Limit theorems for bipower variation of semimartingales

Mathias Vetter

Ruhr-Universität Bochum

Eindhoven, 15 July 2009
Outline of the talk

- Outline of the talk
- Estimation of integrated volatility
- Two central limit theorems
- References
Setting

- We are given an Itô semimartingale $X$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P)$, thus

$$X_t = X_0 + \int_0^t b_u \, du + \int_0^t \sigma_u \, dW_u + (\delta 1_{\{|\delta| \leq 1\}}) * (\mu_t - \nu_t) + (\delta 1_{\{|\delta| > 1\}}) * \mu_t,$$

where $W$ is a Brownian motion and $\mu$ and $\nu$ are a Poisson random measure on $\mathbb{R}_+ \times E$ and its compensator $\nu(du, dx) = du \times \lambda(dx)$.
Setting

- We are given an Itô semimartingale $X$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P)$, thus
  \[ X_t = X_0 + \int_0^t b_u \, du + \int_0^t \sigma_u \, dW_u + (\delta 1_{\{|\delta| \leq 1\}}) \ast (\underline{\mu}_t - \underline{\nu}_t) + (\delta 1_{\{|\delta| > 1\}}) \ast \underline{\mu}_t, \]
  where $W$ is a Brownian motion and $\underline{\mu}$ and $\underline{\nu}$ are a Poisson random measure on $\mathbb{R}_+ \times E$ and its compensator $\underline{\nu}(du, dx) = du \times \lambda(dx)$.

- Standard assumptions:
  1. The drift $b$ is optional and càglàd.
  2. The volatility $\sigma$ is an Itô semimartingale itself and satisfies $\sigma > 0$ almost surely.
  3. The function $\delta$ is predictable and locally bounded by a family $(\gamma_k)$ of non-negative functions such that $\int_E (1 \wedge \gamma_k(z)) \lambda(dz) < \infty$. 

Mathias Vetter

Limit theorems for bipower variation of semimartingales
Setting

- We are given an Ito semimartingale $X$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P)$, thus
  \[ X_t = X_0 + \int_0^t b_u \, du + \int_0^t \sigma_u \, dW_u + (\delta 1_{\{|\delta| \leq 1\}}) \ast (\mu_t - \nu_t) + (\delta 1_{\{|\delta| > 1\}}) \ast \mu_t, \]
  where $W$ is a Brownian motion and $\mu$ and $\nu$ are a Poisson random measure on $\mathbb{R}_+ \times E$ and its compensator $\nu(du, dx) = du \times \lambda(dx)$.

- Standard assumptions:
  1. The drift $b$ is optional and càglàd.
  2. The volatility $\sigma$ is an Ito semimartingale itself and satisfies $\sigma > 0$ almost surely.
  3. The function $\delta$ is predictable and locally bounded by a family $(\gamma_k)$ of non-negative functions such that $\int_E (1 \wedge \gamma_k^s(z)) \, \lambda(dz) < \infty$.

- Aim: Being on a fixed time span $[0, T]$ and observing $X_{\frac{i}{n}}$ for $i = 0, \ldots, \lfloor nT \rfloor$ we are interested in estimating the quadratic variation
  \[ \int_0^t \sigma_u^2 \, ds + \sum_{u \leq t} |\Delta X_u|^2 \]
  or parts thereof, typically the integrated volatility $\int_0^t \sigma_u^2 \, du$. 

Mathias Vetter
Limit theorems for bipower variation of semimartingales
Realized volatility

- In the continuous case

\[ X_t = X_0 + \int_0^t b_u \, du + \int_0^t \sigma_u \, dW_u \]

the standard estimator for the integrated volatility is the realized variance

\[ RV(X)_t^n = \sum_{i=1}^{\lfloor nt \rfloor} |\Delta_i^n X|^2, \text{ where } \Delta_i^n X = X_{n_i} - X_{n_{i-1}}. \]
Realized volatility

- In the continuous case

\[ X_t = X_0 + \int_0^t b_u \, du + \int_0^t \sigma_u \, dW_u \]

the standard estimator for the integrated volatility is the realized variance

\[ RV(X)_t^n = \sum_{i=1}^{\lfloor nt \rfloor} |\Delta_i^n X|^2, \text{ where } \Delta_i^n X = X_{\frac{i}{n}} - X_{\frac{i-1}{n}}.\]

- We have

\[ RV(X)_t^n \xrightarrow{P} \int_0^t \sigma_u^2 \, du \]

and an associated central limit theorem

\[ \sqrt{n} \left( RV(X)_t^n - \int_0^t \sigma_u^2 \, du \right) \xrightarrow{D_{st}} \sqrt{2} \int_0^t \sigma_u^2 \, dW'_u, \]

where \( W' \) is defined on an extension of \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P)\) and independent of \( \mathcal{F} \).
Truncated realized volatility

In the presence of jumps, $RV(X)_t^n$ obviously becomes an estimator for the entire quadratic variation of $X$ and thus has to be modified.
Truncated realized volatility

- In the presence of jumps, $RV(X)_t^n$ obviously becomes an estimator for the entire quadratic variation of $X$ and thus has to be modified.
- Mancini has proposed to use a truncated version of $RV(X)_t^n$, namely

$$TRV(X)_t^n = \sum_{i=1}^{\lfloor nt \rfloor} |\Delta_i^n X|^2 1\{|\Delta_i^n X| \leq \alpha n^{-\varpi}\}$$

for some $\alpha > 0$ and $0 < \varpi < \frac{1}{2}$. 

Intuition: We are cutting off large increments as these are likely due to a jump within $[i-1/n, i/n]$. 

$TRV(X)_t^n$ is consistent for the integrated volatility and we obtain the same central limit theorem as before, as long as $s \leq 4\varpi - \frac{1}{2}\varpi$ (so in particular $s < 1$):

$$\sqrt{n}(TRV(X)_t^n - \int_0^t \sigma^2_u du) \rightarrowD \sqrt{2} \int_0^t \sigma^2_u dW'_u.$$
Truncated realized volatility

- In the presence of jumps, $RV(X)^n_t$ obviously becomes an estimator for the entire quadratic variation of $X$ and thus has to be modified.
- Mancini has proposed to use a truncated version of $RV(X)^n_t$, namely

$$TRV(X)^n_t = \sum_{i=1}^{\lfloor nt \rfloor} |\Delta_i^n X|^2 1\{ |\Delta_i^n X| \leq \alpha n^{-\varpi} \}$$

for some $\alpha > 0$ and $0 < \varpi < \frac{1}{2}$.
- Intuition: We are cutting off large increments as these are likely due to a jump within $[\frac{i-1}{n}, \frac{i}{n}]$. 
Truncated realized volatility

- In the presence of jumps, $RV(X)_{nt}$ obviously becomes an estimator for the entire quadratic variation of $X$ and thus has to be modified.
- Mancini has proposed to use a truncated version of $RV(X)_{nt}$, namely

$$TRV(X)_{nt} = \sum_{i=1}^{\lfloor nt \rfloor} |\Delta_i X|^2 1\{|\Delta_i X| \leq \alpha n^{-\varpi}\}$$

for some $\alpha > 0$ and $0 < \varpi < \frac{1}{2}$.
- Intuition: We are cutting off large increments as these are likely due to a jump within $[\frac{i-1}{n}, \frac{i}{n}]$.
- $TRV(X)_{nt}$ is consistent for the integrated volatility and we obtain the same central limit theorem as before, as long as $s \leq \frac{4\varpi - 1}{2\varpi}$ (so in particular $s < 1$):

$$\sqrt{n} \left( TRV(X)_{nt} - \int_0^t \sigma_u^2 \, du \right) \xrightarrow{D_{st}} \sqrt{2} \int_0^t \sigma_u^2 \, dW'_u.$$
Multipower variation I

Alternatively, one uses multipower variations, which are defined as

$$MV(X, r)_t^n = n^{|r|/2 - 1} \sum_{i=1}^{[nt] - q + 1} \prod_{j=1}^q |\Delta_{i+j-1} X|^{r_j},$$

with \( r = (r_1, \ldots, r_q) \) having non-negative components. We set \(|r| = r_1 + \ldots + r_q\), \( r_+ = \max(r_1, \ldots, r_q) \) and \( r_- = \min(r_1, \ldots, r_q) \).
Multipower variation I

Alternatively, one uses multipower variations, which are defined as

\[
MV(X, r)_t^n = n^{\frac{|r|}{2} - 1} \sum_{i=1}^{\lfloor nt \rfloor - q + 1} q \prod_{j=1}^{|r|} |\Delta_{i+j-1} X |^{r_j},
\]

with \( r = (r_1, \ldots, r_q) \) having non-negative components. We set \(|r| = r_1 + \ldots + r_q, r_+ = \max(r_1, \ldots, r_q) \) and \(|r| = \min(r_1, \ldots, r_q)\).

Intuition: One pairs intervals containing jumps with those that do not contain jumps, and typically (depending on \( r \)) these intervals do not play a role in the asymptotics.
Multipower variation I

- Alternatively, one uses multipower variations, which are defined as

$$MV(X, \mathbf{r})_t^n = n|\mathbf{r}|^{q-1} \sum_{i=1}^{\lfloor nt \rfloor - q + 1} \prod_{j=1}^{q} |\Delta_{i+j-1} X|^{r_j},$$

with $\mathbf{r} = (r_1, \ldots, r_q)$ having non-negative components. We set $|\mathbf{r}| = r_1 + \ldots + r_q$, $r_+ = \max(r_1, \ldots, r_q)$ and $r_- = \min(r_1, \ldots, r_q)$.

- Intuition: One pairs intervals containing jumps with those that do not contain jumps, and typically (depending on $\mathbf{r}$) these intervals do not play a role in the asymptotics.

- Let $m_p$ denote the $p$-th absolute moment of a standard normal distribution and set $m_{\mathbf{r}} = \prod_{j=1}^{q} m_j$. Then

$$MV(X, \mathbf{r})_t^n \xrightarrow{P} m_{\mathbf{r}} \int_0^t \sigma_u |\mathbf{r}| \, du,$$

as long as $r_+ < 2$. 
Central limit theorems have only been shown for specific choices of $r$: Suppose that $\frac{s}{2-s} < r_- \leq r_+ < 1$. Then we have

$$\sqrt{n}(\text{MV}(X,r)_t^n - m_r \int_0^t \sigma_u^{|r|} \, du) \xrightarrow{D_{st}} \sqrt{p(r)} \int_0^t \sigma_u^{|r|} \, dW_u'$$

for some known function $p$. 
Multipower variation II

Central limit theorems have only been shown for specific choices of $r$: Suppose that $\frac{s}{2-s} < r_- \leq r_+ < 1$. Then we have

$$\sqrt{n}\left( MV(X, r)_t^n - m_r \int_0^t \sigma_u |r| \, du \right) \xrightarrow{D_{st}} \sqrt{p(r)} \int_0^t \sigma_u |r| \, dW'_u$$

for some known function $p$.

The two most prominent estimators for the integrated volatility in this context are $MV(X, (1, 1))_t^n$ and $MV(X, (2/3, 2/3, 2/3))_t^n$. For both, we have convergence in probability (of a rescaled version) towards the integrated volatility, but only for the latter one we have a central limit theorem, as long as $\frac{s}{2-s} < \frac{2}{3}$.
Multipower variation II

- Central limit theorems have only been shown for specific choices of \( r \): Suppose that \( \frac{s - s}{2-s} < r_- \leq r_+ < 1 \). Then we have

\[
\sqrt{n} \left( \text{MV}(X, r)_t^n - mn \int_0^t \sigma_u^{|r|} \, du \right) \xrightarrow{D_{st}} \sqrt{p(r)} \int_0^t \sigma_u^{|r|} \, dW'_u
\]

for some known function \( p \).

- The two most prominent estimators for the integrated volatility in this context are \( \text{MV}(X, (1, 1))_t^n \) and \( \text{MV}(X, (2/3, 2/3, 2/3))_t^n \). For both, we have convergence in probability (of a rescaled version) towards the integrated volatility, but only for the latter one we have a central limit theorem, as long as \( \frac{s - s}{2-s} < \frac{2}{3} \).

- Question: Is there no central limit theorem for bipower variation? Or has it simply not been proven yet? And if there is one, does it depend on the jumps?
Multipower variation II

- Central limit theorems have only been shown for specific choices of $r$: Suppose that $\frac{s}{2-s} < r_- \leq r_+ < 1$. Then we have

$$\sqrt{n}\left( MV(X, r)_t^n - mr \int_0^t \sigma_u |r| \, du \right) \xrightarrow{D_{st}} \sqrt{p(r)} \int_0^t \sigma_u |r| \, dW'_u$$

for some known function $p$.

- The two most prominent estimators for the integrated volatility in this context are $MV(X, (1, 1))_t^n$ and $MV(X, (2/3, 2/3, 2/3))_t^n$. For both, we have convergence in probability (of a rescaled version) towards the integrated volatility, but only for the latter one we have a central limit theorem, as long as $\frac{s}{2-s} < \frac{2}{3}$.

- Question: Is there no central limit theorem for bipower variation? Or has it simply not been proven yet? And if there is one, does it depend on the jumps?

- Notation:

$$V(X)_t^n = \sum_{i=1}^{\lfloor nt \rfloor} |\Delta_i^n X| \cdot |\Delta_{i+1}^n X|.$$
Prerequisites

Suppose we have \( s \leq 1 \), thus

\[
X_t = X_0 + B_t + \int_0^t \sigma_u \, dW_u + \sum_{u \leq t} \Delta X_u
\]

with \( B_t = \int_0^t b_u \, du - \delta 1_{\{ |\delta| \leq 1 \}} \ast \nu_t \).
Prerequisites

- Suppose we have $s \leq 1$, thus

\[ X_t = X_0 + B_t + \int_0^t \sigma_u \, dW_u + \sum_{u \leq t} \Delta X_u \]

with $B_t = \int_0^t b_u \, du - \delta 1_{\{|\delta| \leq 1\}} * \nu_t$.

- We define an appropriate extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}^+}, P)$ supporting two independent sequences $(U_{m+})$ and $(U_{m-})$ of standard normally distributed random variables and an independent Brownian motion $W'$, all independent of $\mathcal{F}$. Furthermore, $(T_m)$ is a sequence of stopping times exhausting the jumps of $X$. We set

\[ U'_t = \sum_{m : T_m \leq t} |\Delta X_{T_m}| \cdot \left( \sigma_{T_m} |U_{m-}| + \sigma_{T_m} |U_{m+}| \right) \]

and

\[ U''_t = \sqrt{1 + 2m_1^2 - 3m_1^4} \int_0^t \sigma_u^2 \, dW'_u. \]
CLT for bipower variation I

- Then we have

$$\sqrt{n} \left( V(X)_t^n - m_1^2 \int_0^t \sigma_u^2 \, du \right) \xrightarrow{D_{st}} U'_t + U''_t.$$
CLT for bipower variation I

Then we have

\[ \sqrt{n} \left( V(X)_t^n - m_1^2 \int_0^t \sigma^2_u \, du \right) \xrightarrow{D^{st}} U'_t + U''_t. \]

Intuition: If there are only finitely many jumps on \([0, T]\), all jump times \(T_m\) and \(T'_m\) satisfy \(|T_m - T'_m| > \frac{2}{n}\) for large \(n\). Thus, if a jump lies within \([\frac{i-1}{n}, \frac{i}{n}]\), it occurs in \(V(X)_{\frac{i}{n}}^n\) as

\[ |\Delta_{\frac{i}{n}}^n X| \cdot (|\Delta_{\frac{i-1}{n}} X| + |\Delta_{\frac{i+1}{n}} X|), \]

and the neighbouring increments are not affected by jumps. Using the approximations

\[ \Delta_{\frac{i}{n}}^n X \approx \Delta_{T_m} X \quad \text{and} \quad \Delta_{\frac{i-1}{n}} X \approx \sigma_{T_m} - \Delta_{\frac{i-1}{n}} W, \]

we end up with the finite sum

\[ \sqrt{n} \sum_{m: T_m \leq t} |\Delta_{T_m} X| \cdot (\sigma_{T_m} |\Delta_{\frac{i-1}{n}} W| + \sigma_{T_m} |\Delta_{\frac{i+1}{n}} W|) \rightarrow U'_t. \]
CLT for bipower variation II

- $U_t'$ is not a martingale (unless $X$ is continuous, of course), and thus plays the role of a bias in the limit. How can one get rid of the bias?
CLT for bipower variation II

- $U'_t$ is not a martingale (unless $X$ is continuous, of course), and thus plays the role of a bias in the limit. How can one get rid of the bias?
- By subtracting an estimator for it: Quite naturally, we estimate

$$U'_t = \sum_{m: T_m \leq t} |\Delta X_{T_m}| \cdot \left( \sigma_{T_m-} |U_{m-}| + \sigma_{T_m} |U_{m+}| \right)$$

by $\sqrt{n} V^*(X, \alpha, \varpi)_t^n$ with

$$V^*(X, \alpha, \varpi)_t^n = \sum_{i=1}^{\lfloor nt \rfloor - 1} |\Delta_i^n X||\Delta_{i+1}^n X| \cdot \left( 1\{|\Delta_i^n X| \geq \alpha n^{-\varpi}\} 1\{|\Delta_{i+1}^n X| < \alpha n^{-\varpi}\} + 1\{|\Delta_i^n X| < \alpha n^{-\varpi}\} 1\{|\Delta_{i+1}^n X| \geq \alpha n^{-\varpi}\} \right)$$

and look at the asymptotics of

$$\sqrt{n} \left( (V(X)_t^n - V^*(X, \alpha, \varpi)_t^n) - m_1^2 \int_0^t \sigma_u^2 \, du \right).$$
CLT for truncated bipower variation

- Alternatively, we can look at the direct analogue of $TRV(X)^n_t$, namely

$$TV^*(X, \alpha, \omega)_t^n = \sum_{i=1}^{\lfloor nt\rfloor-1} |\Delta_i^n X| 1\{|\Delta_i^n X| < \alpha n - \omega\} |\Delta_{i+1}^n X| 1\{|\Delta_{i+1}^n X| < \alpha n - \omega\}.$$
CLT for truncated bipower variation

- Alternatively, we can look at the direct analogue of $TRV(X)_t^n$, namely

$$TV^*(X, \alpha, \varpi)_t^n = \sum_{i=1}^{\lfloor nt \rfloor - 1} |\Delta_i^n X| 1\{ |\Delta_i^n X| < \alpha n^{-\varpi} \} |\Delta_{i+1}^n X| 1\{ |\Delta_{i+1}^n X| < \alpha n^{-\varpi} \}.$$ 

- We have

$$\sqrt{n} \left( (V(X)_t^n - V^*(X, \alpha, \varpi)_t^n) - m_1^2 \int_0^t \sigma_u^2 \, du \right) \xrightarrow{Dst} U''_t,$$

and

$$\sqrt{n} \left( TV^*(X, \alpha, \varpi)_t^n - m_1^2 \int_0^t \sigma_u^2 \, du \right) \xrightarrow{Dst} U''_t.$$
CLT for truncated bipower variation

- Alternatively, we can look at the direct analogue of $TRV(X)_t^n$, namely

$$TV^*(X, \alpha, \varpi)_t^n = \sum_{i=1}^{\lfloor nt \rfloor - 1} |\Delta_i^n X| 1_{\{|\Delta_i^n X| < \alpha n^{-\infty}\}} |\Delta_{i+1}^n X| 1_{\{|\Delta_{i+1}^n X| < \alpha n^{-\infty}\}}.$$

- We have

$$\sqrt{n}\left(V(X)_t^n - V^*(X, \alpha, \varpi)_t^n - m_1^2 \int_0^t \sigma_u^2 \ du\right) \xrightarrow{D_{st}} U''_t,$$

and

$$\sqrt{n}\left(TV^*(X, \alpha, \varpi)_t^n - m_1^2 \int_0^t \sigma_u^2 \ du\right) \xrightarrow{D_{st}} U''_t.$$

- Note that this results holds for all $s \leq 1$ and irrespectively of the choice of $\varpi$. 

Mathias Vetter
Limit theorems for bipower variation of semimartingales
References


