On the diameter of random planar graphs

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joint work with

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Planar graphs and maps

- **Planar graph** = (connected) graph on \( V = \{1, 2, \ldots, n\} \) that can be drawn in the plane without edge crossing.
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same graph
different maps
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- Note: the number of embeddings depends on the graph...

**Uniform random planar map \neq Uniform random planar graph!**
Some known results for maps (stated approximately)

- **Thm** [Chassaing-Schaeffer ’04], [Marckert, Miermont ’06], [Ambjörn-Budd ’13]

  In a uniform random map $M_n$ of size $n$, distances are of order $n^{1/4}$.

  For example one has $\frac{\text{Diam}(M_n)}{n^{1/4}} \rightarrow$ some real random variable
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A lot of (very strong) things are known – very active field of research since 2004 [Bouttier, Di Francesco, Gitter, Le Gall, Miermont, Paulin, Addario-Berry, Albenque...]
**Our main result: diameter of random planar GRAPHS**

- **Thm** [C, Fusy, Giménez, Noy 2010+]
  
  Let $G_n$ be the uniform random planar graph with $n$ vertices.

  Then $\text{Diam}(G_n) = n^{1/4+o(1)}$ w.h.p.

  More precisely $\mathbb{P}\left(\text{Diam}(G_n) \not\in \left[n^{1/4-\epsilon}, n^{1/4+\epsilon}\right]\right) = O(e^{-n^{\Theta(\epsilon)}})$. 
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- This is some kind of large deviation result. We also conjecture convergence in law:
  
  $$\frac{\text{Diam}(G_n)}{n^{1/4}} \to \text{some real random variable}$$

- Note: for random trees,
  
  $$\frac{\text{Diam}(T_n)}{n^{1/2}} \to \text{some real random variable}$$

  $$\mathbb{P}\left(\text{Diam}(T_n) \notin \left[n^{1/2-\epsilon}, n^{1/2+\epsilon}\right]\right) = O(e^{-n^{\Theta(\epsilon)}})$$

  [Flajolet et al '93]
(0) Connectivity in graphs

General

Connected (1-connected)

2-Connected

3-Connected

A graph is $k$-connected if one needs to remove at least $k$ vertices to disconnect it.
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\[ \text{[Tutte'66]: } \text{a connected graph decomposes into 2-connected components} \]
\[ \text{a 2-connected graph decomposes into 3-connected components} \]

\[ \text{[Whitney]: A 3-connected planar graph has a UNIQUE embedding} \]

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  - a 2-connected graph decomposes into 3-connected components

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[Tutte 60s], [Bender,Gao,Wormald’02], [Giménez, Noy’05] followed this path carrying counting results along the scheme → exact counting of planar graphs!

Here we follow the same path and carry deviations statements for the diameter.
To simplify the exposition we consider a quadrangular planar map (faces have degree 4)

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```
  i+1  i
 /\    /
 i    i+1
```

```
  i+1  i
 /\    /
 i    i+2
```

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    ![Diagram of Schaeffer rules]

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   ![Schaeffer rules diagram](image)

   - **Fact**: the blue map is a **tree**.
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If one remembers the labels the construction is bijective!
• A well-labelled tree is a plane tree together with a mapping $l : V \to \mathbb{Z}_{>0}$ such that
  - if $v \sim v'$ then $|l(v) - l(v')| \leq 1$
  - $\min_v l(v) = 1$
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• Thm [Cori-Vauquelin’81; Schaeffer’99]
  There is a bijection between quadrangular planar maps with a pointed vertex and $n + 1$ vertices and well-labelled trees with $n$ vertices. The labels in the tree correspond to distances to the root in the map.
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[Chassaing-Schaeffer'04]
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Thm

The largest 2-connected component has size
\[ \frac{n}{3} + \frac{n^{2/3}}{A} \]
where \( A \) converges to an explicit law.

The second-largest component has size \( O(n^{2/3}) \).

[Gao, Wormald’99] [Banderier, Flajolet, Schaeffer, Soria ’01]
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\[ \text{Diam}(B_{n/3}) \sim \text{Diam}(M_n) = n^{1/4+o(1)} \]

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and \( X_n \) is essentially a random 2-conn. map of size \( n/3 \).
(3) Decomposition into 3-connected components

Again one can write everything in terms of generating functions.
→ deduce the g.f. of 3-connected maps from the one of 2-connected maps. [Tutte 60's].
→ deduce the g.f. of 2-connected graphs from the one of 3-connected graphs [Bender, Gao, Wormald'02].

\( T = 3 \)-connected component
\( R = \) series composition
\( M = \) parallel composition
(3) Decomposition into 3-connected components

**Prop** A random 2-connected planar graph with $n$ edges has diameter $n^{1/4+o(1)}$ with high probability.

A 2-connected graph $B_n$

RMT tree
**Prop** A random 2-connected planar graph with $n$ edges has diameter $n^{1/4+o(1)}$ with high probability.

Same idea:
- there exists a $T$-component $Y_n$ of linear size w.h.p.
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**Prop** A random 2-connected planar graph with \( n \) edges has diameter \( n^{1/4+o(1)} \) with high probability.

![Diagram](image)

Same idea:
- there exists a \( T \)-component \( Y_n \) of linear size w.h.p.
- the diameter of the RMT-tree is \( n^{o(1)} \) w.h.p.
- The extra-length due the edge substitution is also \( n^{o(1)} \)
• **Thm** [C, Fusy, Giménez, Noy 2010+]  
  Let $G_n$ be the uniform random planar graph with $n$ vertices.

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  More precisely $\mathbb{P}\left(\text{Diam}(G_n) \notin \left[n^{1/4-\epsilon}, n^{1/4+\epsilon}\right]\right) = O(e^{-n^{\Theta(\epsilon)}})$.

• The proof relies both on exact generating functions and magical bijections: we couldn’t do anything without this (or maybe something much weaker like $O(\sqrt{n})$ ?)

• The general picture is quite clear but the analysis is a bit tedious... (need to work with bivariate generating functions and prove estimates with enough uniformity)

• No way to obtain the convergence of $\frac{\text{Diam}(G_n)}{n^{1/4}}$ - even for planar maps this is very difficult!

• **Same result** for the uniform random graph with $n$ vertices and $\lfloor \mu n \rfloor$ edges for $1 < \mu < 3$. 
Conclusion (II)

- We generalized the Giménez-Noy enumeration result to graphs embeddable on a surface of genus $g \geq 0$

**Thm** [C, Fusy, Giménez, Mohar, Noy 2011] [Bender-Gao 2011]

$$\#\{n\text{-vertex genus } g \text{ graphs}\} \sim c_g \cdot n! \cdot \gamma^n \cdot n^{\frac{5}{2}g-\frac{7}{2}} \quad \gamma \approx 27.\ldots.$$ 

Same kind of proof but Whitney's theorem (uniqueness of embedding) now requires that there is no short non-contractible cycle.
(but we could prove that)
The result on the diameter should be the same but this is not (and won't be) written.

The fact that non-contractible cycles are small imply the following:

**Thm** [C, Fusy, Giménez, Mohar, Noy 2011]

Fix $g \geq 1$. The random graph of genus $g$ and size $n$ has chromatic number in $\{4, 5\}$ and list chromatic number $5$ w.h.p.
Thank you!