A very simple model of a limit order book

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Introduction

A *limit order book* is a pricing mechanism for a single-commodity market.

Other pricing mechanisms:

- Barter
- Haggling in the marketplace
- Auctions
- Walrasian market: send requests to buy and sell to a third party, who will set a single price that maximizes trade

Before limit order books:

- A few large participants publish a “buy” price and a “sell” price, at which they promise to buy (resp. sell) the asset
- The difference (sell)−(buy) is the fee charged for providing guaranteed liquidity
- All other market participants wanting to trade the asset see only the above (small) list of prices

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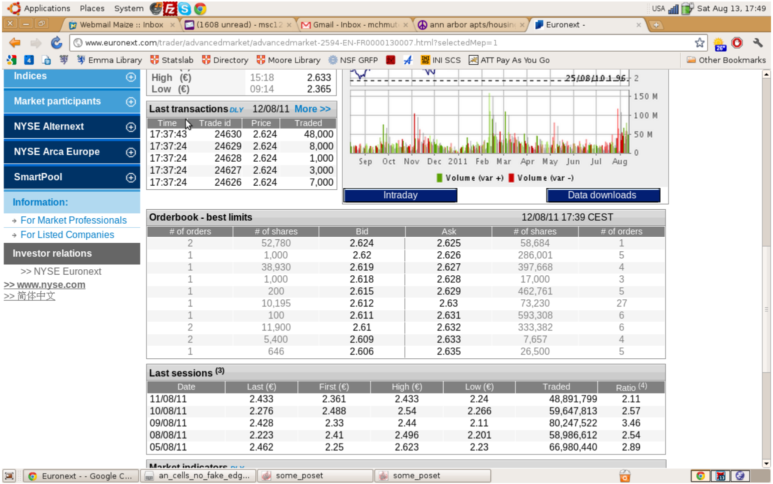
Plus various frills (partially visible orders etc.)



High (€)	15:18	2.633	
Low (€)	09:14	2.365	

Last transactions DLY 12/08/11 More >>

Time	Trade id	Price	Traded
17:37:43	24630	2.624	48,000
17:37:24	24629	2.624	8,000
17:37:24	24628	2.624	1,000
17:37:24	24627	2.624	3,000
17:37:24	24626	2.624	7,000

Orderbook - best limits 12/08/11 17:39 CEST

# of orders	# of shares	Bid	Ask	# of shares	# of orders
2	52,780	2.624	2.625	58,684	1
1	1,000	2.62	2.626	286,001	5
1	38,930	2.619	2.627	397,668	4
1	1,000	2.618	2.628	17,000	3
1	200	2.615	2.629	462,761	5
1	10,195	2.612	2.63	73,230	27
1	100	2.611	2.631	593,308	6
2	11,900	2.61	2.632	333,382	6
2	5,400	2.609	2.633	7,657	4
1	646	2.606	2.635	26,500	5

Last sessions (3)

Date	Last (€)	First (€)	High (€)	Low (€)	Traded	Ratio (4)
11/08/11	2.433	2.361	2.433	2.24	48,891,799	2.11
10/08/11	2.276	2.488	2.54	2.266	59,647,813	2.57
09/08/11	2.428	2.33	2.44	2.11	80,247,522	3.46
08/08/11	2.223	2.41	2.496	2.201	58,986,612	2.54
05/08/11	2.462	2.25	2.623	2.23	66,980,440	2.89

Market indicators DLY

Other work

There is a lot of research looking into pricing mechanisms, in particular limit order books.

- Empirical observations. Often disagree, possibly because study different markets.
  - Distribution of limit order size
  - Distribution of limit order price
  - Shape of the limit order book
  - Order cancellation
  - Mutual dependence between limit order book and order process
  - ...

- Dynamical systems using statistical data from real markets.
- Stochastic simulation using order flow distributions fitted from data.
- A very large repeated game with perfect intelligence.
- Gode, Sunder (1993): a small market with zero-intelligence traders.
- Various options with partial information...

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  - Possible modification: if $P(p_a) < P(p_b)$ (makes a difference when price levels have positive measure)

Is this realistic? Not at all.
But, it's as simple as it could get, and it's clearly not unrelated to the real problem, so it's worth seeing what we can understand about this model.

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(Bids are red, asks are blue)

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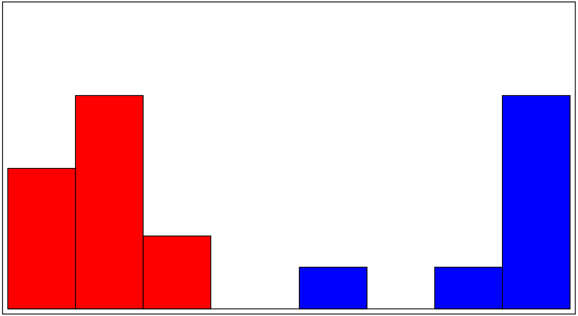

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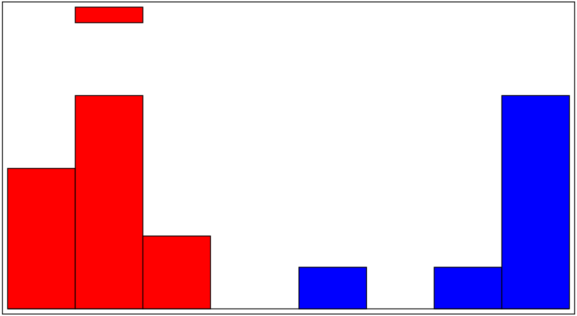

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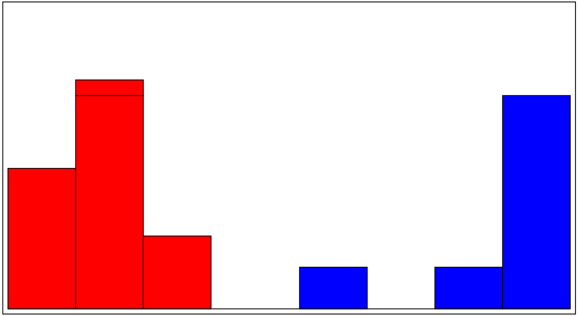

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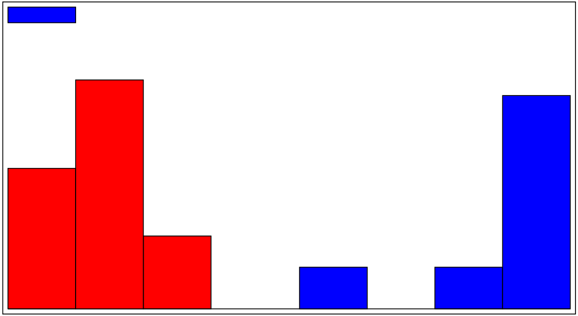

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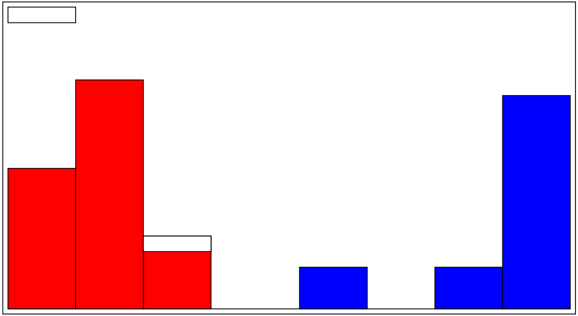

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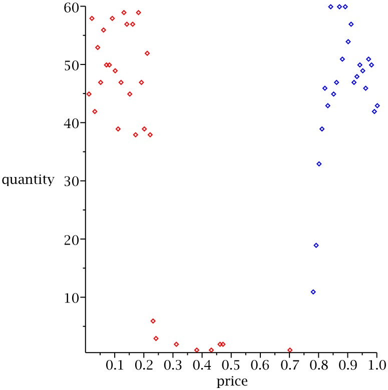
Motivating picture

Number of bids (red) and asks (blue) at a given price level, after a long time



Theorems

Theorem

For any price level function P and any $\epsilon > 0$, there exists a deterministic point κ_b such that the following hold almost surely:

- *Eventually, no bid order at price level $< P(\kappa_b - \epsilon)$ will ever be fulfilled.*
- *The number of unfulfilled bid orders at price levels $> P(\kappa_b + \epsilon)$ hits zero infinitely often.*

(There's also a κ_a with symmetric statements about it.)

Note: don't know if can have no bids right of κ_b and no asks left of κ_a simultaneously.

Sketch of proof.

- A pathwise construction shows that "the number of unfulfilled bids tends to ∞" is a tail event
- If κ_b is the rightmost price level for which this is true, then bids below κ_b will eventually never be the highest bid in the system, so can't leave
- To the right of κ_b, the number of bids does not tend to infinity; whenever the number of bids is $\leq M$ there's a positive (bounded below) probability that over the next M events all bids will leave; Borel-Cantelli lemma says that this will happen infinitely many times
- (In particular, there will be an infinite number of departures from $[P(\kappa_b - \epsilon), P(\kappa_b + \epsilon)]$.)

□

Condition on always having bids at κ_b and asks at κ_a:

- Get a Markov chain in fewer dimensions
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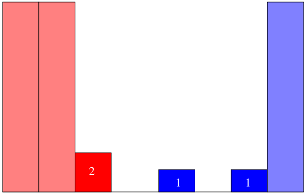


Definition

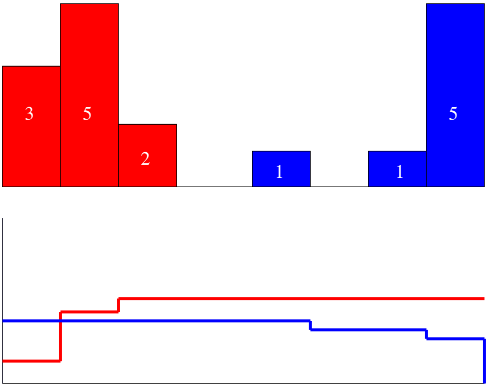
For price level functions P, P' we say P is *coarser* than P' if $P(p) \leq P(q) \implies P'(p) \leq P'(q)$.

E.g., pricing in discrete levels is coarser than continuous pricing.

Definition

The *cumulative bid count* $B_t(p)$ is the number of bids waiting at time t at prices $\leq p$. The cumulative ask count $A_t(q)$ is the number of asks waiting at time t at prices $\geq q$.

Cumulative bid and ask counts:



Theorem

Let L, L' be two limit order books with the same arrival processes and starting states, but two different price level functions P, P'. Let P be coarser than P'. Then $B_t(p) \leq B'_t(p)$ and $A_t(q) \leq A'_t(q)$ at all times t and all prices p, q.

Morally, if we merge price levels, then more orders can leave. This is saying that more orders do leave.

Proof.

Induction. □

Theorem

Let P be coarser than P' as before, but let L and L' be defined using strict inequalities: bid at p and ask at q leave only if $P(q) < P(p)$ (resp. $P'(q) < P'(p)$). Then $B_t(p) \geq B'_t(p)$ and $A_t(q) \geq A'_t(q)$ at all times t and all prices p, q.

In this case, merging price levels means that *fewer* orders can leave. If price levels are equally spaced, this system is very similar to the non-strict-inequality system with one fewer price level.

Theorem (Almost a theorem)

The mysterious constant κ *is bounded below by* $1/9$ *and above by* $1/4$.

Proof.

Analyse some small systems. $1/9$ is the rate at which bids accumulate in a model with 3 price levels.
To get $1/4$:

- Consider Markov chain with edge bins of width a assumed to *always* have waiting orders, and 3 price levels in between.
- For $a > 1$ can construct a Lyapunov function, showing positive recurrence
- Use monotonicity (4-bin system with strict inequalities dominates unbinned chain)

□

The catch: going from limit order book with bins assumed to always have orders to original chain (where this is eventually true w.p.1)

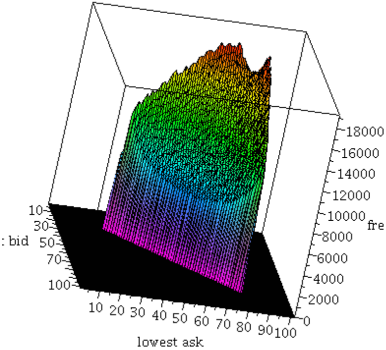
Theorem

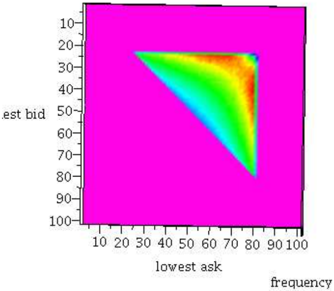
Consider the restricted system as before, with a bid always waiting at κ_b and an ask always waiting at κ_a. Suppose the original limit order book is at least null-recurrent, and the restricted system is positive recurrent. Then we can couple the original limit order book to the restriction.

But, need to know original system is at least null-recurrent first so can't apply to find κ_b, κ_a.

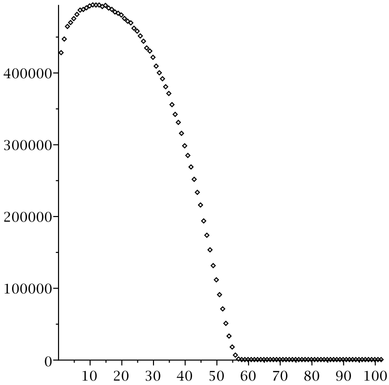
More pretty pictures

Histogram (aka density) of highest bid and lowest ask pairs

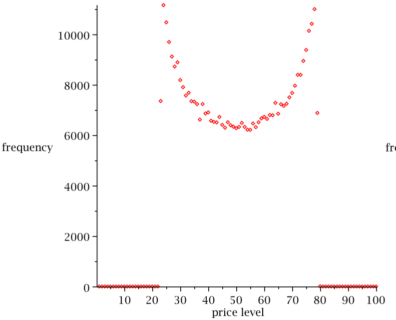




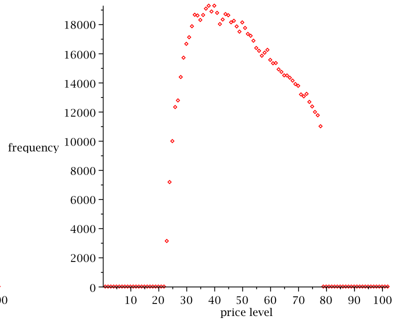
Histogram of spreads (lowest ask−highest bid)



Bid-ask location given zero spread



Highest bid given no asks



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