Invariant distributions and scaling limits for some diffusions in time-varying random environments

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1. Time-inhomogeneous Brox's diffusions and related processes
   - General framework
   - Time-homogenous situation : Brox's diffusion
   - A deterministic case : singular time-inhomogeneous SDEs
2. Study of theses diffusions
   - Existence and uniqueness of solutions
   - Scaling limits and invariant distributions
   - Sketch of the proofs

Settings and assumptions

Diffusion and random environement : $\{B_t : t \geq 0\}$ a standard Brownian motion independant of the Wiener space $(\Theta, \mathscr{B}, \mathscr{W})$ where

$$\Theta := \left\{ \theta \in \mathrm{C}(\mathbb{R};\mathbb{R}) : \theta(0) = 0 \text{ et } \sup_{x \in \mathbb{R}} \frac{|\theta(x)|}{1+x^2} < \infty \right\}.$$

Time-inhomogenous Brox's diffusions

$$\mathrm{d}X_t = \mathrm{d}B_t - \frac{1}{2}\frac{\theta'(X_t)}{t^\beta}\,\mathrm{d}t, \quad t \geq 1, \quad X_1 = x \in \mathbb{R}.$$

$\hookrightarrow$ Meaning (distributional drift) ? Long time behaviour ?

Model and related objects

Time-homogenous Brox's diffusion

$$\mathrm{d}X_t = \mathrm{d}B_t - \frac{1}{2}\theta'(X_t)\,\mathrm{d}t \quad (\beta = 0).$$

Scale function and speed measure :

$$S_\theta(x) := \int_0^x e^{\theta(y)}\,\mathrm{d}y \quad \text{and} \quad m_\theta(\mathrm{d}x) := e^{-\theta(x)}\mathrm{d}x.$$

Infinitesimal generator :

$$\mathscr{L}_\theta := \frac{1}{2}\frac{\mathrm{d}}{m_\theta(\mathrm{d}x)}\frac{\mathrm{d}}{\mathrm{d}S_\theta(x)} = \frac{1}{2}e^{\theta(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-\theta(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right).$$

Ito-McKean's construction

Associated stochastic differential equation (SDE) without drift :

$$\mathrm{d}Y_t = S'_\theta \circ S_\theta^{-1}(Y_t)\,\mathrm{d}B_t \quad \text{where} \quad Y_t := S_\theta(X_t).$$

Proposition (Brownian motion changed in space and time)

There exists a brownian motion B such that

$$X_t = S_\theta^{-1}(Y_t) = S_\theta^{-1}(B_{T^{-1}(t)}),$$

with

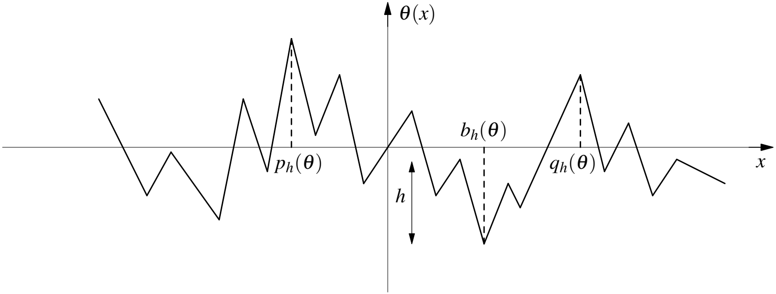
$$T(t) := \int_0^t \exp\left(-2\theta(S_\theta^{-1}(B_u))\right)\mathrm{d}u.$$

Settings

Annealed version and scaling transformation :

$$\widehat{F} := \int_{\Theta} F(\theta)\,\mathscr{W}(\mathrm{d}\theta) \quad \text{and} \quad S_\lambda \theta := \frac{\theta(\lambda \star)}{\sqrt{\lambda}}.$$

Standard Brownian depression of depth h :



Localisation phenomenom and sub-diffusivity

Theorem (Schumacher, Brox, 84-86)

Quenched convergence :

$$\frac{X_t}{(\log t)^2} - b_1\left(S_{(\log t)^2}\theta\right) = \frac{X_t - b_{\log t}(\theta)}{(\log t)^2} \xrightarrow[t\to\infty]{\mathbb{P}_\theta} 0 \quad \mathscr{W}\text{-a.s.}.$$

Annealed convergence :

$$\frac{X_t}{(\log t)^2} \xrightarrow[t\to\infty]{\widehat{\mathbb{P}}} \hat{b}_1 .$$

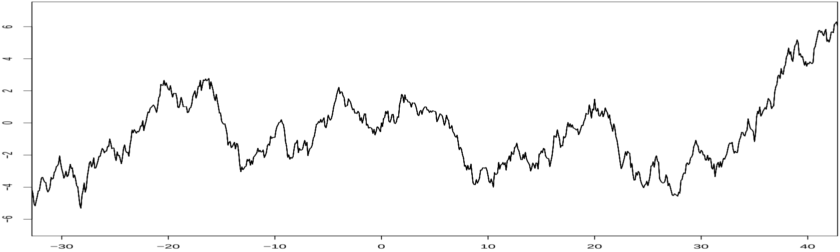
Sketch of the proof :

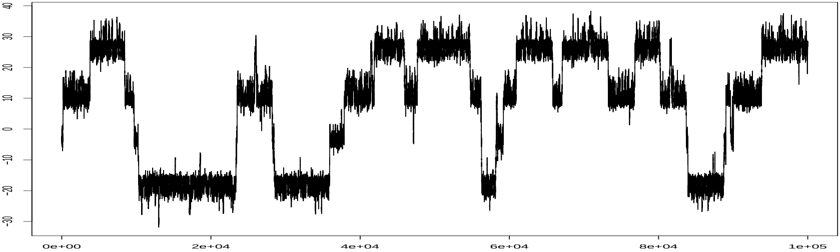
$\longrightarrow$ Ito-McKean's construction.

$\longrightarrow$ Local time of brownian motion.

$\longrightarrow$ Scaling property of the potential θ and the Brownian diffusion.

Numerical illustration





A deterministic counterpart

$$\mathrm{d}X_t = \mathrm{d}B_t - \frac{1}{2}\frac{W'(X_t)}{t^\beta}\,\mathrm{d}t \quad (\theta(x) \equiv W(x) \equiv |x|^{1/2}).$$

Deterministic scaling / Random scaling :

$$S_\lambda W \equiv W \quad / \quad S_\lambda \theta \stackrel{\mathscr{W}}{=} \theta.$$

Theorem (M. Gradinaru, Y. O., 13)

$$\frac{X_t}{\sqrt{t}} \xrightarrow[t\to\infty]{\mathscr{L}} \begin{cases} \gamma := \mathscr{N}(0,1), & \beta > 1/4. \\ \mu := c\, e^{-\left(\frac{x^2}{2}+W(x)\right)}\mathrm{d}x, & \beta = 1/4. \end{cases} \qquad \textit{(Diffusivity)}$$

$$\frac{X_t}{t^{2\beta}} \xrightarrow[t\to\infty]{\mathscr{L}} \pi := c_s\, e^{-W(x)}\,\mathrm{d}x, \quad \beta < 1/4. \quad \textit{(Sub-diffusivity)}$$

General deterministic model

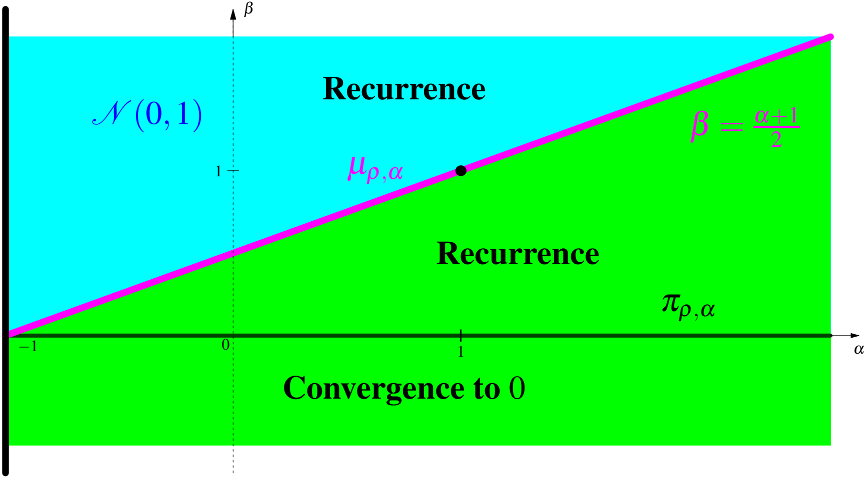
SDEs with homogeneous drift

$$\mathrm{d}X_t = \mathrm{d}B_t + \rho \operatorname{sgn}(X_t) \frac{|X_t|^\alpha}{t^\beta} \mathrm{d}t, \quad X_1 = x, \quad t \geq 1.$$

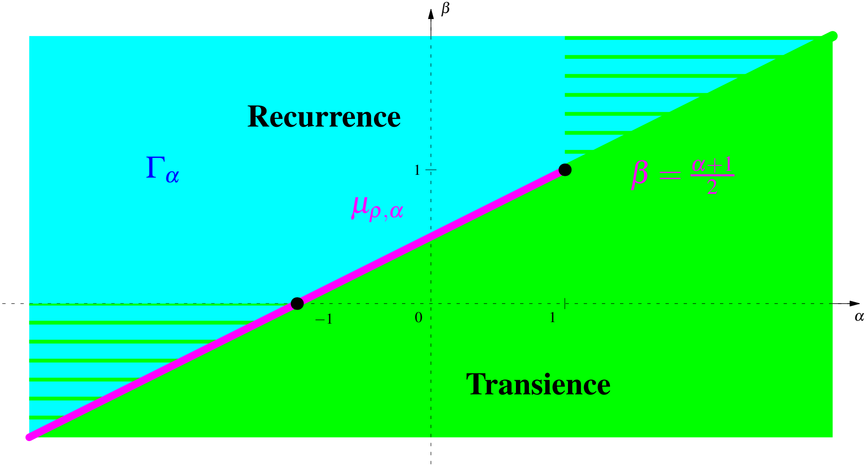
Remarks :

- $\longrightarrow$ Deterministic counterpart : $\alpha = -1/2$.
- $\longrightarrow$ Singular SDEs for $\alpha < 0$.
- $\longrightarrow$ Explosion phenomenon for $\alpha > 1$.
- $\longrightarrow$ Attractive case : $\rho < 0$.
- $\longrightarrow$ Repulsive case : $\rho > 0$.

Attractive case : $\rho < 0$



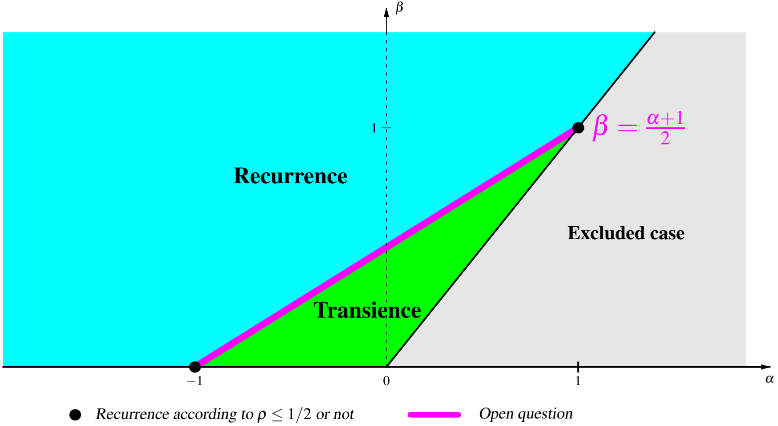
Repulsive case : $\rho > 0$



- *Recurrence according to* $\rho \le 1/2$ *or not*

Menshikov-Volkov model

$$\mathbb{E}[X_{n+1} - X_n | X_n = x] \simeq \rho \frac{x^\alpha}{n^\beta}, \quad \rho > 0.$$



Scaling transformations and heuristic

Exponential transformation :

$$Z_t := \frac{X(e^t)}{\sqrt{e^t}} \quad \xrightarrow{\text{Ito}} \quad \mathrm{d}Z_t = \mathrm{d}B_t - \frac{1}{2} Z_t \,\mathrm{d}t + e^{-(\beta - \frac{1}{4})t} W'(Z_t)\,\mathrm{d}t.$$

Critical case $\beta = 1/4$: μ-ergodic diffusion,

$$\mathrm{d}Z_t = \mathrm{d}B_t - \frac{1}{2} Z_t \,\mathrm{d}t + W'(Z_t)\,\mathrm{d}t.$$

Super-critical case $\beta > 1/4$:

$$\mathrm{d}Z_t = \mathrm{d}B_t - \frac{1}{2} Z_t \,\mathrm{d}t + e^{-(\beta - \frac{1}{4})t} W'(Z_t)\,\mathrm{d}t \quad (\ll 1).$$

$\longrightarrow$ Asymptotic γ-ergodic Ornstein-Uhlenbeck (OU).

$\longrightarrow$ Bounded in probability (by comparison theorem).

Key lemma

Let Z, H be the unique solutions of SDEs with continuous coefficients

$$\mathrm{d}Z_s = \sigma(s, Z_s)\,\mathrm{d}B_s + d(s, Z_s)\,\mathrm{d}s \quad \text{and} \quad \mathrm{d}H_s = \sigma(H_s)\,\mathrm{d}B_s + d(H_s)\,\mathrm{d}s.$$

Assume that (Z, H) is asymptotically homogeneous and μ-ergodic, i.e.

$$\begin{cases} \lim_{s\to+\infty} \sigma(s,z) = \sigma(z) \\ \lim_{s\to+\infty} d(s,z) = d(z) \end{cases} \quad (UC) \quad \text{and} \quad \lim_{t\to+\infty} H_t \overset{\mathscr{L}}{=} \mu,$$

and Z bounded in probability, i.e. for all $\varepsilon > 0$, there exists $r > 0$ s.t.

$$\sup_{s\geq 0} \mathbb{P}(|Z_s| \geq r) < \varepsilon.$$

Then

$$\lim_{t\to+\infty} Z_t \overset{\mathscr{L}}{=} \mu.$$

Scaling transformation and heuristic

General transformation :

$$Z_t := \frac{X(\varphi(t))}{\sqrt{\varphi'(t)}} \to \mathrm{d}Z_t = \mathrm{d}B_t + \rho\, \frac{\varphi'(t)^{\frac{\alpha+1}{2}}}{\varphi(t)^{\beta}}\, \mathrm{sgn}(Z_t)\, |Z_t|^{\alpha}\, \mathrm{d}t - \frac{1}{2}\frac{\varphi''(t)}{\varphi'(t)} Z_t\, \mathrm{d}t.$$

Under-critical case $\beta < 1/4$:

$$\varphi' = \varphi^{\gamma}, \quad \varphi(0) = 1, \quad \gamma := \frac{2\beta}{\alpha+1} \neq 1.$$

$$\mathrm{d}Z_t = \mathrm{d}B_t + \rho\, \mathrm{sgn}(Z_t)\, |Z_t|^{\alpha}\, \mathrm{d}t - \frac{\gamma}{2}\varphi_{\gamma}^{\gamma-1}(t)\, Z_t\, \mathrm{d}t \quad (\ll 1).$$

$\longrightarrow$ Asymptotic time-homogeneous $\pi_{\rho,\alpha}$-diffusion.

$\longrightarrow$ Bounded in probability (by comparison theorem).

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Time-inhomogeneous distributional drift

Let V be a continuous potential and consider

$$\mathrm{d}Z_t = \mathrm{d}B_t - \frac{1}{2}\partial_x V(t, Z_t)\,\mathrm{d}t.$$

Solutions : solution of the martingale problem associated to

$$\mathscr{L}_t := \frac{1}{2} e^{V(t,x)} \frac{\partial}{\partial x}\left(e^{-V(t,x)} \frac{\partial}{\partial x}\right),$$

with domain

$$\mathscr{D}(\mathscr{L}) := \left\{ F \in \mathrm{C}^1 : e^{-V(t,x)} \partial_x F(t,x) \in \mathrm{C}^1 \right\}.$$

$\hookrightarrow$ Existence ? Unicity ? Non-explosion ?

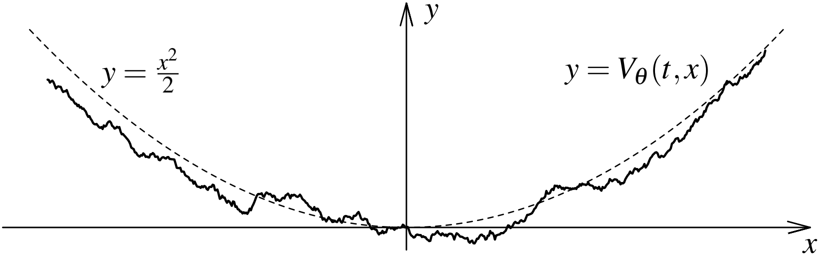
Associated random perturbation of the OU process

Exponential scaling transformation :

$$\frac{\theta(x)}{t^\beta} \xrightarrow{Z_t = \frac{X(e^t)}{\sqrt{e^t}}} V_\theta(t,x) := \frac{x^2}{2} + e^{-(\beta - \frac{1}{4})t} T_t\theta(x),$$

where

$$T_t\theta(x) := S_{e^{t/2}}\theta(x) = \frac{\theta(e^{t/2}x)}{e^{t/4}}.$$



Existence

Theorem (Offret, 14)

For all $\theta \in \Theta$ there exists a unique global solution Z (resp. X) to

$$\mathrm{d}Z_t = \mathrm{d}B_t - \frac{1}{2}\partial_x V_\theta(t, Z_t)\,\mathrm{d}t \quad \left(\textit{resp.}\ \ \mathrm{d}X_t = \mathrm{d}B_t - \frac{1}{2}\frac{\theta'(X_t)}{t^\beta}\,\mathrm{d}t\right),$$

which satisfies the lower local Aronson estimate, i.e. for any $T, R > 0$ there exists $m > 0$ such that for all $0 \leq s \leq t \leq T$ and $|x|, |y| \leq R$,

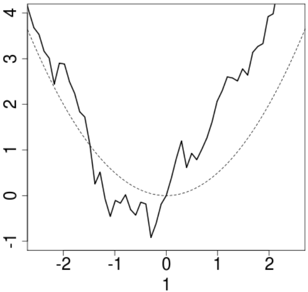
$$p_\theta(s, x; t, y) \geq \frac{1}{\sqrt{m(t-s)}} \exp\left(-m\frac{|y-x|^2}{t-s}\right),$$

where p_θ denotes the transition probability density.

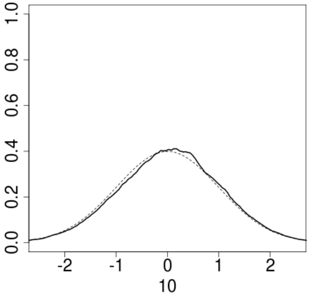
Scaling limits : $\beta > 1/4$

Theorem (CLT, Y. O., 14)

$$\lim_{t\to\infty} Z_t \overset{\mathscr{L}}{=} \mathscr{N}(0,1) \quad \mathscr{W}\text{-a.s.} \quad \left(\lim_{t\to\infty} \frac{X_t}{\sqrt{t}} \overset{\mathscr{L}}{=} \mathscr{N}(0,1)\right).$$



$$V_\theta(t,x) \xrightarrow[t\to\infty]{} \frac{x^2}{2}$$



$$\mathscr{L}(Z_t^\theta) \xrightarrow[t\to\infty]{} \mathscr{N}(0,1)$$

Scaling limits : $\beta = 1/4$

Quenched time-inhomogeneous semi-group and annealed version :

$$\nu P_t(\theta) F := \mathbb{E}_\theta[F(Z_t) | Z_0 \sim \nu] \quad \text{and} \quad \nu \widehat{P}_t F := \int_\Theta \nu P_t(\theta) F \, \mathscr{W}(\mathrm{d}\theta).$$

Weighted total variation norm : for any $F : \mathbb{R} \longrightarrow (0, \infty)$ we set

$$\|\nu\|_F := \sup_{|f| \leq F} \left| \int f \mathrm{d}\nu \right|.$$

In the following we shall consider $F \in \{U, V\}$ where

$$U(x) = \exp\left(\alpha \frac{x^2}{2}\right) \quad \text{and} \quad V(x) = \exp(\alpha |x|) \quad \text{with} \quad \alpha \in [0, 1).$$

Scaling limits : $\beta = 1/4$

Theorem (stationnary random flow, Y. O., 14)

There exists $\lambda > 0$ and a random probability measure μ_θ such that

$$\mu_\theta P_t(\theta) = \mu_{T_t\theta} \quad \text{and} \quad \begin{cases} \|\nu P_t(\theta) - \mu_{T_t\theta}\|_U \leq c_{\theta,\nu}\, e^{-\lambda t}, \\ \lim_{t\to\infty} \|\nu \widehat{P}_t - \hat{\mu}\|_V = 0, \end{cases} \qquad \mathscr{W}\text{-a.s.,}$$

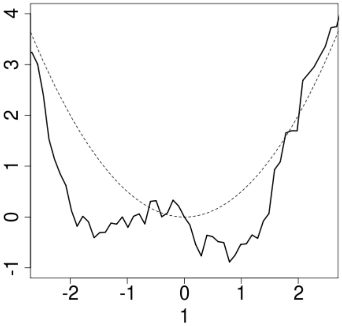
Therefore

$$\left\| \mathscr{L}\left(\frac{X_t}{\sqrt{t}} \middle| \theta, X_1 \sim \nu \right) - \mu_{S_{\sqrt{t}}\theta} \right\|_U \leq \frac{c_{\theta,\nu}}{t^\lambda},$$

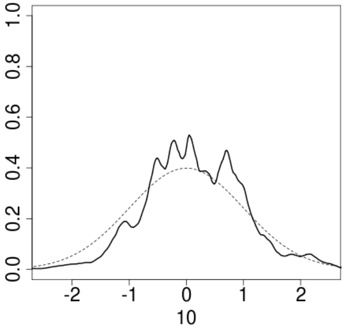
and

$$\lim_{t\to\infty} \left\| \mathscr{L}\left(\frac{X_t}{\sqrt{t}} \middle| X_1 \sim \nu \right) - \hat{\mu} \right\|_V = 0.$$

Numerical illustrations



$$V_\theta(t,x) = \frac{x^2}{2} + T_t\theta(x)$$



$$\mathscr{L}(Z_t^\theta) - \mu_{T_t\theta} \xrightarrow[t\to\infty]{} 0$$

Existence and uniqueness of global solutions

Pseudo-scale function : true SDE with continuous coefficients,

$$S_\theta(t,x) := \int_0^x e^{V_\theta(t,y)}\,\mathrm{d}y \in \mathrm{C}^1,$$

$$Y_t := S_\theta(t,X_t), \quad \mathrm{d}Y_t = \partial_x S_\theta(t, S_\theta^{-1}(t,Y_t))\,\mathrm{d}B_t + \partial_t S_\theta(t, S_\theta^{-1}(t,Y_t))\,\mathrm{d}t.$$

$\hookrightarrow$ Non-explosion ?

$$V_\theta(t,x) \xrightarrow{\text{Girsanov}} \underbrace{a\frac{x^2}{2} + \mathscr{O}(x^2)}_{\text{sufficiently confining } a \gg 1} \xrightarrow{\text{Lyapunov}} \mathscr{A}_\theta U_\theta(t,x) \le \lambda\, U_\theta(t,x),$$

where

$$U_\theta(t,x) :\simeq \int_0^x \exp\left(e^{-(\beta-1/4)t}\, T_t\theta(y)\right) U'(y)\,\mathrm{d}y.$$

Underlying ergodic random dynamical system (RDS)

Ergodic dynamical system $(\Theta, \mathscr{B}, \mathscr{W}, (T_t)_{t\in\mathbb{R}})$:

$\longrightarrow$ Semi-group property :

$$T_{s+t}\theta(x) = \frac{\theta(e^{s/2}e^{t/2}x)}{e^{s/4}e^{t/4}} = T_s T_t \theta(x).$$

$\longrightarrow$ The Wiener mesure $\mathscr{W}$ is invariant under (T_t) by scaling property.

$\longrightarrow$ The dynamical system (T_t) is $\mathscr{W}$-ergodic :

$$t \longmapsto T_t\theta(x) \quad \text{stationnary ergodic OU.}$$

Skew cocycle property for the time-inhomogeneous semi-group :

$$P_{s+t}(\theta) := P_t(\theta)P_s(e^{-(\beta-1/4)t}T_t\theta).$$

$\hookrightarrow$ Convergence ? We need compacity...

Foster-Lyapunov functions

Lemma (contractivity)

For $F \in \{U, V\}$ and $\rho < 1$, there exists B_F such that

$$P_1(\theta)F(x) \leq \rho F(x) + B_F(\theta), \quad \mathscr{W}-a.s.,$$

and for all $s \leq 1$,

$$P_s(\theta)F(x) \leq \rho F(x) + B_F(\theta) \quad \mathscr{W}-a.s.,$$

with

$$\int_\Theta \log(B_U(\theta))\mathscr{W}(\mathrm{d}\theta) < \infty \quad \text{and} \quad \int_\Theta B_V(\theta)\mathscr{W}(\mathrm{d}\theta) < \infty.$$

Main difficulty : U, V independant on θ and do not belong to $\mathscr{D}(\mathscr{L}_\theta)$.

Uniform approximations by Lyapunov functions

Proposition

For $F \in \{U, V\}$ and $\varepsilon > 0$ there exists $F_\theta \in D(\mathscr{L}_\theta)$ such that

$$\|F_\theta - F\|_\infty \leq \varepsilon \quad \text{and} \quad \mathscr{L}_\theta F_\theta(t,x) \leq -\lambda F_\theta(t,x) + B_F(\theta) \quad \mathscr{W}-a.s..$$

Sketch of the proof :

$\longrightarrow$ Uniform polygonal approximations of θ (γ-hölder continuity).

$\longrightarrow$ Gaussian bounds (Fernique theorem).

$$\Longrightarrow \quad \sup_{|x|,|y| \leq n} |\theta(y) - \theta(x)| \leq H_\gamma(\theta) L(n) |y - x|^\gamma,$$

where

$$L(x) := \sqrt{1 + \ln(1 + |x|)} \quad \text{and} \quad \int_\Theta e^{\delta H_\gamma^2(\theta)} \mathscr{W}(\mathrm{d}\theta) < \infty.$$

Scaling limits : $\beta = 1/4$

Random product of Markov operators (RDS)

$$P_n(\theta) = P(\theta)P(T\theta)\cdots P(T^{n-1}\theta)$$

$\hookrightarrow$ Random Perron-Frobenius theorem ?

Known for stochastic matrices (Arnold) and compact space (Kifer).

Coupling methods (Douc-Moulines-Rosenthal, 2004)

$\longrightarrow$ Foster-Lyapunov functions.

$\longrightarrow$ Petite sets (from lower local Aronson estimate) :

$$\inf_{|x|\leq R} P_\theta(x,\star) \geq \varepsilon_\theta \nu_\theta(\star).$$

Scaling limits : $\beta > 1/4$

Associated SDE : $Y_t = S_\theta(t, Z_t)$

$\longrightarrow$ Asymptotically time-homogeneous and $S_\star\gamma$-ergodic with

$$S(x) := \lim_{t\to\infty} S_\theta(t,x) = \int_0^x e^{\lim_{t\to\infty} V_\theta(t,y)}\,\mathrm{d}y = \int_0^x e^{\frac{y^2}{2}}\,\mathrm{d}y.$$

$\longrightarrow$ Bounded in probability (Foster-Lyapunov functions).

Questions

Scaling for $\beta < 1/4$?

$\longrightarrow$ Stronger localization phenomenon is attempded.

$\longrightarrow$ Work in progress with R. Diel, Nice Sophia Antipolis.

General self-similar environment ?

$\longrightarrow$ The proof works for fractional Brownian motion (Gaussian,Hölder).

$\longrightarrow$ It needs to be deeply changed for other.

General self-similar diffusion term ?

$\longrightarrow$ Work in progress with M. Gradinaru, Rennes, for stable Lévy processes in the deterministic situation.

Thanks for your attention