# Stochastic ordering of network throughputs using flow couplings

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# Jyväskylä, Finland


**City**: Population 130,000; area 1,500 km$^2$ (20% lakes)

**University**: 14,500 students; 1,700 academic staff

**Notable people**: Alvar Aalto (architect), Sofi Oksanen (writer), Tommi Mäkinen (rally driver), Minna Kauppi (orienteer), Matti Nykänen (ski jumper)

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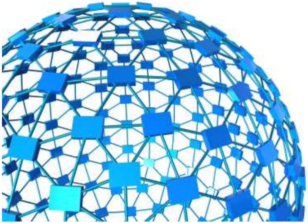
# Stochastic orders in stochastic networks



## Realistic network models

- High-dimensional state vectors
- Network-dependent transition rates
- Finite buffers

$\rightsquigarrow$ Hard to analyze

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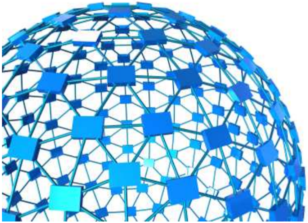

## Realistic network models

- High-dimensional state vectors
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$\rightsquigarrow$ Hard to analyze

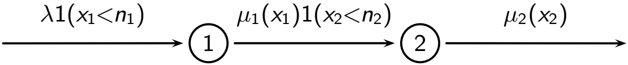
## Stochastic comparison approach

Find a reference model which

- Performs worse than the original
- Can be proven to do so analytically
- Is computationally tractable

$\rightsquigarrow$ *Computable & conservative performance estimates*

# Two-node linear network




## Blocking

- Arrivals blocked when $X_1(t) = n_1$
- 1st server halts when $X_2(t) = n_2$

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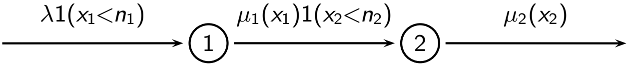


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## Service station models

- Single-server: $\mu_i(x_i) = c_i 1(x_i > 0)$
- Multi-server: $\mu_i(x_i) = c_i x_i$
- More general: $\mu_i = \mu_i(x_1, x_2)$

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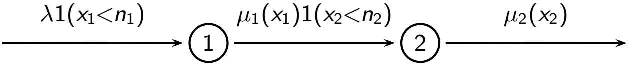


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## Stochastic model

Markov jump process $X = (X_1, X_2)$ with generator

$$Au(x) = \alpha_{0,1}(x)[u(x+e_1) - u(x)] + \alpha_{1,2}(x)[u(x-e_1+e_2) - u(x)] + \alpha_{2,0}(x)[u(x-e_2) - u(x)]$$

where

- $\alpha_{0,1}(x) = \lambda 1(x_1 < n_1)$
- $\alpha_{1,2}(x) = \mu_1(x_1) 1(x_2 < n_2)$
- $\alpha_{2,0}(x) = \mu_2(x_2)$

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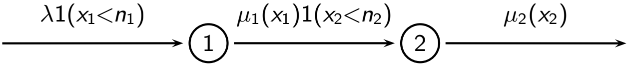


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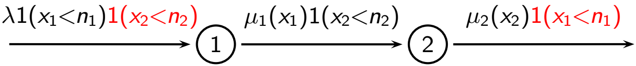
## Performance

- Equilibrium probability distribution $\pi$
- Blocking set $B = \{x : x_1 = n_1\}$
- Loss rate $\lambda\pi(B)$
- Throughput rate $\lambda(1 - \pi(B))$

Computing $\pi$ is hard except for special cases of $\mu_1$ and $\mu_2$

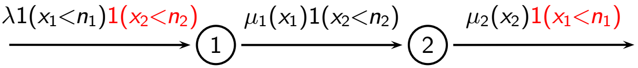
# Balanced system modification




## Balanced operation

- Arrivals blocked when $X_1(t) = n_1$ or $X_2(t) = n_2$
- 1st server halts when $X_2(t) = n_2$
- 2nd server halts when $X_1(t) = n_1$

# Balanced system modification

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- 2nd server halts when $X_1(t) = n_1$

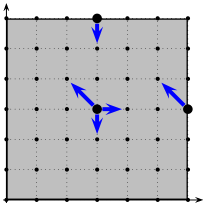
## Performance

- Equilibrium distribution $\pi^{\mathrm{bal}}$
- Blocking set $B^{\mathrm{bal}} = \{x : x_1 = n_1 \text{ or } x_2 = n_2\}$
- Loss rate $\lambda \pi^{\mathrm{bal}}(B^{\mathrm{bal}})$
- Throughput rate $\lambda(1 - \pi^{\mathrm{bal}}(B^{\mathrm{bal}}))$

Balanced system has a product-form equilibrium (van der Wal & van Dijk 1989)

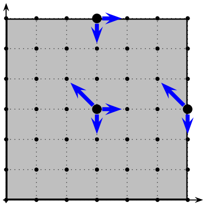
# Balanced vs. original system

Balanced system



$B^{\mathrm{bal}} = \{x : x_1 = n_1 \text{ or } x_2 = n_2\}$

Original system



$B^{\mathrm{orig}} = \{x : x_1 = n_1\}$

## Performance comparison

- Balanced system has more blocking states: $B^{\mathrm{bal}} \supset B^{\mathrm{orig}}$
- $\rightsquigarrow$ Balanced system should have a higher loss rate and smaller throughput
- $\rightsquigarrow$ Conservative & computable performance bound

How to prove the comparison statement?

- Sample path comparison

# Sample path comparison

Heuristic reasoning:

- Balanced system has more blocking states

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- $\rightsquigarrow$ Spends less time in blocking states

# Sample path comparison

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- $\rightsquigarrow$ Has less jobs in the system
- $\rightsquigarrow$ Spends less time in blocking states
- $\rightsquigarrow$ Blocks less jobs?

## How to prove the comparison statement?

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- ~~Sample path comparison~~
- Order-preserving Markov coupling

# Markov couplings


Andrei Markov (1978–)
Montreal Canadiens

Andrei Markov (1856–1922)
St Petersburg University

# Markov couplings

## Coupling of rate matrices

A transition rate matrix $\tilde{Q}$ on $S \times S'$ is a **Markov coupling** of transition rate matrices $Q$ on $S$ and $Q'$ on $S'$ if for all $x \in S$ and $x' \in S'$:

$$\sum_{y' \in S'} \tilde{Q}((x,x'),(y,y')) = Q(x,y) \quad \text{for all } y \neq x,$$

$$\sum_{y \in S} \tilde{Q}((x,x'),(y,y')) = Q'(x',y') \quad \text{for all } y' \neq x'.$$

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## Coupling of stochastic processes

If $\tilde{X} = (\tilde{X}, \tilde{X}')$ is a Markov process with transition rate matrix $\tilde{Q}$, then

- $\tilde{X}$ is Markov with transition rate matrix $Q$
- $\tilde{X}'$ is Markov with transition rate matrix $Q'$

# Approach using order-preserving Markov couplings

A Markov coupling $\tilde{Q}$ on an ordered state space $(S, \leq)$ is order-preserving if

$$x \leq x' \text{ and } \tilde{Q}((x, x'), (y, y')) > 0 \quad \Longrightarrow \quad y \leq y'.$$

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## Comparing the blocking rates

Find an order relation $\leq$ on $[0, n_1] \times [0, n_2]$ such that

- $x \leq x' \implies 1_B(x) \leq 1_{B'}(x')$
- There exists a $\leq$-preserving Markov coupling of the systems

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## Feasible order relations

- Coordinatewise order: $x \leq x'$ if $x_1 \leq x_1'$ and $x_2 \leq x_2'$
- 1st coordinate order: $x \leq x'$ if $x_1 \leq x_1'$

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No order-preserving Markov coupling for these exists (Reason: when $X(t) = x$ and $X'(t) = x$ for some $x$ such that $x_1 = n_1$, the original system spends a longer time in its blocking set.)

How to prove the comparison statement?

- ~~Sample path comparison~~
- ~~Order-preserving Markov coupling~~

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- Relation-preserving Markov coupling

# Relation-preserving Markov couplings

Find a relation $R \subset S \times S'$ such that

- $(x, x') \in R \implies 1_B(x) \leq 1_{B'}(x)$
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A transition rate matrix $\tilde{Q}$ on $S \times S$ is $R$-preserving if

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Does it exist?

- Can be checked using a subrelation algorithm (Leskelä 2010)
- Does not exist.

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How to prove the comparison statement?

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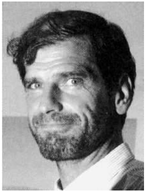
- ~~Order-preserving Markov coupling~~
- ~~Relation-preserving Markov coupling~~
- Markov reward approach



Nico van Dijk 1998:

> *Established comparison, or relatedly monotonicity, proof techniques such as the one-step comparison technique (Keilson & Kester 1977; Whitt 1981, 1986; Massey 1987) and the related sample path technique as in (Shanthikumar & Yao 1986, 1988; van Dijk & Tsoucas & Walrand 1988; Adan & van der Wal 1989), however, do not generally apply.*

# Markov reward approach

Uniformize $\rightsquigarrow$ discrete-time Markov processes

- Original system: transition matrix $P_o$
- Balanced system: transition matrix $P_b$

# Markov reward approach

Uniformize $\rightsquigarrow$ discrete-time Markov processes

- Original system: transition matrix $P_o$
- Balanced system: transition matrix $P_b$

Mean number of departures during first $t$ time steps:

- $V_o^t(x) = \mathrm{E}\{\sum_{s=0}^{t-1} r_o(X(s)) \,|\, X(0) = x\}$, $r_o(x) = \mu_2(x_2)$
- $V_b^t(x) = \mathrm{E}\{\sum_{s=0}^{t-1} r_b(X'(s)) \,|\, X'(0) = x\}$, $r_b(x) = \mu_2(x)1(x_1 < n_1)$

# Markov reward comparison

**Theorem** (Van Dijk 1998)

*Assume that* $r_o(x) + P_o V_o^{t-1}(x) \geq r_b(x) + P_b V_o^{t-1}(x)$ *for all* $x$ *and all* $t \geq 1$.
*Then* $V_o^t(x) \geq V_b^t(x)$ *for all* $x$ *and all* $t \geq 0$.

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## Proof.

Conditioning on the first jump yields the recursive equations:

- $V_o^t = r_o + P_o V_o^{t-1}, \quad t \geq 1$
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By subtracting these, and then using the induction assumption $V_o^{t-1} \geq V_b^{t-1}$,

$$\begin{aligned} V_o^t - V_b^t &= r_o - r_b + P_o V_o^{t-1} - P_b V_b^{t-1} \\ &= r_o - r_b + (P_o - P_b) V_o^{t-1} + P_b (V_o^{t-1} - V_b^{t-1}) \\ &\geq r_o - r_b + (P_o - P_b) V_o^{t-1}. \end{aligned}$$

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The last term on the right is positive by the assumption. □

# Applying the Markov reward comparison

Can we show that

$$r_o(x) + P_o V_o^{t-1}(x) \geq r_b(x) + P_b V_o^{t-1}(x)$$

for the two-node queueing network?

# Applying the Markov reward comparison

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Yes. (van der Wal & van Dijk 1989)

- Uniformize.
- Prove by induction.

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Yes. (van der Wal & van Dijk 1989)

- Uniformize.
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As a consequence, the mean throughputs are ordered by

- $\mathsf{E}\, F_{\mathrm{dep}}^{\mathrm{bal}}(t) \leq \mathsf{E}\, F_{\mathrm{dep}}^{\mathrm{orig}}(t)$ for all $t \geq 0$

whenever both systems are started at the same initial state.

How to prove the comparison statement?

- ~~Sample path comparison~~
- ~~Order-preserving Markov coupling~~
- ~~Relation-preserving Markov coupling~~
- Markov reward approach (OK for mean throughputs)

# Markov reward approach: Pros and cons

## Pros

- Works when Markov couplings don't
- Tailor-made for chosen reward functions
- Time-dependent comparison results

## Cons

- Only a conceptual framework: Requires proving an induction argument
- Hard to tell when works
- Uniformization leads to unintuitive notation
- Results only for the mean rewards

# Hybrid approach

Can we incorporate a reward structure to a coupling construction?

- Use a (non-Markov) coupling
- Embed a reward structure explicitly
- Prove pathwise ordering of rewards

How to prove the comparison statement?

- ~~Sample path comparison~~
- ~~Order-preserving Markov coupling~~
- ~~Relation-preserving Markov coupling~~
- Markov reward approach (OK for mean throughputs)
- Flow coupling

# General Markov network

Traffic flows on a graph $G = (\{0, \dots, n\}, L)$:

- Nodes $1, 2, \dots, n$
- Node 0 represents the external world
- Directed links between nodes $L \subset \{0, \dots, n\}^2$

Network state: Markov jump process $X$ in $S \subset \mathbb{Z}_+^n$ with transitions

$$x \mapsto x - e_i + e_j \text{ at rate } \alpha_{i,j}(x), \quad (i,j) \in L,$$

where $e_i$ is the $i$-th unit vector in $\mathbb{Z}^n$ and $e_0 = 0$
Generator

$$Au(x) = \sum_{(i,j)\in L} \alpha_{i,j}(x)[u(x - e_i + e_j) - u(x)]$$

# State-flow Markov process

Markov jump process $(X, F)$ in $S \times \mathbb{Z}_+^L$ with transitions

$$(x, f) \mapsto (x - e_i + e_j, f + e_{i,j}) \text{ at rate } \alpha_{i,j}(x), \quad (i, j) \in L$$

- $X_i(t)$ is the number of jobs in node $i$ at time $t$
- $F_{i,j}(t) - F_{i,j}(0)$ is the number of transitions over link $(i, j)$ during $(0, t]$

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Redundant process: $F(t)$ may be recovered by observing the path of $X$ up to time $t$ by using the formula

$$F_{i,j}(t) - F_{i,j}(0) \; = \; \# \{ s \in (0, t] : \; X(s) - X(s-) = -e_i + e_j \} ,$$

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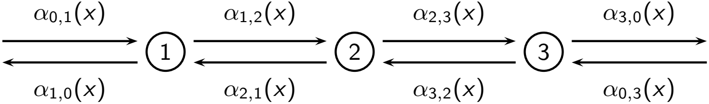
Generator

$$Af(x, f) = \sum_{(i,j) \in L} \alpha_{i,j}(x)[u(x - e_i + e_j, f + e_{i,j}) - u(x, f)]$$

# Netflow ordering in state-flow space




## State-flow relation

- $(x, f)$ has smaller netflow than $(x', f')$ if

$$f_{i,i+1} - f_{i+1,i} \le f'_{i,i+1} - f'_{i+1,i} \quad \text{for all } i = 0, 1, \ldots, d,$$
$$x_i - f_{\mathrm{in},i} + f_{i,\mathrm{out}} = x'_i - f'_{\mathrm{in},i} + f'_{i,\mathrm{out}} \quad \text{for all nodes } i = 1, \ldots, d,$$

## Notation

- $f_{\mathrm{in},i} = \sum_{j \neq i} f_{j,i}, \quad f_{i,\mathrm{out}} = \sum_{j \neq i} f_{i,j}$
- $f_{d,d+1} = f_{d,0},\ f_{d+1,d} = f_{0,d}$

# Flow coupling for linear networks

## Theorem

*Assume that*

$$
\begin{aligned}
x_1 \geq x_1' &\implies \alpha_{0,1}(x) \leq \alpha_{0,1}'(x') \text{ and } \alpha_{1,0}(x) \geq \alpha_{1,0}'(x'), \\
x_i \leq x_i' \text{ and } x_{i+1} \geq x_{i+1}' &\implies \alpha_{i,i+1}(x) \leq \alpha_{i,i+1}'(x') \text{ and } \alpha_{i+1,i}(x) \geq \alpha_{i+1,i}'(x'), \\
x_d \leq x_d' &\implies \alpha_{d,0}(x) \leq \alpha_{d,0}'(x') \text{ and } \alpha_{0,d}(x) \geq \alpha_{0,d}'(x').
\end{aligned}
$$

*Then there exists a Markov coupling of $(X, F)$ and $(X', F')$ which preserves the netflow relation. Especially, the netflow counting processes are ordered by*

$$N_{i,i+1}(t) \leq_{\mathrm{st}} N_{i,i+1}'(t)$$

*for all $t \geq 0$ and $i = 0, \ldots, d$, whenever $X(0) =_{\mathrm{st}} X'(0)$.*

# Flow coupling for linear networks

### Proof: Coupling property.

Let $(\tilde{X}, \tilde{F}, \tilde{X}', \tilde{F}')$ be a Markov process with transitions

$$((x,f),(x',f')) \mapsto \begin{cases} (T_{i,j}(x,f), T_{i,j}(x',f')) & \text{at rate } \alpha_{i,j}(x) \wedge \alpha'_{i,j}(x'), \\ ((x,f), T_{i,j}(x',f')) & \text{at rate } (\alpha'_{i,j}(x') - \alpha_{i,j}(x))_+, \\ (T_{i,j}(x,f),(x,f)) & \text{at rate } (\alpha_{i,j}(x) - \alpha'_{i,j}(x'))_+, \end{cases}$$

where $T_{i,j}(x,f) = (x - e_i + e_j, f + e_{i,j})$

- This is the marching soldiers coupling of $(X, F)$ and $(X', F')$ (Mu-Fa Chen 2005). Why coupling?

# Flow coupling for linear networks

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Let $(\tilde{X}, \tilde{F}, \tilde{X}', \tilde{F}')$ be a Markov process with transitions

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where $T_{i,j}(x,f) = (x - e_i + e_j, f + e_{i,j})$

- This is the marching soldiers coupling of $(X,F)$ and $(X',F')$ (Mu-Fa Chen 2005). Why coupling? Because

$$(a \wedge a') + (a' - a)_+ = a'$$
$$(a \wedge a') + (a - a')_+ = a$$

□

# Flow coupling for linear networks

## Proof: Relation preservation.

Consider the relation $f_{i,i+1} - f_{i+1,i} \leq f'_{i,i+1} - f'_{i+1,i}$ for $i = 1$.

## Flow coupling for linear networks

### Proof: Relation preservation.

Consider the relation $f_{i,i+1} - f_{i+1,i} \leq f'_{i,i+1} - f'_{i+1,i} \quad \text{for } i = 1.$

- Breaking this is possible only when $f_{1,2} - f_{2,1} = f'_{1,2} - f'_{2,1}$

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- Breaking this is possible only when $f_{1,2} - f_{2,1} = f'_{1,2} - f'_{2,1}$
- Flow balance equations at node 1 and node 2:

$$
\begin{aligned}
x_1 - (f_{0,1} + f_{2,1}) + (f_{1,0} + f_{1,2}) &= x'_1 - (f'_{0,1} + f'_{2,1}) + (f'_{1,0} + f'_{1,2}) \\
x_2 - (f_{1,2} + f_{3,2}) + (f_{2,1} + f_{2,3}) &= x'_2 - (f'_{1,2} + f'_{3,2}) + (f'_{2,1} + f'_{2,3})
\end{aligned}
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Consider the relation $f_{i,i+1} - f_{i+1,i} \leq f'_{i,i+1} - f'_{i+1,i} \quad$ for $i = 1$.

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$$

- In light of the equality, these imply

$$
\begin{aligned}
x_1 - (f_{0,1} - f_{1,0}) &= x'_1 - (f'_{0,1} - f'_{1,0}) \\
x_2 + (f_{2,3} - f_{3,2}) &= x'_2 + (f'_{2,3} - f'_{3,2})
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- Therefore, $x_1 \leq x'_1$ and $x_2 \geq x'_2$,

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- The assumption implies $\alpha_{1,2}(x) \leq \alpha'_{1,2}(x)$

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- In light of the equality, these imply

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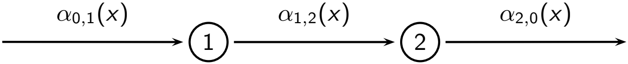
- Therefore, $x_1 \leq x'_1$ and $x_2 \geq x'_2$,
- The assumption implies $\alpha_{1,2}(x) \leq \alpha'_{1,2}(x)$
- The transition $((x, f), (x', f')) \mapsto (T_{1,2}(x, f), (x, f))$ has rate $(\alpha_{1,2}(x) - \alpha'_{1,2}(x'))_+ = 0$

□

# Balanced vs. original two-node network



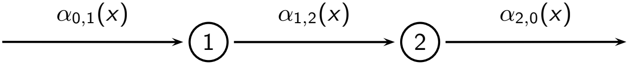

Balanced system

- $\alpha_{0,1}^{\mathrm{bal}}(x) = \lambda 1(x_1{<}n_1)1(x_2{<}n_2)$
- $\alpha_{1,2}^{\mathrm{bal}}(x) = \mu_1(x_1)1(x_2{<}n_2)$
- $\alpha_{2,0}^{\mathrm{bal}}(x) = \mu_2(x_2)1(x_1{<}n_1)$

Original system

- $\alpha_{0,1}^{\mathrm{orig}}(x) = \lambda 1(x_1{<}n_1)$
- $\alpha_{1,2}^{\mathrm{orig}}(x) = \mu_1(x_1)1(x_2{<}n_2)$
- $\alpha_{2,0}^{\mathrm{orig}}(x) = \mu_2(x_2)$

# Balanced vs. original two-node network

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Original system

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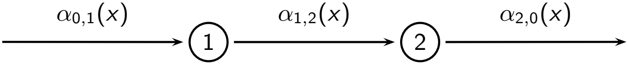
- $\alpha_{1,2}^{\mathrm{orig}}(x) = \mu_1(x_1) 1(x_2 < n_2)$
- $\alpha_{2,0}^{\mathrm{orig}}(x) = \mu_2(x_2)$

$(X^{\mathrm{bal}}, F^{\mathrm{bal}})$ has a stochastically smaller flow than $(X^{\mathrm{orig}}, F^{\mathrm{orig}})$ if

$$\begin{aligned} x_1 \geq x_1' &\implies \alpha_{0,1}^{\mathrm{bal}}(x) \leq \alpha_{0,1}^{\mathrm{orig}}(x') \\ x_1 \leq x_1' \text{ and } x_2 \geq x_2' &\implies \alpha_{1,2}^{\mathrm{bal}}(x) \leq \alpha_{1,2}^{\mathrm{orig}}(x') \\ x_2 \leq x_2' &\implies \alpha_{2,0}^{\mathrm{bal}}(x) \leq \alpha_{2,0}^{\mathrm{orig}}(x'). \end{aligned}$$

# Balanced vs. original two-node network




Balanced system

- $\alpha_{0,1}^{\mathrm{bal}}(x) = \lambda 1(x_1 < n_1) 1(x_2 < n_2)$
- $\alpha_{1,2}^{\mathrm{bal}}(x) = \mu_1(x_1) 1(x_2 < n_2)$
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Original system

- $\alpha_{0,1}^{\mathrm{orig}}(x) = \lambda 1(x_1 < n_1)$
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$(X^{\mathrm{bal}}, F^{\mathrm{bal}})$ has a stochastically smaller flow than $(X^{\mathrm{orig}}, F^{\mathrm{orig}})$ if

$$\begin{aligned} x_1 \ge x_1' &\implies \lambda 1(x_1 < n_1) 1(x_2 < n_2) \le \lambda 1(x_1' < n_1) \\ x_1 \le x_1' \text{ and } x_2 \ge x_2' &\implies \mu_1(x_1) 1(x_2 < n_2) \le \mu_1(x_1') 1(x_2' < n_2) \\ x_2 \le x_2' &\implies \mu_2(x_2) 1(x_1 < n_1) \le \mu_2(x_2') \end{aligned}$$

The above conditions are valid when $\mu_1$ and $\mu_2$ are increasing.

How to prove the comparison statement?

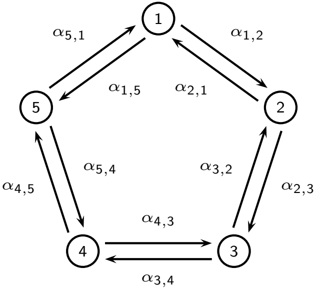
- ~~Sample path comparison~~
- ~~Order-preserving Markov coupling~~
- ~~Relation-preserving Markov coupling~~
- Markov reward approach (OK for mean throughputs)
- Flow coupling (OK for throughput distributions)

# Generalizations

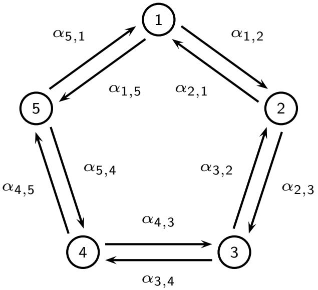
## Other networks structures?

- Closed linear networks (cyclic networks)
- Aggregate flows across linear partitions

# Flow ordering in cyclic networks



# Flow ordering in cyclic networks



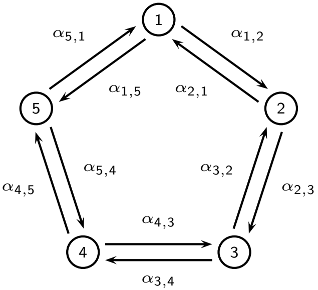

## Theorem

*Assume that for all $i$ and for all $x$ and $x'$,*

$$x_i \le x_i' \text{ and } x_{i+1} \ge x_{i+1}'$$
$$\Longrightarrow$$
$$\alpha_{i,i+1}(x) \le \alpha'_{i,i+1}(x') \text{ and } \alpha_{i+1,i}(x) \ge \alpha'_{i+1,i}(x').$$

*Then $(X, F)$ has stochastically smaller clockwise netflow than $(X', F')$.*

# Flow ordering in cyclic networks




## Theorem

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$$\alpha_{i,i+1}(x) \le \alpha_{i,i+1}'(x') \text{ and } \alpha_{i+1,i}(x) \ge \alpha_{i+1,i}'(x').$$

*Then $(X, F)$ has stochastically smaller clockwise netflow than $(X', F')$.*

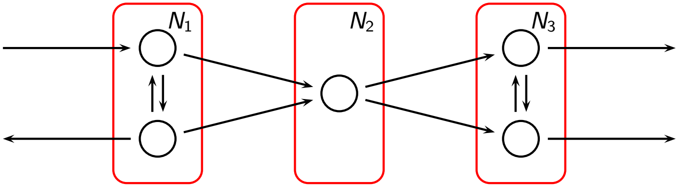
## Proof.

The marching soldiers coupling of $(X, F)$ and $(X', F')$ preserves the state-flow relation

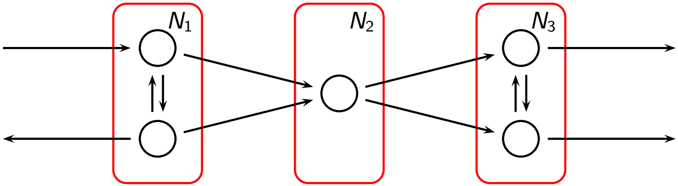
$$f_{i,i+1} - f_{i+1,i} \le f_{i,i+1}' - f_{i+1,i}',$$
$$x_i - f_{\mathrm{in},i} + f_{i,\mathrm{out}} = x_i' - f_{\mathrm{in},i}' + f_{i,\mathrm{out}}'.$$

□

# Aggregate flows through linear partitions



# Aggregate flows through linear partitions




State-flow $(x, f)$ has a smaller netflow through $N_1 \to N_2 \to N_3$ than $(x', f')$ if

$$f_{N_r,N_{r+1}} - f_{N_{r+1},N_r} \leq f'_{N_r,N_{r+1}} - f'_{N_{r+1},N_r} \quad \text{for all clusters } N_r,$$
$$x_i - f_{\mathrm{in},i} + f_{i,\mathrm{out}} = x'_i - f'_{\mathrm{in},i} + f'_{i,\mathrm{out}} \quad \text{for all nodes } i,$$

where

$$f_{N_r,N_s} = \sum_{i \in N_r,\ j \in N_s} f_{i,j}$$

# Aggregate flows through linear partitions

## Theorem

*The system* $(X, F)$ *has a stochastically smaller netflow through* $N_1 \to \cdots \to N_m$ *than* $(X', F')$ *if for all* $x \in S$ *and* $x' \in S'$*:*

(i) $|x_{N_1}| \geq |x'_{N_1}| \implies$

$$\alpha_{0,N_1}(x) \leq \alpha'_{0,N_1}(x') \quad \textit{and} \quad \alpha_{N_1,0}(x) \geq \alpha'_{N_1,0}(x').$$

(ii) $|x_{N_k}| \leq |x'_{N_k}|$ *and* $|x_{N_{k+1}}| \geq |x'_{N_{k+1}}| \implies$

$$\alpha_{N_k,N_{k+1}}(x) \leq \alpha'_{N_k,N_{k+1}}(x') \quad \textit{and} \quad \alpha_{N_{k+1},N_k}(x) \geq \alpha'_{N_{k+1},N_k}(x').$$

(iii) $|x_{N_m}| \leq |x'_{N_m}| \implies$

$$\alpha_{N_m,0}(x) \leq \alpha'_{N_m,0}(x') \quad \textit{and} \quad \alpha_{0,N_m}(x) \geq \alpha'_{0,N_m}(x')$$

## Notation

- $|x_{N_r}| = \sum_{i \in N_r} x_i$
- $\alpha_{N_r,N_s}(x) = \sum_{i \in N_r, j \in N_r} \alpha_{i,j}(x)$

# Aggregate flows through linear partitions

## Theorem

*The system* $(X, F)$ *has a stochastically smaller netflow through* $N_1 \to \cdots \to N_m$ *than* $(X', F')$ *if for all* $x \in S$ *and* $x' \in S'$*:*

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(ii) $|x_{N_k}| \leq |x'_{N_k}|$ *and* $|x_{N_{k+1}}| \geq |x'_{N_{k+1}}| \implies$

$$\alpha_{N_k,N_{k+1}}(x) \leq \alpha'_{N_k,N_{k+1}}(x') \quad \textit{and} \quad \alpha_{N_{k+1},N_k}(x) \geq \alpha'_{N_{k+1},N_k}(x').$$

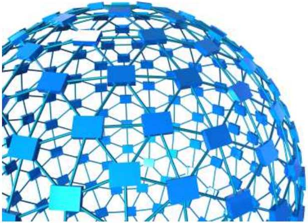
(iii) $|x_{N_m}| \leq |x'_{N_m}| \implies$

$$\alpha_{N_m,0}(x) \leq \alpha'_{N_m,0}(x') \quad \textit{and} \quad \alpha_{0,N_m}(x) \geq \alpha'_{0,N_m}(x')$$

## Proof.

Marching soldiers coupling does not work in general. A netflow-preserving state-flow coupling can be shown to exist (Whitt 1986; Massey 1987; Leskelä 2010). □

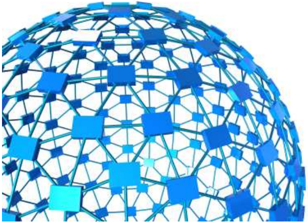
# Conclusion & discussion



## Flow coupling

- State-flow redundant model $\rightsquigarrow$ non-Markov coupling
- Sample paths coupled when both systems started at the same state

# Conclusion & discussion



## Flow coupling

- State-flow redundant model $\rightsquigarrow$ non-Markov coupling
- Sample paths coupled when both systems started at the same state

## Related work

- Generalized semi-Markov processes (Glasserman & Yao 1994)
- Linear bandwidth-sharing networks (Verloop & Ayesta & Borst 2010)
- Chip-firing games (Eriksson 1996)
- Sleepy random walkers (Dickman & Rolla & Sidoravicius 2010)

M.-F. Chen.
*Eigenvalues, Inequalities, and Ergodic Theory.*
Springer, 2005.

P. Diaconis and W. Fulton.
A growth model, a game, an algebra, Lagrange inversion, and characteristic classes.
*Rend. Sem. Mat. Univ. Politec. Torino*, 49(1):95–119 (1993), 1991.

R. Dickman, L. T. Rolla, and V. Sidoravicius.
Activated random walkers: facts, conjectures and challenges.
*J. Stat. Phys.*, 138(1-3):126–142, 2010.

N. M. van Dijk.
Bounds and error bounds for queueing networks.
*Ann. Oper. Res.*, 79:295–319, 1998.

N. M. van Dijk and J. van der Wal.
Simple bounds and monotonicity results for finite multi-server exponential tandem queues.
*Queueing Syst.*, 4(1):1–15, 1989.

K. Eriksson.
Chip-firing games on mutating graphs.
*SIAM J. Discrete Math.*, 9(1):118–128, 1996.

P. Glasserman and D. D. Yao.

*Monotone Structure in Discrete-Event Systems*.
Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics. John Wiley & Sons Inc., New York, 1994.
A Wiley-Interscience Publication.

L. Leskelä.
Computational methods for stochastic relations and Markovian couplings.
In *Proc. 4th International Workshop on Tools for Solving Structured Markov Chains (SMCTools)*, 2009.

L. Leskelä.
Stochastic relations of random variables and processes.
*J. Theor. Probab.*, 23(2):523–546, 2010.

I. Verloop, U. Ayesta, and S. Borst.
Monotonicity properties for multi-class queueing systems.
*Discrete Event Dyn. Syst.*, 20:473–509, 2010.