Hydrodynamic limits for directed traps and systems of independent RWRE

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Joint work with Milton Jara

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Independent RWRE

Environment $\omega = \{\omega_x\}_{x\in\mathbb{Z}}$ i.i.d.

Random walk particles $\{X_\cdot^{x,j}\}_{x\in\mathbb{Z},\, j\geq 1}$

- independently evolve in common environment ω.

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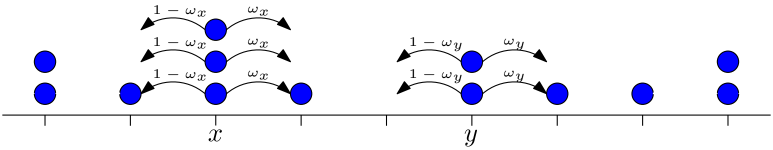

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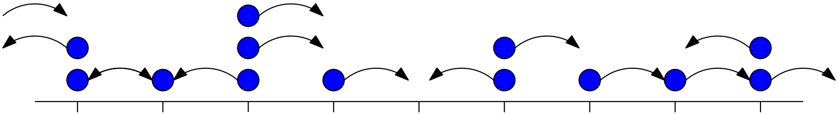

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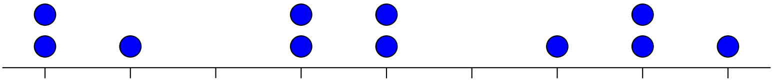

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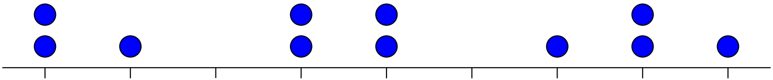

Configuration of particles

- Initial configuration: $\eta_0 = \{\eta_0(x)\}_{x \in \mathbb{Z}}$.
- $\eta_n(x) = \#$(particles at x after n steps).

Hydrodynamic limit - asymmetric SRW

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Hydrodynamic Limit (Asymmetric SRW)

If $\{\eta_0^n\}_{n\geq 1}$ is a sequence of initial configurations such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x\in\mathbb{Z}} \eta_0^n(x)\phi(x/n) = \int u(x)\phi(x)\,dx, \quad \forall \phi \in \mathcal{C}_0,$$

then for any $t \geq 0$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x\in\mathbb{Z}} \eta_{tn}^n(x)\phi(x/n) = \int u(x-\mathrm{v}t)\phi(x)\,dx, \quad \forall \phi \in \mathcal{C}_0,$$

Note: scale time and space by n

Understanding the hydrodynamic limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x\in\mathbb{Z}} \eta^n_{tn}(x)\phi(x/n) = \int u(x-\mathrm{v}t)\phi(x)\,dx, \quad \forall\phi \in \mathcal{C}_0,$$

- (Asymptotic) empirical density of particles

Initial	$u(x)\,dx$
Time tn	$u(x-\mathrm{v}t)\,dx$.

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- Example of initial configurations

$$\eta^n_0 \sim \bigotimes_{x\in\mathbb{Z}} \text{Poisson}(u(x/n)).$$

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$$\eta^n_0 \sim \bigotimes_{x\in\mathbb{Z}} \text{Poisson}(u(x/n)).$$

- $u(t,x) = u(x - \mathrm{v}t)$ solves the PDE

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = -\mathrm{v}\frac{\partial}{\partial x} u(t,x) \\ u(0,x) \equiv u(x) \end{cases}$$

Hydrodynamic limit - symmetric SRW

$$\{t \mapsto X_{tn^2}/n\} \Longrightarrow \text{Brownian Motion} \qquad (\omega_x \equiv p = 1/2)$$

Hydrodynamic Limit (Symmetric SRW)

If $\{\eta_0^n\}_{n\geq 1}$ *is a sequence of initial configurations such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x\in\mathbb{Z}} \eta_0^n(x)\phi(x/n) = \int u(x)\phi(x)\,dx, \quad \forall \phi \in \mathcal{C}_0,$$

then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x\in\mathbb{Z}} \eta_{tn^2}^n(x)\phi(x/n) = \int u(t,x)\phi(x)\,dx, \quad \forall \phi \in \mathcal{C}_0,\ t > 0,$$

where $u(t,x)$ *is a solution to the heat equation*

$$\frac{\partial}{\partial t} u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} u(t,x) \qquad u(0,x) \equiv u(x).$$

RWRE basics - Recurrence/Transience and Speed

$$\rho_x = \frac{1-\omega_x}{\omega_x}, \qquad x \in \mathbb{Z}.$$

Theorem (Solomon '75)

1. *Recurrence/transience is determined by* $E[\log \rho_0]$.
   - $\mathbb{P}(X_n \to \infty) = 1 \iff E[\log \rho_0] < 0$.
   - $\mathbb{P}(X_n \to -\infty) = 1 \iff E[\log \rho_0] > 0$.
   - $\mathbb{P}(X_n \text{ is recurrent}) = 1 \iff E[\log \rho_0] = 0$.

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   - $\mathbb{P}(X_n \text{ is recurrent}) = 1 \iff E[\log \rho_0] = 0$.
2. *If* $E[\log \rho_0] < 0$, *then*

$$\lim_{n\to\infty} \frac{X_n}{n} = \mathrm{v} := \begin{cases} \frac{1-E[\rho_0]}{1+E[\rho_0]} & \text{if } E[\rho_0] < 1 \\ 0 & \text{if } E[\rho_0] \geq 1, \end{cases} \qquad \mathbb{P}\text{-a.s.}$$

RWRE basics - Limiting distributions

Scaling parameter $\kappa > 0$ defined by

$$E[\rho_0^\kappa] = 1.$$

Theorem (Kesten, Kozlov, Spitzer '75)

Assuming $E[\log \rho_0] < 0$ *and some other technical assumptions*

$$\kappa \in (0,1) \qquad \lim_{n\to\infty} \mathbb{P}\left(\frac{X_n}{n^\kappa} \leq x \right) = 1 - L_\kappa(x^{-1/\kappa})$$

$$\kappa \in (1,2) \qquad \lim_{n\to\infty} \mathbb{P}\left(\frac{X_n - n\mathrm{v}}{n^{1/\kappa}} \leq x \right) = 1 - L_\kappa(-x)$$

$$\kappa > 2 \qquad \lim_{n\to\infty} \mathbb{P}\left(\frac{X_n - n\mathrm{v}}{a\sqrt{n}} \leq x \right) = \Phi(x)$$

RWRE basics - Quenched vs. Annealed

Quenched law P_ω - environment ω fixed.
Averaged law $\mathbb{P}$ - averaged over all environments.

$$\mathbb{P}(\cdot) = E[P_\omega(\cdot)]$$

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Theorem (Goldsheid '07, P. '08, P. and Zeitouni '08)

- *If* $\kappa > 2$ *then*

$$\lim_{n\to\infty} P_\omega\left(\frac{X_n - n\mathrm{v} + Z_n(\omega)}{b\sqrt{n}} \leq x \right) = \Phi(x), \quad P\text{-a.s.}$$

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- *If* $\kappa \in (0,2)$, *then for* P*-a.e. environment* ω *there is no quenched limiting distribution.*

If $\kappa \in (0,1)$ $\quad \lim_{n\to\infty} P_\omega\left(\frac{X_n}{n^\kappa} \leq x\right)$ does not converge.

RWRE hydrodynamic limit - ballistic case

Theorem (P. '10)

Assume that $E[\log \rho_0] < 0$ *and* $\kappa > 1$. *If* $\{\eta_0^n\}_{n \geq 1}$ *is a sequence of initial configurations such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x \in \mathbb{Z}} \eta_0^n(x) \phi(x/n) = \int u(x) \phi(x) \, dx, \quad \forall \phi \in \mathcal{C}_0,$$

then for any $t \geq 0$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x \in \mathbb{Z}} \eta_{tn}^n(x) \phi(x/n) = \int u(x - \mathrm{v}t) \phi(x) \, dx, \quad \forall \phi \in \mathcal{C}_0,$$

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Admissible initial conditions

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x\in\mathbb{Z}} \eta_0^n(x)\phi(x/n) = \int u(x)\phi(x)\,dx, \quad \forall \phi \in \mathcal{C}_0,$$

- $\eta_0^n \sim \bigotimes_{x\in\mathbb{Z}}$ Poisson$(u(x/n))$.

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- $\eta_0^n \sim \bigotimes_{x\in\mathbb{Z}} \text{Poisson}(u(x/n))$.
- $\eta_0^n \sim \bigotimes_{x\in\mathbb{Z}} \text{Poisson}(u(x/n)g_\omega(x))$, (locally stationary)

$$g_\omega(x) = E_\omega^x\left[\sum_{n=0}^{\infty} \mathbf{1}_{\{X_n=x\}}\right] = (1+\rho_x)\left(1+\rho_{x+1}+\rho_{x+1}\rho_{x+2}+\cdots\right)$$

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Note: $E[g_\omega(x)] < \infty \iff E[\rho_0] < 1 \iff \kappa > 1$

RWRE hydrodynamic limit - zero speed case

Question What hydrodynamic limit to expect when $\kappa \in (0,1)$?

- Scaling: scale time by $n^{1/\kappa}$ and space by n.

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- No stationary distributions with finite density.

$$\text{If} \quad \eta_0 \sim \bigotimes_{x \in \mathbb{Z}} \text{Poisson}(g_\omega(x)) \quad \text{then} \quad \mathbb{E}[\eta_0(x)] = E_P[g_\omega(x)] = \infty.$$

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- Locally stationary initial configurations are not "smooth."
  If $\eta_0^n \sim \bigotimes_{x\in\mathbb{Z}} \text{Poisson}(u(x/n)g_\omega(x))$

$$\frac{1}{n^{1/\kappa}} \sum_{x\in\mathbb{Z}} \eta_0^n(x)\phi(x/n) \quad \Longrightarrow \quad \int u(x)\phi(x)\,\sigma(dx), \quad \forall \phi \in \mathcal{C}_0,$$

  where σ is a κ-stable subordinator.

RWRE hydrodynamic limit - zero speed case

Theorem (Jara and P. '14)

Assume that $E[\log \rho_0] < 0$ *and* $\kappa \in (0,1)$ *(+ technical conditions).*
If $u \in \mathcal{C}_0$ *and* $\eta_0^n \sim \bigotimes_{x\in\mathbb{Z}} \text{Poisson}(u(x/n)g_\omega(x))$*, then for any* $t \geq 0$

$$\frac{1}{n^{1/\kappa}} \sum_{x\in\mathbb{Z}} \eta^n_{tn^{1/\kappa}}(x)\phi(x/n) \quad \Longrightarrow \quad \int u(t,x)\phi(x)\,\sigma(dx), \quad \forall \phi \in \mathcal{C}_0,$$

where σ *is a* κ*-stable subordinator and* $u(t,x)$ *satisfies*

$$\begin{cases} u(0,x) \equiv u(x) \\ \frac{\partial}{\partial t} u(t,x) = -\frac{d}{d\sigma} u(t,x) \quad \forall t > 0. \end{cases}$$

Interpreting the PDE

$$\begin{cases} u(0,x) \equiv u(x) \\ \frac{\partial}{\partial t} u(t,x) = -\frac{d}{d\sigma} u(t,x) \quad \forall t > 0. \end{cases}$$

- For any point x where $\sigma(x)$ is discontinuous,

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- If $u(x)$ is of bounded variation, then also

$$u(t,b) - u(t,a) = -\int_{(a,b]} \frac{\partial}{\partial t} u(t,x)\, \sigma(dx) \qquad \forall t > 0.$$

Related Results

Systems of independent particles in a random environment

- RW on random conductances (Faggionato, Jara, Landim '09)

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial}{\partial x}\frac{d}{d\sigma}u(t,x)$$

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$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial}{\partial x}\frac{d}{d\sigma}u(t,x)$$

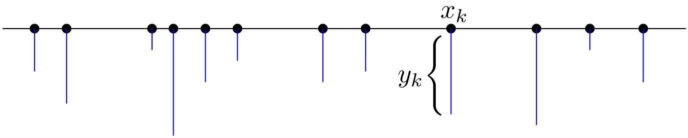
- 1-dim Bouchaud trap model (Jara, Landim, Teixeira '11)

$$\frac{\partial}{\partial t}u(t,x) = \frac{d}{d\sigma}\frac{\partial}{\partial x}u(t,x)$$

Directed Trap Process

Trap environment $W = \sum_k \delta_{(x_k, y_k)}$.

- x_k - spatial location of trap
- y_k - "depth" of the trap



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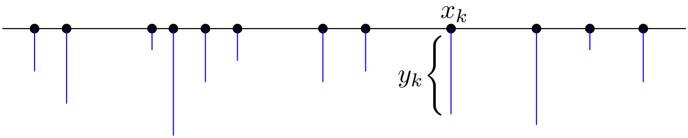

Directed trap process $Z_W(t)$

- Stays at x_k for Exp$(1/y_k)$
- then jumps to the "next" trap to the right.

Directed Traps and RWRE

Theorem (P. and Samorodnitsky '12)

If $E[\log \rho_0] < 0$ *and* $\kappa \in (0,1)$, *then*

$$P_\omega\left(\frac{X_{tn}}{n^\kappa} \le x\right) \implies P_W\left(Z_W(t) \le x\right),$$

where W *is a Poisson point process*($\lambda y^{-\kappa-1}\,dx\,dy$).

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Idea of proof:

- Implicit "trapping structure" in ω: $\mathfrak{B} = \mathfrak{B}(\omega) = \sum_k \delta_{(\nu_k,\beta_k)}$.

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- Rescaled trap environments $W_n = \sum_k \delta_{\left(\frac{\nu_k}{n}, \frac{\beta_k}{n^{1/\kappa}}\right)}$ converge *in distribution* to W.

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- Rescaled trap environments $W_n = \sum_k \delta_{\left(\frac{\nu_k}{n}, \frac{\beta_k}{n^{1/\kappa}}\right)}$ converge *in distribution* to W.
- Couple random walk X_n with directed trap process $Z_{W_n}(t)$.

Directed Traps Hydrodynamic Limit

$\mathcal{T}$ = trap environments where $Z_W(t)$ is well defined.
$\mathcal{T}'$ = trap environments with traps dense in $\mathbb{R}$.

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- Sequence $W_n = \sum_k \delta_{(x_k^n, y_k^n)} \in \mathcal{T}$.
- $W_n \to W \in \mathcal{T}'$ (vague convergence)
- initial configurations: $\{\eta_0^n(x_k^n)\}_k$ product Poisson with

$$\eta_0^n(x_k^n) \sim \text{Poisson}(a_n y_k^n u(x_k^n)).$$

  for some $a_n \to \infty$.

Directed Traps Hydrodynamic Limit

Theorem (Jara and P. '14)

Under the previous assumptions, for any $t > 0$ *and* $\phi \in \mathcal{C}_0$,

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_k \eta_t^n(x_k^n)\phi(x_k^n) = \int u_W(t,x)\phi(x)\,\sigma_W(dx), \quad \textit{in probability}$$

where $\sigma_W(dx) = \int_0^\infty y\, W(dx\, dy)$ *and* $u_W(t,x)$ *satisfies*

$$\begin{cases} u_W(0,x) \equiv u(x) \\ \frac{\partial}{\partial t} u_W(t,x) = -\frac{d}{d\sigma_W} u_W(t,x) \quad \forall t > 0. \end{cases}$$

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$$u_W(t,x) = E[u(Z_W^*(t;x))],$$

where $Z_W^*(\cdot;x)$ is the *left*-directed trap process started at x.

Sketch of proof

$$E[\eta_t^n(x_k^n)] = \sum_m E[\eta_0^n(x_m^n)] P\left(Z_W(t; x_m^n) = x_k^n\right)$$

Sketch of proof

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Sketch of proof

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Sketch of proof

$$E[\eta_t^n(x_k^n)] = a_n y_k^n u_{W_n}(t, x_k^n)$$

Therefore

$$\begin{aligned} E\left[\frac{1}{a_n}\sum_k \eta_t^n(x_k^n)\phi(x_k^n)\right] &= \sum_k y_k^n u_{W_n}(t, x_k^n)\phi(x_k^n) \\ &= \int u_{W_n}(t,x)\phi(x)\,\sigma_{W_n}(dx) \end{aligned}$$

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