,\lambda}(T_{L/\lambda} < T_{-L/\lambda}) \geq \frac{2}{3}$$

for all $L \geq L_0, 0 < \lambda \leq \lambda_0$ *and for all* ω.

Lemma

There exists a $\delta = \delta(L) > 0$ *such that for all* ω, *for all* $\lambda \in (0,1)$, *and* $x \in \mathbb{Z}^d$ *with* $-\frac{L}{\lambda} \leq x \cdot e_1 \leq \frac{L}{\lambda}$ *we have*

$$P^x_{\omega,\lambda}(T_{L/\lambda} < \left(4\tfrac{L}{\lambda}\right)^2) \geq \delta.$$

Lemma

For any $p \geq 1$ *there exist positive constants* C *and* η *depending only on* p, *the ellipticity constant* κ *and the dimension, such that for all* $\lambda \in (0,\eta)$ *and* $t \geq 1/\lambda^2$ *and for any* ω,

$$E_{\omega,\lambda}\left[\max_{0\leq s\leq t} |X_s|^p\right] \leq C(\lambda t)^p.$$

The Einstein relation is conjectured to hold for many models, but it is proved for few. Apart from the results mentioned, examples include:

- Random walks in balanced random environments (Xiaoqin Guo)
- Random walks on Galton-Watson trees (Gérard Ben Arous, Yueyun Hu, Stefano Olla, Ofer Zeitouni)
- Tagged particle in asymmetric exclusion (Michail Loulakis)

The following examples are in progress:

- Random walks on percolation clusters of ladder graphs (NG, Matthias Meiners, Sebastian Müller)
- Mott random walks (Alessandra Faggionato, NG, Michele Salvi)

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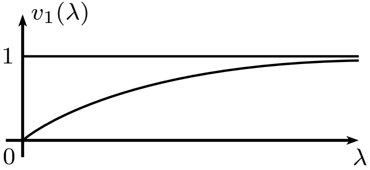
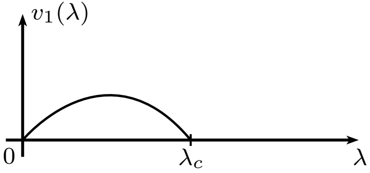
Question:
Is $v_1(\lambda) = v(\lambda) \cdot \ell$ increasing as a function of λ?

Back to the homogeneous medium: in this case, $v(\lambda)$ can be computed and $v_1(\lambda) = v(\lambda) \cdot \ell$ looks as follows:



For the speed of the random walk on an infinite percolation cluster, the following picture is conjectured:
for each $p \in (p_c, 1)$ we have, with $v_1(\lambda) = v(\lambda) \cdot \ell$



Reason for the zero speed regime: "traps" in the percolation cluster!
Alexander Fribergh and Alan Hammond showed recently that there is, for each $p \in (p_c, 1)$, a critical value λ_c such that $v_1(\lambda) > 0$ for $\lambda < \lambda_c$ and $v_1(\lambda) = 0$ for $\lambda > \lambda_c$. Quoting from their paper:

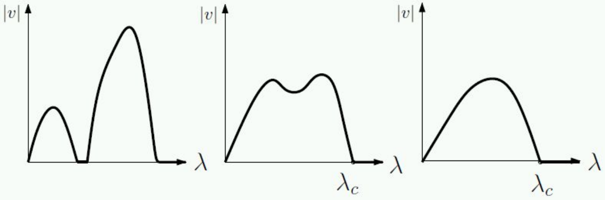
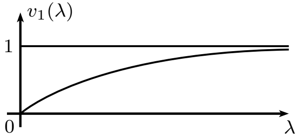


FIGURE 3. The speed as a function of the bias. Sznitman and Berger, Gantert and Peres established positive speed at low λ, but their works left open the possibility depicted in the first sketch. Our work rules this out, though the behavior of the speed in the ballistic regime depicted in the second sketch remains possible. The third sketch shows the unimodal form predicted physically.

How does $v_1(\lambda)$ depend on λ for the random walk among uniformly elliptic random conductances?

For the homogeneous medium, we have



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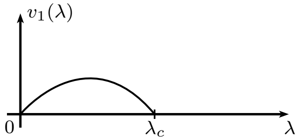
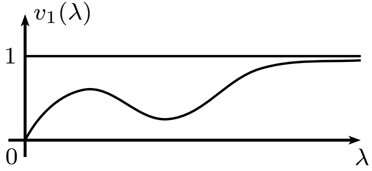


For the random walk among random conductances, we believe that the picture can be



We show (Noam Berger, NG, Jan Nagel, in progress): the speed in the favourite direction is in general *not* increasing. More precisely, assume the conductances take the values 1 (with probability $> p_c$) and δ with probability $1 - p$. Then, for δ small enough, there are $0 < \lambda < \lambda'$ such that $v_1(\lambda) > v_1(\lambda')$.

Thanks for your attention!