Plan

1. Gaussian processes on trees
2. Branching Brownian motion
3. The extremal process of BBM
4. Variable speed BBM
5. Universality

A. Bovier (IAM Bonn)

Extremal Processes of Gaussian Processes Indexed by Trees
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5. Universality
Motivation

- **Spin glasses**: What is the structure of ground states for (mean field) spin glasses? 
- **Extreme value theory**: What are the extreme values and the extremal process of dependent random processes? 
- **Spatial branching processes**: Describe the cloud of spatial branching processes, in particular near their propagation front! 
- **Reaction diffusion equations**: Characterise convergence to travelling wave solutions in certain non-linear pdes!

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A time-homogeneous tree. Label individuals at time $t$ as $i_1(t), \ldots, i_n(t)$.

Canonical tree-distance: $d(i_\ell(t), i_k(t)) \equiv$ time of most recent common ancestor of $i_\ell(t)$ and $i_k(t)$.

For fixed time horizon $t$, define Gaussian process, $(x_t(k(s)), k \leq n(t), s \leq t)$, with covariance $E x_t(k(r)) x_t(\ell(s)) = A(t - 1) d(i_k(r), i_\ell(s)))$ for $A: [0, 1] \rightarrow [0, 1]$, increasing.

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Gaussian processes on trees

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- General $A$: variable speed BBM [Derrida-Spohn '88, Fang-Zeitouni '12]
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- Is there a rescaling $u_t(x)$, such that
  \[ P(M(t) \leq u_t(x)) \to F(x)? \]
- Is there a limiting extremal process, $\mathcal{P}$, such that
  \[ \sum_{k \leq n(t)} \delta_{u_t^{-1}(x_k(t))} \to \mathcal{P}? \]
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With \( u_t(x) = t \sqrt{2r} - \frac{\ln(rt)}{2\sqrt{2r}} + \frac{x}{\sqrt{r}} + \frac{\ln(n(t)/\mathbb{E}n(t))}{\sqrt{2r}} \), where \( n(t)/\mathbb{E}n(t) \to RV \), a.s.

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\mathbb{P}(M(t) \leq u_t(x)) \to \exp \left(-\frac{1}{4\pi} e^{-\sqrt{2}x}\right)
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\[ \mathbb{P}(M(t) \leq u_t(x)) \to \exp \left( -\frac{1}{4\pi} e^{-\sqrt{2}x} \right) \]

\[ \sum_{k \leq n(t)} \delta_{u_t^{-1}(x_k(t))} \to \text{PPP}\left( \frac{1}{4\pi} e^{-\sqrt{2}x} dx \right) \]

where \( \text{PPP}(\mu) \) denotes the Poisson Point Process with intensity \( \mu \).
Universality 1: the order of the maximum

In all models, it has been shown (or can be shown easily) that the order of the maximum is only a function of the concave hull of the function $A$ (and on the growth rate of $n(t)$):

If $\bar{A}$ denotes the concave hull of $A$, then:

$$\lim_{t \to \infty} t^{-1} M(t) = \sqrt{2} \lim_{t \to \infty} t^{-1} \ln n(t) \int_{0}^{1} \sqrt{ds} \bar{A}(s)$$

[B-Kurkova 01, for binary tree, Fang-Zeitouni 11, GW tree]

Note in particular that as long as $A(s) \leq s$, for all $s \leq 1$, then $\bar{A}(s) = s$, and the order of the maximum is the same as in the REM.
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If $\tilde{A}$ denotes the concave hull of $A$, then:

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The full picture is known (or easy to get) if $A$ is a step function. In that case:

- If $A(s) < s$, for all $s \in (0, 1)$, then all results are the same as in the corresponding REM!
- If $A(s) \leq s$, with equality in a finite number of points, the REM picture holds, but a prefactor appears in front of the $e^{-x}$'s.
- If $\bar{A}(s) \neq s$, then the leading order and the logarithmic correction are changed and depend on $\bar{A}$; the extremal process is a Poisson cascade process.

This is all proven for the binary tree, but extension to general trees are straightforward.

Note the special role of the linear function $A(s) = s$. A. Bovier (IAM Bonn)
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Branching Brownian motion

(BBM) is a classical object in probability, combining the standard models of \textit{random motion} and \textit{random genealogies} into one: Each particle of the Galton-Watson process performs Brownian motion independently of any other. This produces an immersion of the Galton-Watson process in space.
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BBM is the canonical model of a spatial branching process.
The F-KPP

One of the simplest reaction-diffusion equations is the Fisher-Kolmogorov-Petrovsky-Piscounov (F-KPP) equation:

$$ \frac{\partial v(x,t)}{\partial t} = \frac{1}{2} \frac{\partial^2 v(x,t)}{\partial x^2} + v(x,t) - v^2(x,t) $$

Fischer used this equation to model the evolution of biological populations. It accounts for:

- birth: $v$, 
- death: $-v^2$, 
- diffusive migration: $\frac{\partial^2 v}{\partial x^2}$.
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**Lemma (McKeane ’75, Ikeda, Nagasawa, Watanabe ’69)**

Let $f : \mathbb{R} \to [0, 1]$ and $\{ x_k(t) : k \leq n(t) \}$ BBM.

$$ u(t, x) = \mathbb{E} \left[ \prod_{k=1}^{n(t)} f(x - x_k(t)) \right] $$

Then $v \equiv 1 - u$ is the solution of the F-KPP equation with initial condition $v(0, x) = 1 - f(x)$. 
Travelling waves

\[ \omega'' + \sqrt{2} \omega' - \omega^2 + \omega = 0. \]

has a unique solution satisfying

\[ 0 < \omega(x) < 1, \quad \omega(x) \to 0 \text{ as } x \to +\infty, \] and

\[ \omega(x) \to 1 \text{ as } x \to -\infty, \] up to translation, i.e. if \( \omega, \omega' \) are two solutions, then there exists an \( a \in \mathbb{R} \) s.t.

\[ \omega'(x) = \omega(x + a). \]

For suitable initial conditions,

\[ u(t, x + m(t)) \to \omega(x), \]

where \( m(t) = \sqrt{2t - \frac{3}{2}} \sqrt{2 \ln t}, \) where \( \omega \) is one of the stationary solutions.
Travelling waves

Theorem (Bramson '78)

The equation

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where $m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \ln t$, where $\omega$ is one of the stationary solutions.
Examples

Choosing suitable initial conditions, this theorem applies to
\[ u(t, x) = P(\max_{k \leq n(t)} x_k(t) \leq x), \]
and the Laplace functional
\[ u(t, x) = E\exp(-\sum_{k \leq n(t)} \phi(x_k(t))) \]

Needs a bit extra work...

In particular, it gives Bramson's celebrated result
\[ \lim_{t \to \infty} P(\max_{k \leq n(t)} x_k(t) - m(t) \leq x) = \omega(x). \]
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In particular, it gives Bramson’s celebrated result

\[
\lim_{t \to \infty} \mathbb{P}\left( \max_{k \leq n(t)} x_k(t) - m(t) \leq x \right) = \omega(x)
\]
The Lalley-Sellke representation

The derivative martingale

\[ \omega(x) = E \left[ e^{-C Z e^{-\sqrt{2}x}} \right], \] \((\ast)\)

\[ Z(t) = \lim_{t \to \infty} Z(t), \text{ where } Z(t) \text{ is the derivative martingale,} \]

\[ Z(t) = \sum_{k \leq n(t)} \left\{ \sqrt{2}t - x_k(t) \right\} e^{-\sqrt{2} \left\{ \sqrt{2}t - x_k(t) \right\}} \]

The form \((\ast)\) seems universal, but \(Z\) is particular.

For the REM on the GW tree \((\ast)\) holds with \(Z\) a standard exponential.
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Lalley-Sellke, 1987: $\omega(x)$ is random shift of Gumbel-distribution

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\]

\( Z \overset{(d)}{=} \lim_{t \to \infty} Z(t) \), where \( Z(t) \) is the derivative martingale,

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Z(t) = \sum_{k \leq n(t)} \{ \sqrt{2}t - x_k(t) \} e^{-\sqrt{2}\{\sqrt{2}t-x_k(t)\}}
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The form $(\ast)$ seems universal, but $Z$ is particular. For the REM on the GW tree $(\ast)$ holds with $Z$ a standard exponential.
The Lalley-Sellke representation

Description of the extremal process

Poisson Point Process:

$$\mathbb{P} \equiv \sum_{i \in \mathbb{N}} \delta_{p_i} \equiv \text{PPP}(C \mathbb{Z}_e - \sqrt{2} \mathbb{X})$$

Cluster process:

$$\Delta(t) \equiv \sum_{k} \delta_{x_k}(t) - \max_{j \leq n(t)} x_j(t).$$

conditioned on the event

$$\{ \max_{j \leq n(t)} x_j(t) > \sqrt{2} t \}$$

converges

in law to point process, $\Delta$.

$$\mathbb{E} \equiv \sum_{i, j \in \mathbb{N}} \delta_{p_i + \Delta(i)}(j), \Delta(i) \text{ iid copies of } \Delta.$$

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Extremal Processes of Gaussian Processes Indexed by Trees
Description of the extremal process

Poisson Point Process: \( P_Z = \sum_{i \in \mathbb{N}} \delta_{p_i} \equiv \text{PPP} \left( CZ e^{-\sqrt{2}x} dx \right) \)
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Cluster process:

\[ \Delta(t) \equiv \sum_k \delta_{x_k(t) - \max_{j \leq n(t)} x_j(t)}. \]

Conditioned on the event \( \{\max_{j \leq n(t)} x_j(t) > \sqrt{2}t\} \)

converges in law to point process, \( \Delta \).

[Chauvin, Rouault '90]

\[ \mathcal{E} \equiv \sum_{i,j \in \mathbb{N}} \delta_{p_i + \Delta_j^{(i)}}, \quad \Delta^{(i)} \text{ iid copies of } \Delta \]
The extremal process

\[ E_t \equiv \sum_{i=1}^{n(t)} \delta_{x_i(t)} - m(t) \rightarrow E. \]

Interpretation:
\[ p_i \]: positions of maxima of clusters with recent common ancestors.
\[ \Delta(i) \]: positions of members of clusters seen from their maximal one.
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Theorem (Arguin-B-Kistler ’11, Aidékon, Brunet, Berestycki, Shi ’11)

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\[ \mathbb{E}\left[ \exp \left( -\int \phi(y)\mathcal{E}_t(dy) \right) \right] \rightarrow \mathbb{E}\left[ \exp \left( -C(\phi)Z \right) \right] \]

for any \( \phi \in C_c(\mathbb{R}) \) non-negative, where

\[ C(\phi) = \lim_{t \to \infty} \sqrt{\frac{2}{\pi}} \int_0^\infty \left( 1 - u(t, y + \sqrt{2}t) \right) ye^{\sqrt{2}y} dy \]

\( u(t, y) \): solution of F-KPP with initial condition \( u(0, y) = e^{-\phi(y)} \).
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Then show that the limit is the Laplace functional of the process \( \mathcal{E} \) described above.
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Chose an embedding \( \gamma : \{1, \ldots, n(t)\} \rightarrow \mathbb{R}_+ \), such that

\[
|\gamma(i_k(t)), -\gamma(i_j(t))| \sim e^{-d(i_k(t), i_j(t))}
\]

Define for \( x \in \mathbb{R}_+ \),

\[
Z(r, t, u) \equiv \sum_{k: \gamma(i_k(r)) \leq u} \{\sqrt{2}t - x_k(t)\} e^{-\sqrt{2}\{\sqrt{2}t-x_k(t)\}}
\]

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\lim_{r \uparrow \infty} \lim_{t \uparrow \infty} Z(r, t, u) \rightarrow Z(u)
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**Theorem (B, Hartung ’14)**

The point process $\mathcal{E}_t \equiv \sum_{k=1}^{n(t)} \delta(\gamma(i_k(t)), x_i(t) - m(t)) \to \tilde{\mathcal{E}}$ on $\mathbb{R}_+ \times \mathbb{R}$, where

$$\tilde{\mathcal{E}} \equiv \sum_{i,j} \delta(q_i, p_i) + (0, \Delta_j^{(i)})'$$

with $(q_i, p_i)$ atoms of a Cox process on $\mathbb{R}_+ \times \mathbb{R}$ with intensity measure $Z(du) \times Ce^{-\sqrt{2}x} dx$, and $\Delta_j^{(i)}$ as before.
Adding another dimension
Variable speed BBM.....below the straight line...

Assume that $A(x) < x$, $\forall x \in (0, 1)$, $A'(0) = a_2 < 1$, $A'(1) = b_2 > 1$.

Then $\exists C(b)$ and a r.v. $Y_a$ such that

$$P(M(t) - \tilde{m}(t) \leq x) \to Ee^{-C(b)Y_a} - \sqrt{2} \sum_{k \leq n(t)} \delta x_k(t) - \tilde{m}(t) \equiv \sqrt{2} t^{1/2} \ln t.$$
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1. $\mathbb{P}(M(t) - \tilde{m}(t) \leq x) \rightarrow \mathbb{E}e^{-C(b)Y_ae^{-\sqrt{2x}}}$

2. $\sum_{k \leq n(t)} \delta_{x_k(t) - \tilde{m}(t)} \rightarrow \mathcal{E}_{a,b} = \sum_{i,j} \delta_{p_i + b\Delta_j}$

3. $\tilde{m}(t) \equiv \sqrt{2}t - \frac{1}{2\sqrt{2}} \ln t$.

4. $p_i$: e the atoms of a PPP($C(b)Y_ae^{-\sqrt{2x}} \, dx$),

5. $Y_s \equiv \sum_{i=1}^{n(s)} e^{-s(1+\sigma_1^2)+\sqrt{2}x_i(s)}$
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- $\Delta$: are as in BBM but with the conditioning on the event $\{\max_k x_k(t) \geq \sqrt{2}bt\}$. 

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A. Bovier (IAM Bonn) Extremal Processes of Gaussian Processes Indexed by Trees
Elements of the proof:
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1) Explicit construction for the case of two speeds:
2) Gaussian comparison for general $A$.

For details, go to Lisa’s talk (Friday, 9h)!!
Above the straight line

When the concave hull of $A$ is above the straight line, everything changes. If $A$ is piecewise linear, it is quite easy to get the full picture: Cascade of BBM processes. If $A$ is strictly concave, Fang and Zeitouni ’12 and Maillard and Zeitouni ’13 have shown that the correct rescaling is

$$m(t) = C \sigma t^{1/3} - D \sigma t^{2/3} \ln t$$

(with explicit constants $C$ and $D$) but there are no explicit limit laws or limit processes available.
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Universality

The new extremal processes should not be limited to BBM: Branching random walk [Bramson ’78, Addario-Berry, A´ıdek’13 (law of max), Madaule ’13 (full extremal process), ...] Gaussian free field in $d=2$ [Bolthausen, Deuschel, Giacomin ’01, Bramson-Ding-Zeitouni ’13, Biskup-Louidor ’13 [Poisson cluster extremes] ....]

Cover times of random walks [Lawler ’9,3 Dembo-Peres-Rosen-Zeitouni ’06, Belius-Kistler ’14 ....]

Spin glasses with log-correlated potentials [Fyodorov, Bouchaud ’08, Arguin, Zindy ’12 ..]

Statistics of zeros of Riemann zeta-function [Fyodorov, Keating ’12]
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references


Thank you for your attention!