Phase transitions for the long-time behaviour of interacting diffusions

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Abstract

Let $(\{X_i(t)\}_{i\in\mathbb{Z}^d})_{t\geq 0}$ be the system of interacting diffusions on $[0,\infty)$ defined by the following collection of coupled stochastic differential equations:

$$dX_i(t) = \sum_{j\in\mathbb{Z}^d} a(i,j)[X_j(t) - X_i(t)]\, dt + \sqrt{bX_i^2(t)}\, dW_i(t), \qquad i\in\mathbb{Z}^d, t\geq 0.$$

Here, $a(\cdot,\cdot)$ is an irreducible random walk transition kernel on $\mathbb{Z}^d\times\mathbb{Z}^d$, $b\in(0,\infty)$ is a diffusion parameter, and $(\{W_i(t)\}_{i\in\mathbb{Z}^d})_{t\geq 0}$ is a collection of independent standard Brownian motions on $\mathbb{R}$. The initial condition is chosen such that $\{X_i(0)\}_{i\in\mathbb{Z}^d}$ is a shift-invariant and shift-ergodic random field on $[0,\infty)$ with 1-st moment $\Theta\in(0,\infty)$ (during the evolution, the 1-st moment is preserved). We show that the long-time behaviour of this system is the result of a delicate interplay between $a(\cdot,\cdot)$ and b, in contrast to systems where the diffusion function is subquadratic. In particular, let $\widehat{a}(i,j) = \frac{1}{2}[a(i,j) + a(j,i)]$, $i,j\in\mathbb{Z}^d$, denote the symmetrised transition kernel. We show that:

(A) If $\widehat{a}(\cdot,\cdot)$ is recurrent, then for any $b>0$ the system locally dies out.

(B) If $\widehat{a}(\cdot,\cdot)$ is transient, then there exist $b_* > b_2 > 0$ such that:
  (B1) The system converges to an equilibrium ν_Θ if $0<b<b_*$.
  (B2) The system locally dies out if $b>b_*$.
  (B3) ν_Θ has a finite 2-nd moment if and only if $0<b<b_2$.
  (B4) The 2-nd moment diverges exponentially fast if and only if $b>b_2$.

For the case where $a(\cdot,\cdot)$ is symmetric and transient we further show that:

(C) There exists a sequence $b_2\geq b_3\geq b_4\geq\cdots>0$ such that:
  (C1) ν_Θ has a finite m-th moment if and only if $0<b<b_m$.
  (C2) The m-th moment diverges exponentially fast if and only if $b>b_m$.
  (C3) $b_2\leq(m-1)b_m<2$.
  (C4) $\lim_{m\to\infty}(m-1)b_m$ exists.

The proof of these results is based on self-duality and on a representation formula through which the moments of the components are related to exponential moments of the intersection local time of random walks. Via large deviation theory, the latter lead to variational

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expressions for b_* and the b_m's, from which sharp bounds are deduced. The critical value b_* arises from a representation formula for the Palm distribution of the system. The equilibrium ν_Θ is shown to be associated and mixing for all $0 < b < b_*$.

The special case where $a(\cdot,\cdot)$ is simple random walk is commonly referred to as the parabolic Anderson model with Brownian noise. This case was studied in the memoir by Carmona and Molchanov (1994), where some of our results were already established.

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