Strategic Customers in a Transportation Station: When is it Optimal to Wait?

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The motivation

to wait or not?

- large waiting time
- high probability of being served

- small waiting time
- high probability of not being served
The model

- Infinite waiting space (transportation station)
- Poisson($\lambda$) customers’ arrival process
- 1 server (bus)
- Renewal server’s visiting process $\{M(t)\}$
  - Times between two visits: $X_1, X_2, X_3, \ldots \sim F(x)$
- Random server’s capacities: $C_1, C_2, C_3, \ldots \sim (g_k, k = 1, 2, \ldots)$
- When a server with capacity $k$ visits the system:
  - serves $k$ customers instantaneously,
  - the others abandon the system
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State description

\[
\begin{align*}
N(t) & : \text{number of customers at time } t \\
R(t) & : \text{remaining time until the next server’s visit at time } t
\end{align*}
\]
\Rightarrow \{ (N(t), R(t)) \} \text{ C.T.M.P.}
The problem: economic analysis of customer behavior

join or balk

symmetric game among customers
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⇓

symmetric game among customers

Information level:

- Unobservable case: observes nothing
- Observable case: observes $N(t)$
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\[ \Downarrow \]

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**Reward-Cost structure:**

- Customers’ reward $R$ for completing service
- Customers’ waiting cost $K$ per time unit
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*join* or *balk*

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symmetric game among customers

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Decisions:

- Upon arrival a customer decides to join or to balk
- Decisions are irrevocable
- Customers’ purpose: maximization of individual expected net benefit
Symmetric non-cooperative game

$S$: Set of strategies

$U(s_1, s_2)$: Payoff function of a tagged player, who follows the $s_1$ strategy, when all other players follow the $s_2$ strategy
Symmetric non-cooperative game

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**Definition (Best Response)**

A strategy \( s_1^* \) is said to be a best response against a strategy \( s_2 \), iff

\[
U(s_1^*, s_2) \geq U(s_1, s_2), \quad \forall s_1 \in S
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**Definition (Symmetric Nash Equilibrium)**
A strategy \( s_1^* \) is said to be a symmetric Nash equilibrium iff it is a best response against itself, i.e.

\[
U(s_1^*, s_1^*) \geq U(s_1, s_1^*), \quad \forall s_1 \in S
\]
OBSERVABLE CASE

↓

customers observe $N(t)$

↓

strategies: $\mathbf{q} = (q_0, q_1, q_2, \ldots)$, $q_n \in [0, 1]$, $n = 0, 1, \ldots$

$(q_n =$ probability of joining, when $N(t) = n)$
OBSERVABLE CASE

\[ \downarrow \]

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\( (q_n = \text{probability of joining, when } N(t) = n) \)

\( S_n(q) \): expected net benefit of a tagged customer, who finds \( n \) present customers and decides to join, given that all other customers follow strategy \( q \).

\[
S_n(q) = RP[\text{service}|n, q] - KE[\text{sojourn time}|n, q]
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\[
S_n(q) = R P[\text{service}|n, q] - KE[\text{sojourn time}|n, q] \\
\sum_{k=n+1}^{\infty} g_k
\]
$E[\text{sojourn time}|n, q]$  

expected residual service time at the arrival instant of a customer, who finds $n$ present customers, given that all other customers follow strategy $q$
Let

$R_q(t)$: residual service time at time $t$, when the customers follow a strategy $q$,

$N_q(t)$: number of customers in the system at time $t$, when the customers follow a strategy $q$,

${\{P(t), t \geq 0\}}$: Poisson process at rate $\lambda$,

then

$$
\begin{cases}
{(N_q(u), R_q(u)), 0 \leq u \leq t}, \\
{P(t + u) - P(t), u \geq 0}
\end{cases}
$$

: Lack of Anticipation assumption

\[\Downarrow\text{(PASTA)}\]

residual service time at the arrival instant of a customer, given that he finds $n$ present customers and that all other customers follow strategy $q$

\[\Downarrow\]

residual service time at arbitrary instant, given that there are $n$ present customers in the system and that all customers follow strategy $q$
Let
\(\bar{n}(\mathbf{q}) = \inf\{n \geq 0 : q_i > 0 \text{ for } i < n \text{ and } q_n = 0\}\)

and
\(\{P_n(t), \ t \geq 0\}: \text{Poisson process at rate } \lambda q_n, \ n = 0, 1, \ldots, \bar{n}(\mathbf{q}) - 1\)

then

\[
\begin{cases}
(N_{\mathbf{q}}(u), R_{\mathbf{q}}(u)), & 0 \leq u \leq t, \\
\{P_n(t + u) - P_n(t), & u \geq 0\}
\end{cases}
\]

: Lack of Anticipation assumption

\(\downarrow\) (Conditional PASTA)

residual service time at the arrival instant of a customer, who joins, given that he finds \(n\) present customers and that all customers follow strategy \(\mathbf{q}\)

\(\parallel\)

residual service time at arbitrary instant, given that there are \(n\) present customers in the system and that all customers follow strategy \(\mathbf{q}\)
For $0 \leq n < \bar{n}(q)$

residual service time at the arrival instant of a customer, given that he finds $n$

present customers and that all other customers follow strategy $q$

\[ R_{n,q} \]
For $0 \leq n = \bar{n}(q)$

residual service time at the arrival instant of a customer, given that he finds $n$ present customers and that all other customers follow strategy $q$

\[ \| \]

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\[ \| \]

$R_{n,q}$
Recursive scheme for $R_{n,q}$

- $R_{0,q} \overset{d}{=} R(X), \quad \bar{n}(q) = 0$

- $R_{0,q} \overset{d}{=} (X - T_{\lambda q_0} | X \geq T_{\lambda q_0}), \quad \bar{n}(q) > 0,$
  where $T_{\lambda q_0} \sim \text{Exp}(\lambda q_0)$

- $R_{n,q} \overset{d}{=} R(R_{n-1,q}), \quad \bar{n}(q) = n > 0$

- $R_{n,q} \overset{d}{=} (R_{n-1,q} - T_{\lambda q_n} | R_{n-1,q} \geq T_{\lambda q_n}), \quad \bar{n}(q) > n \geq 1,$
  where $T_{\lambda q_n} \sim \text{Exp}(\lambda q_n)$
Recursive scheme for LSTs of $R_{n,q}$

Lemma

Let $T_1$, $T_2$ and $Y$ be independent random variables, with $T_1$ and $T_2$ being exponentially distributed with parameters $\lambda_1$ and $\lambda_2$, respectively, and $Y$ being a non-negative generally distributed random variable with LST $\tilde{F}_Y(s)$. Then we have the following formulas.

\[
\Pr[Y \leq T_1] = \tilde{F}_Y(\lambda_1), \quad (1)
\]
\[
\Pr[Y \leq T_1 + T_2] = \frac{\lambda_2}{\lambda_2 - \lambda_1} \tilde{F}_Y(\lambda_1) + \frac{\lambda_1}{\lambda_1 - \lambda_2} \tilde{F}_Y(\lambda_2), \quad \lambda_1 \neq \lambda_2 \quad (2)
\]
\[
\Pr[Y \leq T_1 + T_2] = \tilde{F}_Y(\lambda_1) - \lambda_1 \tilde{F}_Y'(\lambda_1), \quad \lambda_1 = \lambda_2. \quad (3)
\]
Recursive scheme for LSTs of $R_{n,q}$

$F_{n,q}(s)$: the LST of $R_{n,q}$
$F(s)$: the LST of $X$

- If $\bar{n}(q) = 0$,
  \[ R_{0,q} \overset{d}{=} R(X) \Rightarrow \tilde{F}_{0,q}(s) = \frac{-1 + \tilde{F}(s)}{s \tilde{F}'(0)} \quad (4) \]

- If $\bar{n}(q) > 0$,
  $T_s \sim \text{Exp}(s)$
  \[ R_{0,q} \overset{d}{=} (X - T_{\lambda q_0} \mid X \geq T_{\lambda q_0}) \Rightarrow \]
  \[ \Pr[T_s \geq R_{0,q}] = \Pr[T_s \geq (X - T_{\lambda q_0}) \mid (X \geq T_{\lambda q_0})] \Rightarrow \]
  \[ \Pr[T_s \geq R_{0,q}] = \frac{\Pr[T_s \geq X - T_{\lambda q_0}, X \geq T_{\lambda q_0}]}{\Pr[X \geq T_{\lambda q_0}]} \Rightarrow \]
  \[ \Pr[T_s \geq R_{0,q}] = \frac{\Pr[X \leq T_{\lambda q_0} + T_s] - \Pr[X < T_{\lambda q_0}]}{\Pr[X \geq T_{\lambda q_0}]} \quad (5) \]

if $s \neq \lambda q_0$, $(5) \overset{(1),(2)}{\Rightarrow} \tilde{F}_{0,q}(s) = \frac{\lambda q_0 (\tilde{F}(\lambda q_0) - \tilde{F}(s))}{(s - \lambda q_0)(1 - \tilde{F}(\lambda q_0))} \quad (6)$

if $s = \lambda q_0$, $(5) \overset{(1),(3)}{\Rightarrow} \tilde{F}_{0,q}(\lambda q_0) = -\frac{\lambda q_0 \tilde{F}'(\lambda q_0)}{1 - \tilde{F}(\lambda q_0)} \quad (7)$
Recursive scheme for LSTs of $R_{n,q}$

- If $\bar{n}(q) = n \geq 1$,
  \[ R_{n,q} \overset{d}{=} R(R_{n-1,q}) \Rightarrow \tilde{F}_{n,q}(s) = \frac{-(1-\tilde{F}_{n-1,q}(s))}{sF'_{n-1,q}(0)} \quad (8) \]

- If $\bar{n}(q) > n \geq 1$ and $s \neq \lambda q_n$,
  \[ R_{n,q} \overset{d}{=} (R_{n-1,q} - T_{\lambda q_n} | R_{n-1,q} \geq T_{\lambda q_n}) \Rightarrow \]
  \[ \tilde{F}_{n,q}(s) = \frac{\lambda q_n(\tilde{F}_{n-1,q}(\lambda q_n) - \tilde{F}_{n-1,q}(s))}{(s-\lambda q_n)(1-\tilde{F}_{n-1,q}(\lambda q_n))} \quad (9) \]

- If $\bar{n}(q) > n \geq 1$ and $s = \lambda q_n$,
  \[ R_{n,q} \overset{d}{=} (R_{n-1,q} - T_{\lambda q_n} | R_{n-1,q} \geq T_{\lambda q_n}) \Rightarrow \]
  \[ \tilde{F}_{n,q}(\lambda q_n) = \frac{-\lambda q_n \tilde{F}'_{n-1,q}(\lambda q_n)}{1-\tilde{F}_{n-1,q}(\lambda q_n)} \quad (10) \]

$\tilde{F}_{n,q}(s)$ depends on $q$ only through $q_n = (q_0, q_1, q_2, \ldots, q_n)$. So, we can write
$\tilde{F}_{n,q}(s) = \tilde{F}_{n,q_n}(s)$
$\tilde{F}_{n,q_n}(s)$ is continuous in $q_n$
Recursive scheme for $E[R_{n,q}]$

**Corollary (Expected sojourn times)**

Consider the observable model of a transportation station, where customers join the system according to a strategy $q = (q_0, q_1, q_2, \ldots)$. For the expected conditional residual service times, $E[R_{n,q}]$, we have the following recursive scheme

$$E[R_{n,q}] = \frac{E[R_{n-1,q_{n-1}}]}{1 - \tilde{F}_{n-1,q_{n-1}}(\lambda q_n)} - \frac{1}{\lambda q_n}$$

$$, q_i \neq 0, \ i = 0, 1, \ldots, n, \ n \geq 1,$$

$$E[R_{n,q}] = \frac{E[R_{n-1,q_{n-1}}]}{2E[R_{n-1,q_{n-1}}]}$$

$$, q_i \neq 0, \ i = 0, 1, \ldots, n-1, \ q_n = 0, \ n \geq 1,$$

with initial condition

$$E[R_{0,q_0}] = \frac{E[X]}{1 - \tilde{F}(\lambda q_0)} - \frac{1}{\lambda q_0}$$

$$, q_0 \neq 0$$

$$E[R_{0,q_0}] = \frac{E[X^2]}{2E[X]}$$

$$, q_0 = 0.$$
Proposition (Expected net benefit)

Consider the observable model of a transportation station, where the customers join the system according to a strategy $q = (q_0, q_1, q_2, \ldots)$. Then, the expected net benefit $S_n(q)$ of an arriving customer, who finds $n$ present customers in the system and decides to join, is given by the formulas

$$
S_n(q) = R \sum_{k=n+1}^{\infty} g_k - K \left[ \frac{E[R_{n-1},q_{n-1}]}{1 - \tilde{F}_{n-1}(\lambda q_n)} - \frac{1}{\lambda q_n} \right], \quad q_i \neq 0, \quad i = 0, 1, \ldots, n, \quad n \geq 1,
$$

$$
S_n(q) = R \sum_{k=n+1}^{\infty} g_k - K \frac{E[(R_{n-1},q_{n-1})^2]}{2E[R_{n-1},q_{n-1}]} , \quad q_i \neq 0, \quad i = 0, 1, \ldots, n - 1,
$$

$q_n = 0, \quad n \geq 1,$

$$
S_0(q) = R - K \left[ \frac{E[X]}{1 - \tilde{F}(\lambda q_0)} - \frac{1}{\lambda q_0} \right], \quad q_0 \neq 0,
$$

$$
S_0(q) = R - K \frac{E[X^2]}{2E[X]}, \quad q_0 = 0.
$$
Equilibrium strategies

Recursive scheme for the computation of equilibrium probabilities:
• Computation of $q_0^e$

Theorem (Equilibrium probability $q_0^e$)
Consider the observable model of a transportation station. Then, an equilibrium probability $q_0^e$ for joining when finding the system empty exists. Specifically, we have the following comprehensive (but not necessarily mutually exclusive) cases:

**Case I:** $\frac{R}{K} \leq \frac{E[X^2]}{2E[X]}$. Then, $q_0^e = 0$.

**Case II:** $\frac{R}{K} \geq \frac{E[X]}{1-F(\lambda)} - \frac{1}{\lambda}$. Then, $q_0^e = 1$.

**Case III:** $\frac{E[X^2]}{2E[X]} < \frac{R}{K} < \frac{E[X]}{1-F(\lambda)} - \frac{1}{\lambda}$. Then, there exists a $q_0'$ such that $0 < q_0' < 1$ and $\frac{E[X]}{1-F(\lambda q_0')} - \frac{1}{\lambda q_0'} = \frac{R}{K}$. The equilibrium joining probability is $q_0^e = q_0'$. 

- Computation of $q^e_n$, given the $q^e_{n-1}$

**Theorem (Equilibrium probability $q^e_n$)**

Consider the observable model of a transportation station. Then, assuming that an equilibrium joining probability vector $q^e_{n-1}$ is known, an equilibrium probability $q^e_n$ for joining when finding $n$ present customers in the system exists. Specifically, we have the following cases:

**Case I:**

$$
\frac{R \sum_{k=n+1}^{\infty} g_k}{K} \leq \frac{E[(R_{n-1},q^e_{n-1})^2]}{2E[R_{n-1},q^e_{n-1}]}.
$$

Then, $q^e_n = 0$.

**Case II:**

$$
\frac{R \sum_{k=n+1}^{\infty} g_k}{K} \geq \frac{E[R_{n-1},q^e_{n-1}]}{1-F_{n-1},q^e_{n-1}(\lambda)} - \frac{1}{\lambda}.
$$

Then, $q^e_n = 1$.

**Case III:**

$$
\frac{E[(R_{n-1},q^e_{n-1})^2]}{2E[R_{n-1},q^e_{n-1}]} < \frac{R \sum_{k=n+1}^{\infty} g_k}{K} < \frac{E[R_{n-1},q^e_{n-1}]}{1-F_{n-1},q^e_{n-1}(\lambda)} - \frac{1}{\lambda}.
$$

Then, there exists a $q'_n$ such that $0 < q'_n < 1$ and

$$
\frac{E[R_{n-1},q^e_{n-1}]}{1-F_{n-1},q^e_{n-1}(\lambda q'_n)} - \frac{1}{\lambda q'_n} = \frac{R \sum_{k=n+1}^{\infty} g_k}{K}.
$$

The equilibrium joining probability is $q^e_n = q'_n$. 
BIBLIOGRAPHY


Thank you!