Extremal Processes of Gaussian Processes Indexed by Trees

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PROBABILISTIC STRUCTURES
IN EVOLUTION
DFG SPP 1590

Plan

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PSE PROBABILISTIC STRUCTURES IN EVOLUTION

Plan

1. Gaussian processes on trees
2. Branching Brownian motion
3. The extremal process of BBM
4. Variable speed BBM
5. Universality

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This is too hard in general, but we will look at a setting where these questions have a chance to be answered. Branching Brownian motion is at the heart of this setting.

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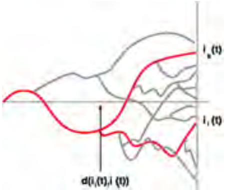

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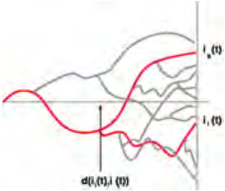
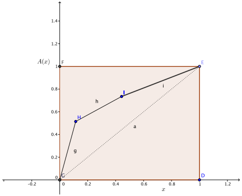
- For fixed time horizon t, define Gaussian process, $(x_k^t(s), k \leq n(t), s \leq t)$, with covariance

$$\mathbb{E} x_k^t(r) x_\ell^t(s) = tA(t^{-1} d(\mathbf{i}_k(r), \mathbf{i}_\ell(s)))$$

for $A : [0,1] \to [0,1]$, increasing.





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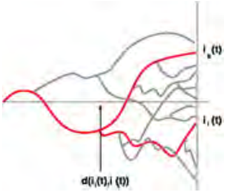
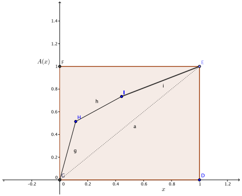
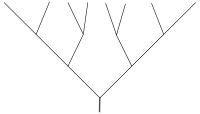




Can be constructed as time change of branching Brownian motion

Examples



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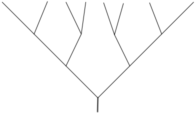

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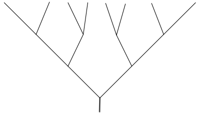

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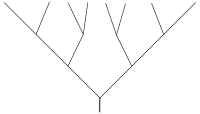

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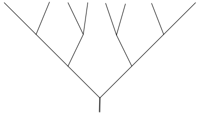

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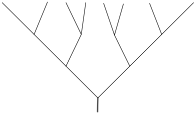

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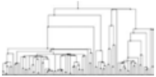

Supercritical Galton-Watson tree

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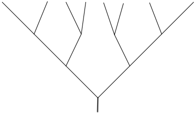

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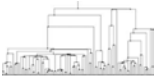

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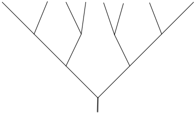

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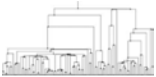

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- General A: variable speed BBM [Derrida-Spohn '88, Fang-Zeitouni '12]

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- Is there a limiting extremal process, $\mathcal{P}$, such that

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- $$\sum_{k\leq n(t)} \delta_{u_t^{-1}(x_k(t))} \to \mathrm{PPP}\left(\frac{1}{4\pi} e^{-\sqrt{2}x} dx\right)$$

where $\mathrm{PPP}(\mu)$ denotes the Poisson Point Process with intensity μ.

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Note in particular that as long as $A(s) \leq s$, for all $s \leq 1$, then $\bar{A}(s) = s$, and the order of the maximum is the same as in the REM.

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Note the special role of the linear function $A(s) = s$

Branching Brownian motion

(BBM) is a classical object in probability, combining the standard models of random motion and random genealogies into one: Each particle of the Galton-Watson process performs Brownian motion independently of any other. This produces an immersion of the Galton-Watson process in space.

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Picture by **Matt Roberts**, Bath

BBM is the canonical model of a spatial branching process.

The F-KPP

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One of the simplest reaction-diffusion equations is the Fisher-Kolmogorov-Petrovsky-Piscounov (F-KPP) equation:

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Fischer used this equation to model the evolution of biological populations. It accounts for:

- birth: v,
- death: $-v^2$,
- diffusive migration: $\partial_x^2 v$.

F-KPP equation and BBM

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Lemma (McKeane '75, Ikeda, Nagasawa, Watanabe '69)

Let $f : \mathbb{R} \to [0,1]$ *and* $\{x_k(t) : k \leq n(t)\}$ *BBM.*

$$u(t,x) = \mathbb{E}\left[\prod_{k=1}^{n(t)} f(x - x_k(t))\right]$$

Then $v \equiv 1 - u$ *is the solution of the F-KPP equation with initial condition* $v(0,x) = 1 - f(x)$.

Travelling waves

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Theorem (Bramson '78)

The equation

$$\frac{1}{2}\omega'' + \sqrt{2}\omega' - \omega^2 + \omega = 0.$$

has a unique solution satisfying $0 < \omega(x) < 1$, $\omega(x) \to 0$, *as* $x \to +\infty$, *and* $\omega(x) \to 1$, *as* $x \to -\infty$, *up to translation, i.e. if* ω, ω' *are two solutions, then there exists* $a \in \mathbb{R}$ *s.t.* $\omega'(x) = \omega(x + a)$.

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For suitable initial conditions,

$$u(t, x + m(t)) \to \omega(x),$$

where $m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \ln t$, *where* ω *is one of the stationary solutions.*

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In particular, it gives Bramson's celebrated result

$$\lim_{t\to\infty} \mathbb{P}(\max_{k\leq n(t)} x_k(t) - m(t) \leq x) = \omega(x)$$

The derivative martingale

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Lalley-Sellke, 1987: $\omega(x)$ is random shift of Gumbel-distribution

$$\omega(x) = \mathbb{E}\left[e^{-C\mathbf{Z}e^{-\sqrt{2}x}}\right], \quad (*)$$

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$Z \overset{(d)}{=} \lim_{t\to\infty} Z(t)$, where $Z(t)$ is the derivative martingale,

$$Z(t) = \sum_{k\leq n(t)} \{\sqrt{2}t - x_k(t)\} e^{-\sqrt{2}\{\sqrt{2}t - x_k(t)\}}$$

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The form $(*)$ seems universal, but Z is particular.
For the REM on the GW tree $(*)$ holds with Z a standard exponential.

Description of the extremal process

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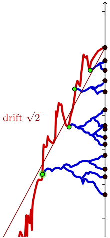
Cluster process:

$$\Delta(t) \equiv \sum_k \delta_{x_k(t)-\max_{j\le n(t)} x_j(t)}.$$



conditioned on the event $\left\{\max_{j\le n(t)} x_j(t) > \sqrt{2}t\right\}$ converges in law to point process, Δ.

[Chauvin, Rouault '90]

$$\mathcal{E} \equiv \sum_{i,j\in\mathbb{N}} \delta_{p_i+\Delta_j^{(i)}}, \qquad \Delta^{(i)} \text{ iid copies of } \Delta$$

The extremal process

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Theorem (Arguin-B-Kistler '11, Aidékon, Brunet, Berestycki, Shi '11)

The point process $\mathcal{E}_t \equiv \sum_{i=1}^{n(t)} \delta_{x_i(t)-m(t)} \to \mathcal{E}$.

The extremal process

Theorem (Arguin-B-Kistler '11, Aidékon, Brunet, Berestycki, Shi '11)

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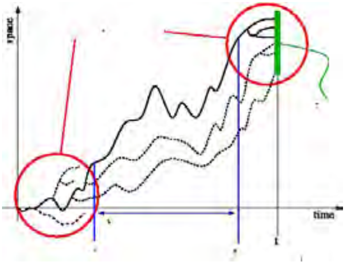

Interpretation:

p_i: positions of maxima of clusters with recent common ancestors.

$\Delta^{(i)}$: positions of members of clusters seen from their maximal one

The extremal process

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for any $\phi \in \mathcal{C}_c(\mathbb{R})$ non-negative, where

$$C(\phi) = \lim_{t\to\infty} \sqrt{\frac{2}{\pi}} \int_0^\infty \left(1 - u(t, y + \sqrt{2}t)\right) y e^{\sqrt{2}y} dy$$

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Then show that the limit is the Laplace functional of the process $\mathcal{E}$ described above.

Adding an extra dimension...

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Chose an embedding $\gamma : \{1, \dots, n(t)\} \to \mathbb{R}_+$, such that

$$|(\gamma(i_k(t)), -\gamma(i_j(t))| \sim e^{-d(i_k(t), i_j(t))}$$

Define for $x \in \mathbb{R}_+$,

$$Z(r, t, u) \equiv \sum_{k:\gamma(i_k(r)) \leq u} \{\sqrt{2}t - x_k(t)\} e^{-\sqrt{2}\{\sqrt{2}t - x_k(t)\}}$$

$$\lim_{r \uparrow \infty} \lim_{t \uparrow \infty} Z(r, t, u) \to Z(u)$$

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Adding an extra dimension

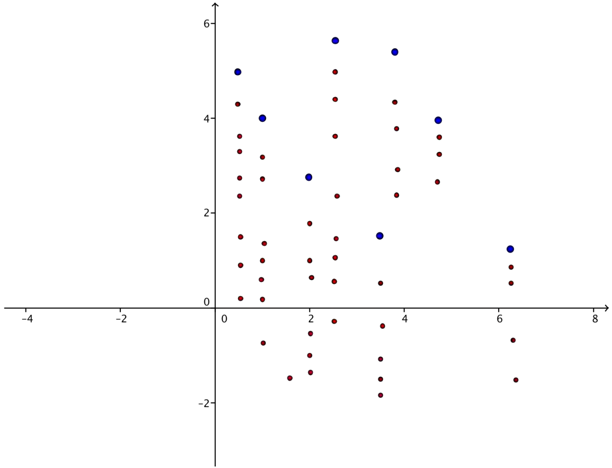
Theorem (B, Hartung '14)

The point process $\mathcal{E}_t \equiv \sum_{k=1}^{n(t)} \delta_{(\gamma(i_k(t)),x_i(t)-m(t))} \to \widetilde{\mathcal{E}}$ *on* $\mathbb{R}_+ \times \mathbb{R}$*, where*

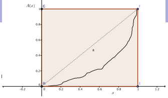
$$\widetilde{\mathcal{E}} \equiv \sum_{i,j} \delta_{(q_i,p_i)+(0,\Delta_j^{(i)})},$$

with (q_i, p_i) *atoms of a Cox process on* $\mathbb{R}_+ \times \mathbb{R}$ *with intensity measure* $Z(du) \times Ce^{-\sqrt{2}x}dx$*, and* $\Delta_j^{(i)}$ *as before.*

Adding another dimension



Variable speed BBM.....below the straight line...



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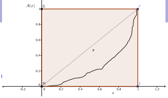

Theorem (B-Hartung '13,'14)

Assume that $A(x) < x, \forall x \in (0,1)$*,* $A'(0) = a^2 < 1$*,* $A'(1) = b^2 > 1$*.*

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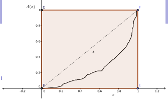

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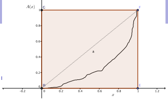

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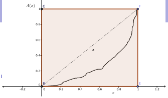

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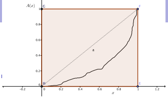

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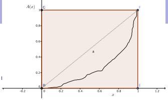

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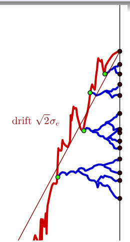
- Δ: are as in BBM but with the conditioning on the event $\{\max_k x_k(t) \geq \sqrt{2}bt\}$.



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Elements of the proof:

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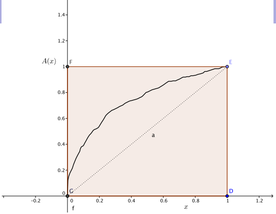
1) Explicit construction for the case of two speeds:
2) Gaussian comparison for general A.

For details, go to Lisa's talk (Friday, 9h)!!

Above the straight line



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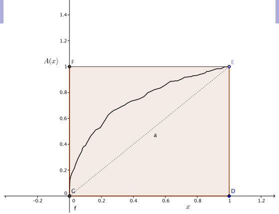
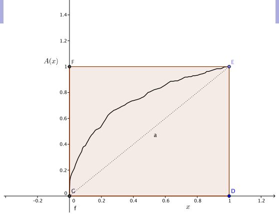


When the concave hull of A is above the straight line, everything changes.

Above the straight line



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- If A is piecewise linear, it is quite easy to get the full picture: Cascade of BBM processes.

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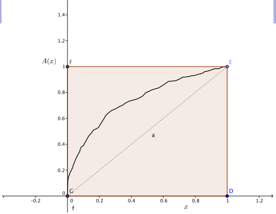

When the concave hull of A is above the straight line, everything changes.

- If A is piecewise linear, it is quite easy to get the full picture: Cascade of BBM processes.
- If A is strictly concave, Fang and Zeitouni '12 and Maillard and Zeitouni '13 have shown that the correct rescaling is

$$m(t) = C_\sigma t - D_\sigma t^{1/3} - \sigma^2(1) \ln t$$

(with explicit constants C_σ and D_σ) but there are no explicit limit laws or limit processes available.

Universality

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- Branching random walk [Bramson '78, Addario-Berry, Aídékon '13 (law of max), Madaule '13 (full extremal process),...]
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- Statistics of zeros of Riemann zeta-function [Fyodorov, Keating '12]

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Thank you for your attention!

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