Daphnia revisited: an example of local stability and bifurcation analyses for physiologically structured population models

NDNS+ workshop
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@Eindhoven TU
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Based on the paper
Odo Diekmann, Mats Gyllenberg, Hans Metz, SN, Andre de Roos
Daphnia revisited: local stability and bifurcation theory for physiologically structured population models explained by way of an example.
Journal of Mathematical Biology in press.
Delay equations

A couple system of renewal and delay differential equation

Space

\[ X := L^1([-h, 0]; \mathbb{R}), \quad Y := C(([[-h, 0]; \mathbb{R}). \]

Initial condition

\[ (\varphi, \psi) \in X \times Y, \quad b(t) = \varphi(t) \text{ and } S(t) = \psi(t), \quad -h \leq t \leq 0 \]

A general form for a finite delay case

\[
\begin{cases}
  b(t) = F_1(b_t, S_t), \\
  \frac{dS}{dt}(t) = F_2(b_t, S_t)
\end{cases}
\]

where \( S_t(\sigma): \sigma \mapsto S(t + \sigma), \quad \sigma \leq 0 \)
Mathematical theory


A finite delay case


An infinite delay case


O. Diekmann *et al*., Equations in infinite delay, submitted.
Resource-consumer system

\[ S(t): \text{ food (algae) concentration (resource)} \]

\[ b(t): \text{ } Daphnia \text{ population birth rate (consumer)} \]

\[ b(t) = \int_{0}^{\infty} b(t - a)\beta(\Xi(a; S_t), S(t))\mathcal{F}(a; S_t)da, \]

\[ \frac{dS}{dt}(t) = f(S(t)) - \int_{0}^{\infty} b(t - a)\gamma(\Xi(a; S_t), S(t))\mathcal{F}(a; S_t)da. \]

\[ \Xi(a; S_t): \text{ The current body size of an individual with age } a \]

\[ \beta(\Xi(a; S_t), S(t)): \text{ the probability per unit of time of giving birth} \]

\[ \gamma(\Xi(a; S_t), S(t)): \text{ the rate of food consumption of an individual} \]

\[ \mathcal{F}(a; S_t): \text{ the survival probability of an individual} \]
Model ingredients

☑️ Individual body size growth and survival

An individual has age $a$ at the current time $t$

$\Xi(a; S_t)$: The current body size of an individual with age $a$

\[
\frac{d\xi}{d\tau}(\tau) = g(\xi(\tau), \psi(-a + \tau)), \quad \xi(0) = \xi_b.
\]

$\Xi(a; \psi) := \xi(a; a, \psi)$

$\mathcal{F}(a; S_t)$: The survival probability of an individual

\[
\frac{d\mathcal{G}}{d\tau}(\tau) = -\mu(\xi(\tau; a, \psi), \psi(-a + \tau))\mathcal{G}(\tau), \quad \mathcal{G}(0) = 1
\]

$\mathcal{F}(a; \psi) = \mathcal{G}(a; a, \psi)$
Linearised stability

Steady state \((\bar{b}, \bar{S})\)

\(R_0: \) the basic reproduction number of the \textit{Daphnia}

\[
R_0(\bar{S}) = \int_0^\infty \beta(\Xi(a; \bar{S}), \bar{S}) \mathcal{F}(a; \bar{S}) da
\]

A steady \textit{Daphnia} population

\[
R_0(\bar{S}) = 1
\]

\[
\bar{b} = f(\bar{S}) / \int_0^\infty \gamma(\Xi(a; \bar{S}), \bar{S}) \mathcal{F}(a; \bar{S}) da
\]

We can derive steady state age-size relation and survival probability (not shown)
Linearised stability

\[\begin{aligned}
  y(t) &= c_1 z(t) + \int_0^\infty (k_{11}(a)y(t-a) + k_{12}(a)z(t-a))\,da, \\
  \frac{dz}{dt}(t) &= c_2 z(t) + \int_0^\infty (k_{21}(a)y(t-a) + k_{22}(a)z(t-a))\,da
\end{aligned}\]

Characteristic equation

\[
\left(1 - \hat{k}_{11}(\lambda)\right) \left(\lambda - c_2 - \hat{k}_{22}(\lambda)\right) = \hat{k}_{21}(\lambda) \left(c_1 + \hat{k}_{12}(\lambda)\right)
\]

where

\[
\hat{k}_{ij}(\lambda) := \int_0^\infty e^{-\lambda a} k_{ij}(a)\,da
\]
A special case: stage structure

Stage transition from juvenile to adult

Maturation

- juvenile non-reproductive
- adult reproductive

\[ \bar{a} = \bar{a}(\psi) \]: the age of the individuals that mature exactly now

\[ b(t) = \int_{\bar{a}(S_t)}^{\infty} b(t - a) \beta(\Xi(a; S_t), S(t)) \mathcal{F}(a; S_t) da. \]
A special case: stage structure

☑ Variable maturation delay

Assumption

Model ingredients \( g, \mu, \beta \) and \( \gamma \) are independent of \( \xi \)

\[
\int_{-\tau}^{0} g(S_t(\theta)) d\theta = \xi_A - \xi_b
\]

variable maturation delay

\( \tau = \tau_m(S_t) \)

\( \xi_b \) juvenile \( \xi_A \) adult

size at born size at maturation

Numerical bifurcation analysis

- Hopf bifurcation and population cycles

Characteristic equation: \( P(\lambda; a, b) = 0 \quad a, b \in \mathbb{R} \)

\( \lambda = i\omega: \) a parametrized curve \((a(\omega), b(\omega))\)

Example:

\[
g(S) = g_0((1 - \alpha)A + \alpha S')
\]

\(\alpha: A\) fraction of juveniles to exploit \(S\)
Numerical simulation

By Escalator Boxcar Train method

De Roos et al, Am Nat 1992
Future work

Numerical analysis


Numerical equilibrium analysis for structured consume-resource models

Future work: numerical continuation of periodic orbits etc...

Another direction

Application of the theory of physiologically structured population models and delay equations TO

epidemiological model, cell biological context etc...