Remarks on a homogenization problem for stochastic HJB equations with a non-convex Hamiltonian

Elena Kosygina

Baruch College and the CUNY Graduate Center

"April 1", 2015

- *Environment:* probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{R}^d$ acts on Ω by shifts $\tau_x : \Omega \to \Omega$, $x \in \mathbb{R}^d$. We assume that $\mathbb{P}$ is invariant under the shifts and ergodic.
- *Hamiltonian:* $H(p, x, \omega) := \tilde{H}(p, \tau_x \omega)$. Examples:
  - (i) $\tilde{H}(p, \omega) = |p|^\alpha + b(\omega) \cdot p - V(\omega)$, $\alpha > 1$, b, V are bounded and locally Lipschitz under the shifts, $V \geq 0$ a.s.. This is a convex example. We can define the corresponding Lagrangian,
  $$\tilde{L}(q, \omega) := \sup_{p \in \mathbb{R}^d} (p \cdot q - \tilde{H}(p, \omega)).$$
  - (ii) $\tilde{H}(p, \omega) = (|p|^2 - 1)^2 - V(\omega)$.
  - (iii) $\tilde{H}(p, \omega) = \frac{1}{2}|p|^2 - |p| + b(\omega) \cdot p - V(\omega)$.

  All these examples are of the type $\tilde{H}(p, \omega) = \tilde{H}_1(p, \omega) - \tilde{H}_2(p, \omega)$, where $\tilde{H}_i$ are convex, $\tilde{H}_1$ is super-linear and grows faster than $\tilde{H}_2$ as $|p| \to \infty$. Let us assume that this is the case for the rest of the talk.

- *HJB equation:* $a > 0$ - viscous case, $a = 0$ - inviscid case; we shall assume $a = 1$. Let $u^\varepsilon = u^\varepsilon(t, x, \omega)$ be the (unique viscosity) solution of

$$\text{(HJB)} \quad u_t^\varepsilon = \frac{\varepsilon}{2}\, a\, \Delta u^\varepsilon + H\left(\nabla u^\varepsilon, \frac{x}{\varepsilon}, \omega\right), \quad (t, x) \in (0, \infty) \times \mathbb{R}^d;$$

$$u^\varepsilon|_{t=0} = g(x), \quad x \in \mathbb{R}^d, \quad g \text{ is Lipschitz continuous.}$$

- *Problem:* Show that (under suitable assumptions) $u^\varepsilon(t, x, \omega)$ converges as $\varepsilon \to 0$ in some sense (in probability or a.s.) to a deterministic $u(t, x)$, where $u(t, x)$ is the solution of the effective equation $u_t = \bar{H}(\nabla u)$ with the same initial condition g.

- *References for (qualitative) convex homogenization for viscous (HJB):* P.-L. Lions, P. E. Souganidis (2005, 2010); EK, F. Rezakhanlou, S.R.S. Varadhan (2006); EK, S.R.S. Varadhan (2008); S. Armstrong, P. E. Souganidis (2012), S. Armstrong, H. Tran (2015) and references therein.
- *Closely related results on large deviations of RW in random environments and random potential:* F. Rassoul-Agha, T. Seppäläinen, A. Yilmaz (2013) and references therein.

A word about correctors

Corrector problem: For every $\theta \in \mathbb{R}^d$ find a constant $\lambda = \lambda(\theta)$ such that the equation

$$\frac{a}{2}\Delta v + H(\theta + \nabla v, x, \omega) = \lambda, \ x \in \mathbb{R}^d$$

has a viscosity solution $v(x,\omega)$ which is Lipschitz with stationary, mean zero gradient. These conditions guarantee that $v(x,\omega)$ is strictly sub-linear at infinity. The latter also ensures that $\lambda(\theta)$ is unique for each θ.[1] Then $\lambda(\theta)$ is the effective Hamiltonian.

Just to indicate the connection: for v as above the function $U^\varepsilon(t,x,\omega) = \lambda(\theta)t + \theta \cdot x + \varepsilon v(x/\varepsilon,\omega)$ solves (HJB) with the initial data $\theta \cdot x + \varepsilon v(x/\varepsilon,\omega)$ and converges to $\lambda(\theta)t + \theta \cdot x$, which is a solution to $u_t = \lambda(\nabla u)$.

[1] P.-L. Lions, P. E. Souganidis, CPAM, 2003, Prop. 2.1.

References about correctors

- F. Rezakhanlou, J. E. Tarver (2000) proved, in particular, that the existence of correctors implies homogenization for general initial data ($a = 0$).
- P.-L. Lions and P. E. Souganidis (CPAM, 2003) have shown that correctors for stochastic HJ equations in general need not exist.
- A. Davini and A. Siconolfi (Mat. Annalen, 2009) have shown that for level-set convex H and $d = 1$ approximate correctors for the inviscid problem do exist and derived homogenization for this case.
- For further developments see A. Marigonda and A. Siconolfi (Adv. Diff. Equat., 2011).

Recent progress in homogenization of non-convex HJ equations

Results for $a = 0$:

- S. Armstrong, P. E. Souganidis (IMRN, 2013). Stochastic homogenization of level-set convex HJ equations.
- S. Armstrong, H. Tran, Y. Yu (Calc. Var. PDE, 2015). Stochastic homogenization of a non-convex HJ equation. Treated $H(p,\omega) = (|p|^2 - 1)^2 - V(\omega)$ (Example (ii)).
- S. Armstrong, H. Tran, Y. Yu (arxiv, 2014). Stochastic homogenization of nonconvex Hamilton-Jacobi equations in one space dimension.
- B. Fehrman (arxiv, 2014). A partial homogenization result for nonconvex viscous HJ equations. (Proof for $a = 0$, changes are sketched for $a > 0$.)

Basic facts

- *Linear initial data:* $g(x) = \theta \cdot x$. Observe that the function $u(t,x) = t\bar{H}(\theta) + \theta \cdot x$ solves the equation $u_t = \bar{H}(\nabla u)$ with this initial data. In particular, if the homogenization takes place then $\bar{H}(\theta) = \lim_{\varepsilon \to 0} u_\theta^\varepsilon(1, 0, \omega)$, where u_θ^ε solves the original HJB equation with $g(x) = \theta \cdot x$. Homogenizing $u_\theta^\varepsilon(1, 0, \omega)$ for all $\theta \in \mathbb{R}^d$ would *identify the candidate* for the effective Hamiltonian.
- *Homogenization as a scaling limit:* Define v^ε by $u^\varepsilon(t, x, \omega) = \varepsilon\, v^\varepsilon\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \omega\right)$ then v^ε solves (with $a = 1$)

$$v_t^\varepsilon = \frac{1}{2}\Delta v^\varepsilon + H\left(\nabla u^\varepsilon, x, \omega\right), \quad (t,x) \in (0,\infty) \times \mathbb{R}^d;$$

$$v^\varepsilon|_{t=0} = \varepsilon^{-1} g(\varepsilon x) (= \theta \cdot x \text{ if } g(x) = \theta \cdot x), \quad x \in \mathbb{R}^d.$$

Thus if $g(x) = \theta \cdot x$ then v^ε does not depend on ε.

- Can it be that $\bar{H}$ is convex when the original H is not? Yes. For $a = 0$ see the cited above [ATY]. But not in general. This can be seen by comparison: solutions of the HJB equation with the following Hamiltonians (and the same g)
  - $|p|^2 - 2|p|$;
  - $|p|^2 - 2|p| - V(\omega)$, $0 \le V \le 1/2$;
  - $|p|^2 - 2|p| - 1/2$.

  are ordered. The first and the third are non-random so the effective Hamiltonians are the same as the original ones. By comparison, the effective Hamiltonian for the second one has to be in between which implies that it can not be convex.
- Question in a different direction: suppose that $\tilde{H}(p, \omega) = H_1(p) - \gamma H_2(p) - V(\omega)$, where $\mathbb{P}(V \not\equiv const) > 0$ and $\gamma > 0$. Is it true that there is $\gamma_c > 0$ such that for all $\gamma < \gamma_c$ the effective Hamiltonian is convex? The example from [ATY] suggests that this might be the case ($a = 0$). Can one prove an a priori convexity of $\bar{H}$ for small γ and use it to derive homogenization?

Case $\tilde{H}(p,\omega) = \frac{1}{2}|p|^2 - |p| + b(\omega)\cdot p - V(\omega)$

- *Hopf-Cole transformation:*
  $h^\varepsilon(t,x,\omega) = e^{v^\varepsilon(t,x,\omega)} = e^{\varepsilon^{-1}u(\varepsilon t,\varepsilon x,\omega)}$ so that
  $u^\varepsilon(t,x,\omega) = \varepsilon \log h^\varepsilon\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\omega\right)$, where h^ε solves
  $$h_t^\varepsilon = \frac{1}{2}\Delta h^\varepsilon - |\nabla h^\varepsilon| + b(\tau_x\omega)\cdot\nabla h^\varepsilon - V(\tau_x\omega)h^\varepsilon,$$
  $h^\varepsilon(0,x) = \exp(\varepsilon^{-1}g(\varepsilon x))(= e^{\theta\cdot x}$ if $g(x) = \theta\cdot x)$.
- Note that $|p| = \sup_{|q|\le 1}(p\cdot q)$, so
  $-|\nabla h^\varepsilon(t,x,\omega)| \le c(t,x,\omega)\cdot\nabla h^\varepsilon(t,x,\omega)$ for all $c(t,x,\omega)$: $\|c\|_\infty \le 1$. By comparison, $h^\varepsilon \le h_c^\varepsilon$, where h_c^ε solves the linear heat equation with drift $c(t,x,\omega) + b(\tau_x\omega)$ and potential V. In fact,
  $$h^\varepsilon(t,x,\omega) = \inf_{\|c\|_\infty\le 1} E_x \exp\left(\varepsilon^{-1}g(\varepsilon X(t)) - \int_0^t V(\tau_{X(s)}\omega)\,ds\right),$$
  where $dX(s) = (b(\tau_{X(s)}\omega) + c(s,X(s),\omega))\,ds + dW(s)$, $X(0) = x,\ 0\le s\le t$.

- *Problem:* Show the existence of

$$\lim_{\varepsilon \to 0} \varepsilon \inf_{\|c\|_\infty \le 1} \log E^\omega_{x/\varepsilon} \exp\left(\varepsilon^{-1} g(\varepsilon X(t/\varepsilon)) - \int_0^{t/\varepsilon} V(\tau_{X(s)}\omega)\, ds\right),$$

where $dX(s) = (b(\tau_{X(s)}\omega) + c(s, X(s), \omega))\, ds + dW(s)$, $X(0) = x/\varepsilon$. As the first step take $g(x) = \theta \cdot x$ and $x = 0$.

- RWRE version: Let the environment be stationary and ergodic with respect to the shifts on $\mathbb{Z}^d$. Assume uniform ellipticity: $\mathbb{P}\left(\min_{e \in \mathbb{Z}^d, |e|=1} \omega(0, e) \ge \kappa\right) = 1$ for some $\kappa > 0$. Let $d(x, \omega) := \sum_{e:|e|=1} \omega(x, e)e$ be the drift at x in environment ω. For each n and x let $\omega'(n, x, \cdot, \omega)$ be a probability measure on $\{e \in \mathbb{Z}^d : |e| = 1\}$ and $d'(n, x, \omega)$ be the corresponding drift. Fix a small $c > 0$. Let $P_0^{\omega,\omega'}(X_{n+1} = X_n + e | \mathcal{F}_n) = \omega'(n, X_n, e, \omega)$. For each $\theta \in \mathbb{R}^d$ show the existence of

$$\lim_{n \to \infty} \frac{1}{n} \inf_{\omega' : |d' - d| \le c} \log E_0^{\omega,\omega'} \exp\left(\theta \cdot X_n - \sum_{i=0}^{n-1} V(\tau_{X_i}\omega)\right).$$

- *Case* $d = 1$, $V \equiv 0$. To identify the candidate for $\bar{H}$ consider $g(x) = \theta \cdot x$ and the diffusion which starts at 0. Then we are interested in the limit of

$$u_\theta^\varepsilon(1,0) = \varepsilon \inf_{\|c\|_\infty \le 1} \log E_0^\omega \exp(\theta \cdot X(1/\varepsilon)).$$

In this case the infimum is attained at $c \equiv = \mathrm{sign}\theta$ if $\theta \neq 0$ ($u_0^\varepsilon(1,0) \equiv 0$) by the comparison of solutions of 1d SDE,

$$dX(s) = (b(\tau_{X(s)}\omega) + c(s, X(s), \omega))ds + dW(s), \quad X(0) = 0;$$
$$dY(s) = (b(\tau_{Y(s)}\omega) - \mathrm{sign}\,\theta)ds + dW(s), \quad Y(0) = 0.$$

We conclude that $\bar{H}(0) = 0$ and

$\bar{H}(\theta) = \lim_{\varepsilon \to 0} \varepsilon \log E_0^\omega(\theta \cdot X(1/\varepsilon))$, where $dX(s) = (b(\tau_{X(s)}\omega) - \mathrm{sign}\,\theta)\, ds + dW(s)$ for $\theta \neq 0$.

Let $\theta \neq 0$. Note that we have a convex homogenization problem for $\tilde{H}_\theta(p,\omega) = \frac{1}{2}|p|^2 + (b(\omega) - \operatorname{sign}\theta)p$. The limit exists and can be expressed as follows:

$$\text{(I)}\quad \bar{H}(\theta) = \sup_{(B,\Phi)\in\mathcal{E}} \mathbb{E}[(B(\omega)\theta - \tilde{L}_\theta(B(\omega),\omega))\Phi(\omega)],\ \text{where}$$

$$\mathcal{E} = \{(B,\Phi):\ \mathbb{E}(\Phi) = 1,\ \Phi,\ \Phi^{-1},\ \Phi',\Phi'',B$$
$$\text{are essentially bounded, } \frac{1}{2}\Phi'' = (B\Phi)' \text{ in the weak sense}\}$$

and $\tilde{L}_\theta(q,\omega) = \frac{1}{2}(q - b(\omega) + \operatorname{sign}\theta)^2$. Thus, for $\theta \neq 0$

$$\bar{H}(\theta) = \sup_{(B,\Phi)\in\mathcal{E}} \mathbb{E}[(B(\omega)\theta - \frac{1}{2}(B(\omega) - b(\omega) + \operatorname{sign}\theta)^2)\Phi(\omega)].$$

But can one see from this formula that if b is a constant (i.e. there is nothing to homogenize) then $\bar{H}(\theta) = \frac{1}{2}\theta^2 - |\theta| + b\theta$ as it should be (i.e. that $\Phi \equiv 1$ and constant B are optimal)?

There is another way to express the effective Hamiltonian:

$$\text{(II)} \quad \bar{H}(\theta) = \inf_{\Psi} \operatorname*{ess\,sup}_{\omega} [\tilde{H}_\theta(\theta + \nabla\Psi, \omega) + \frac{1}{2}\Delta\Psi],$$

where the infimum is taken over functions $\Psi(x, \omega)$ which have essentially bounded, stationary, mean zero gradient. Hence,

$$\begin{aligned}\bar{H}(\theta) &= \sup_{(B,\Phi)\in\mathcal{E}} \mathbb{E}[(B(\omega)\theta - \frac{1}{2}(B(\omega) - b(\omega) + \operatorname{sign}\theta)^2)\Phi(\omega)] \\ &= \inf_{\Psi} \operatorname*{ess\,sup}_{\omega} \left[\frac{1}{2}|\theta + \Psi'|^2 + (b(\omega) - \operatorname{sign}\theta)(\theta + \Psi') + \frac{1}{2}\Psi''\right].\end{aligned}$$

Now to complete the check in the constant drift case: putting $\Phi \equiv 1$, and optimizing over constant B we get

$\bar{H}(\theta) \geq \frac{1}{2}|\theta|^2 - |\theta| + b\theta$. Choosing $\Psi \equiv 1$ we see that

$\bar{H}(\theta) \leq \frac{1}{2}|\theta|^2 - |\theta| + b\theta$.

The "main equation" in the convex case

Set $\mathcal{A}_B = \frac{1}{2}\Delta + B(\omega)\cdot\nabla$ and $\mathbb{E}_\Phi(X) = \int_\Omega X\Phi\, d\mathbb{P}$. Formally, if $\nabla\Psi$ is essentially bounded, stationary, and $\mathbb{E}(\nabla\Psi) = 0$, then

$$
\begin{aligned}
\bar{H}(\theta) &= \sup_{(B,\Phi)\in\mathcal{E}} \mathbb{E}[(\theta\cdot B(\omega) - L(B(\omega),\omega))\Phi(\omega)] \\
&= \sup_\Phi \sup_B \inf_\Psi \mathbb{E}[(\theta\cdot B(\omega) - L(B(\omega),\omega) + (\mathcal{A}_B\Psi)(\omega))\Phi(\omega)] \\
&= \sup_\Phi \inf_\Psi \sup_B \mathbb{E}_\Phi[\theta\cdot B(\omega) - L(B(\omega),\omega) + \frac{1}{2}\Delta\Psi + \nabla\Psi\cdot B(\omega)] \\
&= \sup_\Phi \inf_\Psi \sup_B \mathbb{E}_\Phi[(\theta+\nabla\Psi)\cdot B(\omega) - L(B(\omega),\omega) + \frac{1}{2}\Delta\Psi] \\
&= \sup_\Phi \inf_\Psi \mathbb{E}_\Phi[H(\theta+\nabla\Psi,\omega) + \frac{1}{2}\Delta\Psi] \\
&= \inf_\Psi \sup_\Phi \mathbb{E}_\Phi[H(\theta+\nabla\Psi,\omega) + \frac{1}{2}\Delta\Psi] \\
&= \inf_\Psi \operatorname*{ess\,sup}_\omega [H(\theta+\nabla\Psi,\omega) + \frac{1}{2}\Delta\Psi].
\end{aligned}
$$

Observe that $\inf_\Psi \mathbb{E}[(\mathcal{A}_B\Psi)\Phi] = -\infty$ unless $\Phi\, d\mathbb{P}$ is an invariant measure for $\mathcal{A}_B$, in which case it is 0.

Return to $H = H_1 - H_2$. There is a representation formula for solutions of (HJB) in the differential games theory.[2] But the formula does not seem to be usable.

One could start with "better" representations for solutions of $u_t = \bar{H}(\nabla u)$. Recall formulas for solutions of $u_t + H(\nabla u) = 0$. When H is convex and $g \in \mathrm{UC}$ we have a Hopf-Lax formula[3]

$$u(t,x) = \inf_z \sup_y [g(z) + y \cdot (x - z) - tH(y)]$$
$$= \inf_y [g(z) + tL((x-z)/t)] = \min_y [g(y - tk) + tL(k)].$$

If $H \in C$ and g is convex and Lipschitz then the second Hopf formula states

$$u(t,x) = \sup_y \inf_z [g(z) + y \cdot (x - z) - tH(y)]$$
$$= \max_y [x \cdot y - g^*(y) - tH(y)].$$

[2] H as above satisfies *Isaacs condition*, and the game has a value.

[3] M. Bardi, S. Faggian, 1998, and references therein.

M. Bardi, S. Faggian, 1998, have shown, in particular, that for $H = H_1 - H_2$

$$\max_m \min_k f(t,x,k,m) \le u(t,x) \le \min_k \max_m f(t,x,k,m),$$

where $f(t,x,k,m) = g(x - t(k-m)) + tL_1(k) - tL_2(m)$. The equalities hold for linear initial data for all t. They also gave examples where the inequalities are strict. All examples are non-coercive but one can give a coercive example ($H(p) = |p|^2/2 - |p|$, $g(x) = -2|x|$) when for all $t > 0$ upper and lower bounds do not match.
Also note that the decomposition $H = H_1 - H_2$ is not unique. Can this be used to improve the bounds?
L.C. Evans, 2014, proposed another representation formula (and more) for solutions of HJ equations with general H by using the adjoint method.