Maximum-Likelihood-Estimation of Lévy driven Ornstein-Uhlenbeck Processes

Hilmar Mai
Humboldt-Universität zu Berlin

Workshop on Statistical Inference for Lévy processes
EURANDOM

16 July 2009
Ornstein-Uhlenbeck (OU) Process

Let \((L_t, t \geq 0)\) be a Lévy process on \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\). For every \(a \in \mathbb{R}\)
\[
dX_t = -aX_t \, dt + dL_t, \quad t \in \mathbb{R}_+, \quad X_0 = x, \tag{1}
\]
defines an Ornstein-Uhlenbeck process driven by the Lévy process \(L\) with initial distribution \(\pi = \mathcal{L}(X_0)\).
Equivalently,
\[
X_t = e^{-at}X_0 + \int_0^t e^{-a(t-s)} \, dL_s. \tag{2}
\]
Sample path from compound Poisson plus Wiener process driver
Recent Literature on Lévy OU Inference

- Stochastic volatility modelling: Barndorff-Nielsen and Shephard [2001]
- (Non-)Parameteric estimation for driving subordinators Jongbloed, van der Meulen and van der Vaart [2005]
- Maximum-likelihood-estimation from discrete observations Valdivieso, Schoutens and Tuerlinckx [2009]
- Least squares estimation from discrete observations for $\alpha$-stable driver Hu and Long [2009]
Problem: Estimation of $a$ from continuous observations $X_t$, $0 \leq t \leq T$ and know Lévy-Khintchine triplet of $L$.

We work throughout in the canonical setting:

- $\Omega = D(\mathbb{R}_+) = \{ f : \mathbb{R}_+ \to \mathbb{R}; \text{ f càdlàg} \}$
- $X(\omega, t) = \omega(t)$ for all $\omega \in \Omega$ coordinate process
- Filtration generated by $X$

$$\mathcal{F}_t = \bigcap_{s > t} \sigma(X_u : u \leq s) \text{ and } \mathcal{F} = \bigvee_{t} \mathcal{F}_t$$

For every $a \in \mathbb{R}$ we obtain a solution measure $P^a$ of the OU equation on $D(\mathbb{R}_+)$. 
Absolute Continuity/Singularity (ACS)
Problem

\[ P^a_t := P^a \mid \mathcal{F}_t \] denotes the restriction of \( P^a \) to \( \mathcal{F}_t \).

Local absolute continuity:

\[ P^a' \overset{loc}{\ll} P^a \iff P^a'_t \ll P^a_t \quad \forall t \in \mathbb{R}_+ \]

In order to define an MLE for the statistical experiment \((\Omega, \mathcal{F}, (\mathcal{F}_t), (P^a)_{a \in \mathbb{R}})\) we need:

1. Does \( P^a' \overset{loc}{\ll} P^a \) hold for all \( a, a' \in \mathbb{R} \)?

2. Can we derive \( Z_t = \frac{dP^a'}{dP^a_t} \) explicitly?
Let $P, P'$ be two probability measures on $(\Omega, \mathcal{F}, (\mathcal{F}_t))$.

**Theorem (Jacod and Mémin (1979))**

Let $h(\alpha)$, $\alpha \in (0, 1)$ be a version of the Hellinger process $h(\alpha; P, P')$. Then for every stopping time $T$ there is equivalence between

1. $P_T' \ll P_T$;
2. $\exists \alpha \in (0, 1)$ such that $P'(h(\alpha)_T < \infty) = 1$ and $P_0' \ll P_0$ and $P'(h(0)_T = 0) = 1$. 

Semimartingale characteristics of $X$

Let $(b, \sigma^2, \mu)$ denote the Lévy-Khintchine triplet of $L$.

Then the semimartingale characteristics $(B, C, \nu)$ of $X$ are given by

$$B(\omega, t) = bt - a \int_0^t X_s(\omega) \, ds,$$

$$C(\omega, t) = \sigma^2 t,$$

$$\nu(\omega, dt, dx) = \mu(dx) \lambda(dt),$$

where $\lambda$ denotes the Lebesgue measure on $\mathbb{R}$.
The Hellinger Process

Proposition

A version of the Hellinger process of two solution measures $P^a, P^{a'}$ is

$$h_t(\alpha; a, a') = \frac{\alpha(1 - \alpha)}{2\sigma^2} \int_0^t \left[ \int_0^u \left( a' e^{-a' (u-s)} - a e^{-a (u-s)} \right) L(ds) \right]^2 du.$$

for $a, a' \in \mathbb{R}$. 
Theorem

Let $P^a, P^{a'}$ be two solution measures of the OU equation for the driving Lévy process $L$ with characteristic triplet $(b, \sigma^2, \rho)$ and initial distributions $\pi$ and $\pi'$. Suppose that $\sigma^2 > 0$ and $\pi' \ll \pi$, then we have

$$P^{a'} \ll P^a.$$
Proposition

There exists a $P$-local martingale $N : \Delta \to \mathbb{R}$ on a random interval $\Delta \subset \Omega \times \mathbb{R}_+$ such that the density process is given by

$$Z_t = \frac{dP_{t}^{a'}}{dP_{t}^{a}} = Z_0 \exp \left( N_t - \frac{(a' - a)^2}{2\sigma^2} \int_0^t X_s^2 ds \right).$$

Furthermore, for every stopping time $S$ such that $[0, S] \subset \Delta$ the stopped process $N^S$ is of the form

$$N^S = \left( \frac{(a' - a)}{\sigma^2} X_{t-1}^{[0,S]} \right) \cdot X^c,$$

where $X^c$ denotes the continuous martingale part of $X$ under $P^{a}$.
Maximum-Likelihood-Estimator (MLE)

For continuous observations of the Ornstein-Uhlenbeck process $X$ the likelihood function $L$ for the statistical experiment $(\Omega, \mathcal{F}, (\mathcal{F}_t), (P^a)_{a \in \mathbb{R}})$ takes the form

$$L(a, X^T) = \frac{dP^a_t}{dP^0_t} = \exp \left( \frac{a}{\sigma^2} \int_0^T X_{s-} \, dX^c_s - \frac{a^2}{2\sigma^2} \int_0^T X^2_{s-} \, ds \right),$$

Hence, the MLE for $a$ is explicitly given by

$$\hat{a}_T = \frac{\int_0^T X_{s-} \, dX^c_s}{\int_0^T X^2_{s-} \, ds}.$$
The continuous martingale part $X^c$

By the Lévy-Itô decomposition of $L$ we can write $X$ as

$$X_t = X_0 - a \int_0^t X_s^- \, ds + W_t + J_t \quad , \quad t \geq 0,$$

where $W$ is a Wiener Process and $J$ a quadratic pure jump process given by

$$J_t = \int_{\{|x|<1\}} x(N_t(dx) - t\mu(dx)) + bt + \sum_{0 \leq s \leq t} \Delta X_s 1_{\{|\Delta X_s| \geq 1\}}.$$

$N$ is the jump measure of $L$ with compensator $\mu$. 
Under $P^0$ it follows that $X^c = W$, but under $P^a$

$$\tilde{W}_t = W_t - a \int_0^t X_{s-} \, ds$$

defines a Wiener process such that $X^c = \tilde{W}$ under $P^a$. Hence, given observations $(X_t(\omega), t \in [0, T])$

$$X^c_t = X_t - \int_{\{|x|<1\}} x(N_t(dx) - t\mu(dx)) - bt - \sum_{0 \leq s \leq t} \Delta X_s 1\{|\Delta X_s| \geq 1\}.$$ 

which can be reconstructed from continuous observations. Hence, the MLE can be rewritten as

$$\hat{a}_T = \frac{\int_0^T X_{s-}(dW_s - aX_{s-}ds)}{\int_0^T X_{s-}^2 ds} = \frac{\int_0^T X_{s-}dW_s}{\int_0^T X_{s-}^2 ds} - a.$$ 

under $P^a$. 
Let \( \{ P^\theta, \theta \in \Theta \} \) be a family of measures on \((\Omega, \mathcal{F}, (\mathcal{F}_t))\).

**Definition (Küchler and Sørensen (1997))**

*We say that a statistical experiment \( \{ P^\theta, \theta \in \Theta \} \) forms a curved exponential family* if the likelihood function exists and is of the form

\[
\frac{dP^\theta_t}{dP^\theta_0} = \exp \left( \theta^t A_t - \kappa(\theta) S_t \right)
\]

where \( \kappa : \Theta \to \mathbb{R} \), for \( \theta_0 \in \Theta \) arbitrary but fixed and \( A : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d \) is a càdlàg process. Moreover, \( S : \Omega \times \mathbb{R}_+ \to \mathbb{R} \) is assumed to be a non-decreasing continuous process with \( S_0 = 0 \) and \( S_t \xrightarrow{t \to \infty} \infty \).
Strong Consistency

**Theorem**

Under the condition $\sigma^2 > 0$ the MLE $\hat{a}_T$ for any $a \in \mathbb{R}$ based on continuous observations $X_t$, $t \in [0, T]$ exists and is given by

$$
\hat{a}_T = \frac{\int_0^T X_s dX_s^c}{\int_0^T X_s^2 ds}.
$$

Furthermore, under $P_a$ the MLE is unique for $T$ sufficiently large and

$$
\hat{a}_T \to a \quad \text{almost surely}
$$

as $T \to \infty$. 
Asymptotic Normality

**Theorem**

Let $X$ be a stationary OU process and $a > 0$ such that $c = E[X^2] < \infty$. Then under $P^a$

$$\sqrt{T}(\hat{a}_T - a) \to N(0, \frac{\sigma^2}{c}) \quad \text{weakly}$$

as $T \to \infty$. 
Summary and Outlook

- Under $\sigma^2 > 0$ the solution measures $\{P^a, a \in \mathbb{R}\}$ are locally equivalent.
- The MLE takes an explicit form and is consistent and asymptotically normal as well as efficient.
- Computation from discrete observations is straightforward.
- Asymptotics for $a < 0$ and without second moments of $X$?
- Delay estimation

