Perfect simulation of loss networks and statistical mechanical models with exclusions

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Loss networks with fixed routing

Setting

- Graph $\mathbb{L}$ (e.g. $\mathbb{Z}^d$) with a countable family of links
- Countable family Γ of routes. Each route γ is defined by
  - A subset of links of $\mathbb{L}$
  - (Flow, current) numbers associated to each of these links

Process

- Calls request routes γ at independent Poissonian rate $w(\gamma)$
- A requested call is established if a test is passed
  - *Deterministic:* predefined link capacities not exceeded
  - *Stochastic:* large uses of a link discouraged
- Once established there is an independent holding period

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Issues addressed

(I) For the process:

- **(i)** Existence (on the full $\mathbb{L}$) for finite times
- **(ii)** Existence of infinite-time limits

(II) For the invariant measures:

- **(i)** Existence
- **(ii)** Uniqueness
- **(iii)** Properties: mixing, finite-region corrections, CLT

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Stat-mech models: the Ising model

Ingredients

- (Spin) Configurations: $\{-1,1\} \ni \sigma = (\sigma_x)_{x\in\mathbb{Z}^d}$
- Interaction $-\sigma_x\sigma_y$, for x, y nearest-neighbor (n.n.)
- Hamiltonian:

$$H_\Lambda(\sigma \mid \omega) \;=\; -\sum_{\{x,y\}\subset\Lambda\,\text{n.n.}} \sigma_x\sigma_y - \sum_{\{x,y\}\,\text{n.n.},\, x\in\Lambda,\, y\notin\Lambda} \sigma_x\omega_y$$

- (Conditional) probabilities:

$$\text{Prob}_\Lambda(\sigma \mid \omega) \;=\; \exp\{-\beta\, H_\Lambda(\sigma \mid \omega)\}/\text{Norm.}$$

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Ising model as a model with exclusions

Peierls representation

For "+" or "−" boundary conditions, map

$$\sigma_\Lambda \longleftrightarrow \Gamma_\sigma = \{\gamma\}$$

- Place a plaquette orthogonal to each link with $\sigma_x \sigma_y = -1$
- This yields closed surfaces (curves)
- A *contour* γ is a maximally connected closed surface
- Each contour has a weight $w(\gamma) = \mathrm{e}^{-2\beta|\gamma|}$

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Contour measures

The map leads to measures

$$\mu_\Lambda(\Gamma) = \prod_{\gamma\in\Gamma} w(\gamma) \prod_{\gamma,\theta\in\Gamma} \big[1 - I(\gamma,\theta)\big]/\text{Norm.} \tag{1}$$

with

$$I(\gamma,\theta) = \begin{cases} 1 & \text{if } \gamma \text{ incompatible with } \theta \\ 0 & \text{otherwise} \end{cases}$$

Incompatible: Share endpoints of links

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As $\Lambda \to \mathbb{Z}^d$

- Do infinite contours develop?
- Are there increasing sequences of nested contours?

If the answer to both questions is *no*, then

- Typically: sea of one spin value with islands of the other
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Loss network representation

Key: μ_Λ is invariant for a loss-network-like process:

- Each γ attempts birth at indep. Poissonian rate $w(\gamma)$
- Birth is successful in the absence of incompatibilities
- Born γ's live independent $\exp(1)$ times

Random-cluster model

Ising with q colors = Potts

Fortuin-Kasteleyn representation yields μ_Λ as in (1) with

$$\gamma \;=\; \text{connected sets of bonds}$$

and

$$w(\gamma) \;=\; \Big(\frac{p}{1-p}\Big)^{B(\gamma)} \Big(\frac{1}{q}\Big)^{V(\gamma)}$$

with

- $p = 1 - \mathrm{e}^{-\beta}$
- $B(\gamma) = \#$ links in γ
- $V(\gamma) = \#$ vertices in links in γ

Compatibility = no vertex sharing

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- Low-T expansions
- High-T expansion
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Point process = random subset of $\mathbb{L}$ (e.g. $\mathbb{Z}^d$ or $\mathbb{R}^d$)

- Each *seed* x planted with independent Poissonian rate
- x carries a *grain* G_x of deterministic or random shape
- The planting is successful if the emerging seed
  - satisfies a (deterministic) compatibility constraint, or
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Examples with stochastic conditions:

- *Area-interacting processes* (Baddeley - van Lieshout, 1995):

$$\text{Prob} \propto \exp(\text{overlapping area})$$

- *Strauss processes* (Strauss, 1975):

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Finite-region forward simulation

Natural simulation for a finite region Λ

- Choose an initial call-lifetime configuration (e.g. empty)
- Run clocks, at each ring check compatibility of perform test
- If passed, generate lifetime
- Calls dissapear when lifetime exhausted
- Continue until desired endtime

Alternative: Independent death times; pick next one

Above: more economical; leads to cylinders

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Forward-forward simulation

Alternative two-step construction

Step 1: The free process

- All calls are established
- With each call γ two variables are generated:
  - Lifetime: $S_\gamma \approx \exp(1)$
  - Test variable: $Z_\gamma \approx U(0,1)$

Free process: $(\gamma^{(i)}, S_\gamma^{(i)}, Z_\gamma^{(i)})$ $[\gamma^{(i)} = i\text{-th occurrence of } \gamma]$

Visualized as marked cylinders

$$\left(\gamma^{(i)} \times [t_i, t_i + S_{\gamma^{(i)}}], Z_{\gamma^{(i)}}\right)$$

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Step 2: Cleaning (or thining)

- Keep 1st generation (eg. initial) cylinders
- Test or check 2nd generation and keep survivors.
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Features:

- Many mathematical properties can be directly derived from the simpler free process
- Free process = coupling between loss networks with same rates but different compatibility or survival conditions

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Previous construction useless in infinite volume (no first ring)

Backwards-constructed free process:

- Time zero: start with empty finite window Λ
- Run $w(\gamma)$-Poissonian cloks towards the past for $\gamma \cap \Lambda \neq \emptyset$
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Result: first-ring cylinder $(\gamma_1 \times [-t_1, 0], Z_{\gamma_1})$

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- Properties of inv. measure related to backwards cylinders

Big question: Conditions of feasibility

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Clan of ancestors

Must track cylinders alive at moment of birth

Definition
$\widetilde{C} = \widetilde{\gamma} \times [-\widetilde{t}, -\widetilde{t} + \widetilde{s}]$ is an **ancestor** of $C = \gamma \times [-t, -t+s]$ if

$$\begin{aligned} -\widetilde{t} &\le -t \\ -\widetilde{t} + \widetilde{s} &\ge -t \\ \widetilde{\gamma} &\text{ incomp. } \gamma \end{aligned}$$

Then

$$\begin{aligned} A_1^C &= \{\text{ancestors of } C\} \\ A_2^C &= A_1^{A_1^C} \\ &\cdots \\ A_\infty^C = \bigcup_n A_n^C &= \textbf{Clan of ancestors of } C \end{aligned}$$

[Likewise A_∞^Λ]

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Consequences of finiteness of clans

Finite-clan condition of the free process: for each cylinder C,

$$|A_\infty^C| < \infty \quad \text{a.a.}$$

If this condition holds,

- Back-forth construction works
- There exists a unique invariant measure
- The back-forth construction is a perfect-simulation scheme
- Time-length of clans → speed of convergence
- Space-time size of clans →
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Clans and mixing properties

More precisely

Time mixing

$$\Big|E\big[f(\eta^t)\big] - \mu(f)\Big| \;\propto \sum_{x\in \operatorname{supp} f} P\big[TL(A_\infty^{(x,0)} > t\big]$$

E.g. $w(\gamma) \sim \mathrm{e}^{-2\beta|\gamma|} \Rightarrow$ exponential speed

Space mixing

$$\Big|\mu(fg) - \mu(f)\mu(g)\Big| \;\propto\; P\Big[A_\infty^{\operatorname{supp} f} \neq \widehat{A}_\infty^{\operatorname{supp} g}\Big]$$

[A, $\widetilde{A}$ independent realizations]

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$$\Big|\mu(fg) - \mu(f)\mu(g)\Big| \;\propto\; P\Big[A_\infty^{\operatorname{supp} f} \neq \widehat{A}_\infty^{\operatorname{supp} g}\Big]$$

[A, $\widetilde{A}$ independent realizations]

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Clans and mixing properties

More precisely

Time mixing

$$\Big|E\big[f(\eta^t)\big] - \mu(f)\Big| \;\propto\; \sum_{x\in \operatorname{supp} f} P\big[TL(A_\infty^{(x,0)} > t\big]$$

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Condition for finiteness of clans

Deterministic case (incompatibilities). Standard approach:

"Ancestor of" $\rightarrow$ Backwards oriented percolation (BAP) of cylinders

Finite clan $\Longleftrightarrow$ no percolation

BAP dominated by multitype branching

- Independent branches
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$C = \gamma \times [t_B, t_D]$ produces a branch $\theta \times [t, t+s]$ $(t \le t_B)$at rate

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Total rate of branches of "basis" θ:

$$m(\gamma,\theta) = w(\theta)\mathbb{1}[\gamma \not\sim \theta] \int_{t_B}^{\infty} \mathrm{e}^{-(t_B - t)} dt = w(\theta)\mathbb{1}[\gamma \not\sim \theta]$$

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Mean total number of grand-branches of each cylinder of base γ

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By Borel Cantelli

$$\text{finite branching} \iff \sum_n \sum_\theta m^n(\gamma,\theta) < \infty \; \forall \gamma \impliedby \alpha < 1$$

Extinction $\Rightarrow$ lack of BAP $\Rightarrow$ back-forth OK if

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Theoretical tool

Allows proof, for the invariant measure, of

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- Mixing properties
- Typical configurations

in a region larger than the default cluster-expansion approach (which, however, yields analyticity)

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- Perfect simulation scheme without monotonicity
- Fast convergence, without metastable traps
- Samples directly from full volume (no finite-volumen corrections)

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