Tree-valued spatial Lambda-Fleming-Viot processes: The finite system scheme

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Offen im Denken

Outline

1. The **finite system scheme** and classical examples
2. The **measure-valued Λ-Fleming-Viot** process
3. Encoding genealogies as **ultra-metric measure spaces**
4. **Evolving genealogies of** the **Λ-Fleming-Viot** process
5. The **finite system scheme** for the evolving genealogies

Finite System Scheme (FSS)

Idea behind the finite system scheme

This scheme compares the long-time behavior of **large finite** interacting **systems with** the corresponding **infinite** interacting **systems**. Assume that the **finite** interacting **systems and infinite** interacting **systems have <u>different</u> ergodic behavior**. We are interested in the question:

How does the finite system realized its finiteness?

- **Macroscopic point.** At which **macroscopic time scale** does the finite system realizes its finiteness?
- **Microscopic point.** What do we see when let the finite system **evolve for a long time** (according to the macroscopic time scale) **and then** start observing it **at the regular time scale**?

The voter model and its ergodic theory

The **voter model** $(\eta_t)_{t\geq 0}$ is the Markov process on $\{0,1\}^{\mathbb{Z}^d}$ determined by

$$\eta_t(x) \mapsto 1-\eta_t(x), \qquad \text{at rate} \qquad \frac{1}{2d}\#\{y\sim x:\, \eta_t(y)\neq\eta_t(x)\}.$$

Assume that the initial distribution is **translation invariant** and **shift ergodic with density** $\theta\in[0,1]$.

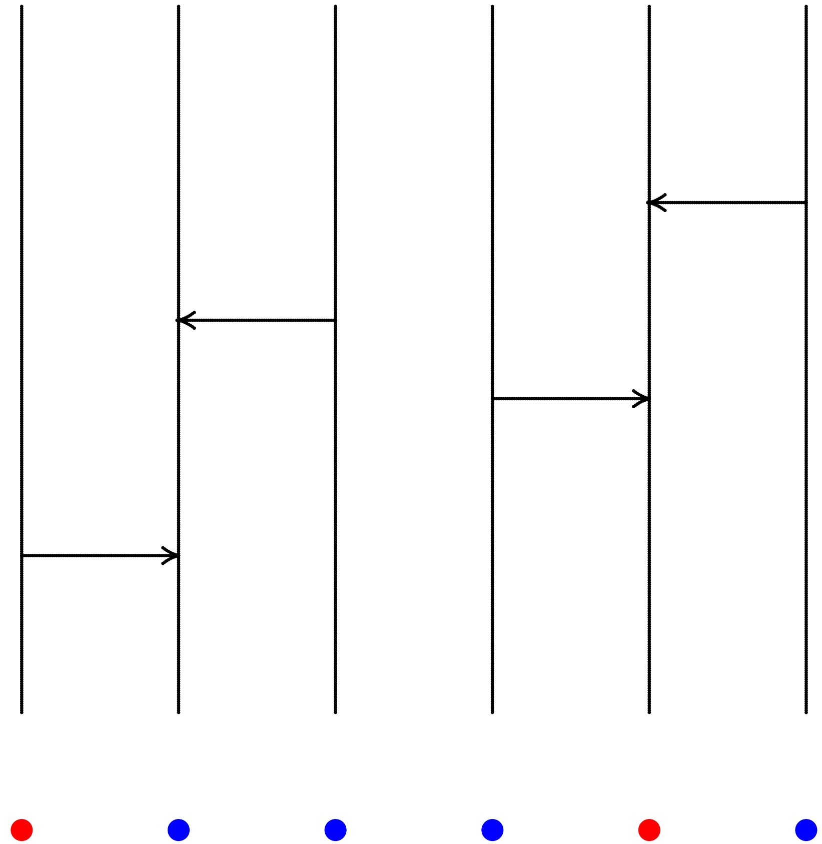
Non-trivial equilibria on $\mathbb{Z}^d$. If $d\geq 3$, then there is a translation invariant and shift ergodic measure ν_θ with density $\theta\in[0,1]$ such that

$$\mathcal{L}\big[\eta_t\big] \underset{t\to\infty}{\Longrightarrow} \nu_\theta.$$

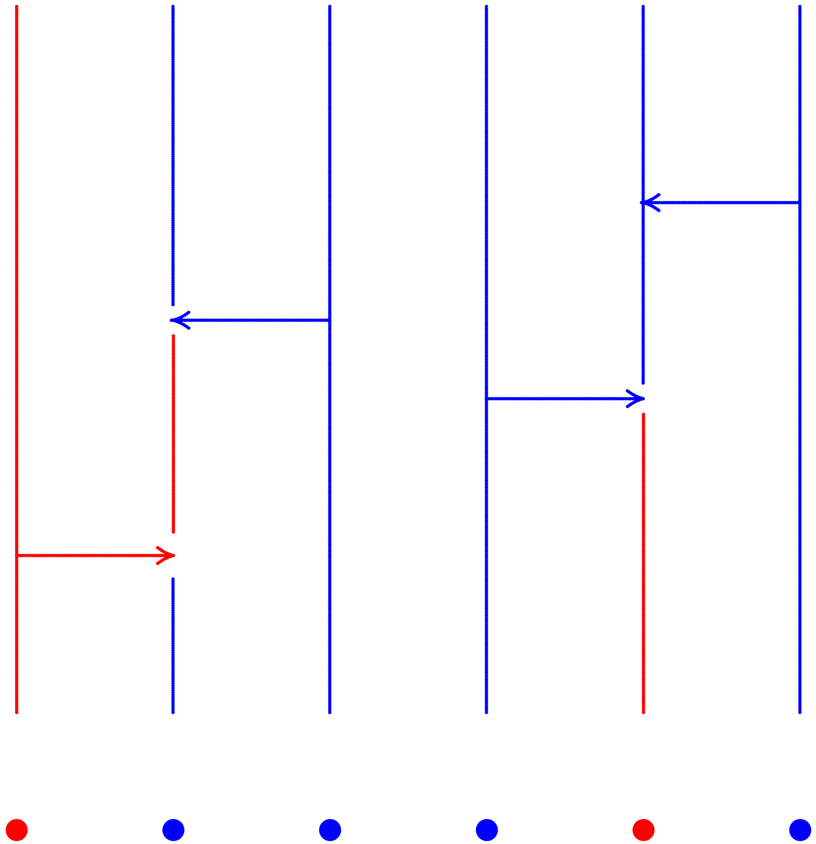
Monotype equilibria on G_N. If $G_N := [-N,N]^d\cap\mathbb{Z}^d$ and $(\eta^N_t)_{t\geq 0}$ be the Markov process on $\{0,1\}^{G_N}$ obtained by restricting in some natural way the dynamics of $(\eta_t)_{t\geq 0}$ to G_N, then

$$\mathcal{L}\big[\eta^N_t\big] \underset{t\to\infty}{\Longrightarrow} (1-\theta)\delta_{\underline{0}}+\theta\delta_{\underline{1}}.$$

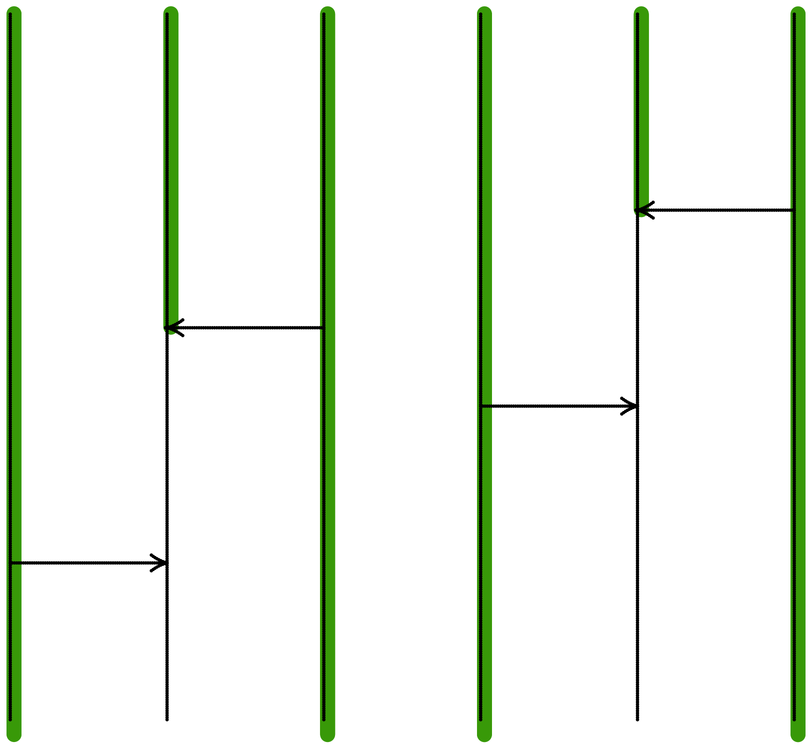
Graphical Representation



Graphical Representation



Tracing back lines: Duality to coalescing random walks



Coalescing RWs

ancestral lines perform SRW, and merge whenever they meet

The finite system scheme: the global perspective

Ted Cox (1989); Coalescing random walks and voter model consensus times on the torus in $\mathbb{Z}^d$, Ann. Probab.

Ted Cox and Andreas Greven (1990); On the long term behavior of some finite particle systems, PTRF.

- **Estimator of density.** $\hat{\theta}^N(\eta) := \frac{1}{\#G_N} \sum_{x \in G_N} \eta(x)$.
- **Macroscopic time scale.** $T_N(s) := \#G_N \cdot s,\ s \in [0, \infty]$.
- **Limiting density process.** $\mathrm{d}Z_t = \sqrt{\gamma Z_t (1 - Z_t)}\mathrm{d}B_t$ with $Z_0 = \theta$, where

  $\gamma :=$*probability that two independent SRW started in* 0 *never meet*

  .

Global perspective. Assume that the initial distribution is **translation invariant** and **shift ergodic with density** $\theta \in [0, 1]$. Then

$$\big(\hat{\theta}^N(\eta^N_{T_N(s)})\big)_{s \geq 0} \underset{N \to \infty}{\Longrightarrow} (Z_s)_{s \geq 0}.$$

The finite system scheme: the local perspective

Ted Cox & Andreas Greven (1990); On the long term behavior of some finite particle systems, PTRF.

Fix a finite window $W \subset \mathbb{Z}^d$. Then $W \subset G_N$ for large enough $N \in \mathbb{N}$.

- **Restriction of a configuration to the window W.** $\eta^W := \eta\big|_W$.
- **Estimator of the configuration's law.**

$$\widehat{\Sigma}^N(\eta) := \frac{1}{\# G_N} \sum_{x \in G_N} \delta_{\sigma_x^N \eta},$$

where σ_x is the **shift** by x on the torus G_N, i.e.,

$$\left(\sigma_x^N \eta\right)_y := (x+y) \bmod 2N.$$

Local perspective. Assume that the initial distribution is **translation invariant** and **shift ergodic with density** $\theta \in [0,1]$. Then for fixed $t > 0$,

$$\mathcal{L}\Big[\big(\widehat{\Sigma}^N(\eta^{N,W}_{T_N(t)+s})\big)_{s \geq 0} \big| \widehat{\theta}^N\big(\eta^N_{T_N(t)}\big) = \theta'\Big] \underset{N \to \infty}{\Longrightarrow} \mathcal{L}^{\nu_{\theta'}}\big[(\eta^W_s)_{s \geq 0}\big].$$

Some proof heuristics

1. **Not feeling the finiteness yet.** Two coalescing particles on G_N move **until one of them hits the boundary** which takes time

$$\mathcal{O}(\sqrt{N}) \ll T_N(t), \qquad t > 0.$$

2. **Loosing a feeling for the geographic structure.** If the have not coalescent by that time, they (independently) **wrap around the torus**. For any $\varepsilon > 0$, by time of order $T_N(\varepsilon)$ they have forgotten their initial distances.

3. **Coalescing with delay on a complete graph.** From now on the geographic structure does not play a role anymore, and it takes the 2 particles an

$$\textit{exponential time with mean } \#G_N$$

to meet (and coalesce).

Finite system schemes: More examples

- **Branching RW; Contact process.** Cox & Greven (1990); On the long term behavior if some finite particle system, PTRF
- **Interacting diffusions.** Cox, Greven & Shiga (1995), Finite and infinite systems of interacting diffusions, PTRF
- **Interacting measure-valued processes.** Dawson, Greven & Vaillancourt (1995); Equilibria and quasi-equilibria for infinite collections of interacting Fleming-Viot processes, Trans. AMS
- **Interacting mutually catalytic branching.** Cox, Dawson & Greven (2004); Mutually catalytic branching random walks: large finite systems and renormalization analysis, Memoirs AMS
- **Historical processes.** Greven, Limic & Winter (2005); Representation theorems for interacting Moran models, interacting Fisher-Wright diffusions and applications, EJP
- **Interacting state-dependent multi-type branching.** Pfaffelhuber (2006); The finite system scheme for state-dependent multi-type branching models, ALEA

Main goal

Establish the finite system scheme for evolving genealogies of Λ-Fleming-Viot processes.

Measure-valued interacting Λ-Fleming-Viot

Our model: Interacting Λ-Cannings model

Geographic space. G, discrete

We consider a multi-type asexual population of fixed size which individuals placed at a site $x \in G$

- **Migration.** The individuals perform independently rate 1 random walks with transition kernel $a(x, y)$
- **Reproduction.** At each site $x \in G$, if there are currently n individuals then for each $k \in \{2, ..., n\}$ and k-individuals $\{i_1, ..., i_k\}$ at rate $\lambda_{n,k}$,
  - the k-individuals $\{i_1, ..., i_k\}$ currently situated in G are killed, and
  - **replaced by k copies of the individual i_ℓ** chosen at random among $\{i_1, ..., i_k\}$. That is, the offspring inherits the type from i_ℓ.

Consistency. (= same dynamics is observed in any sample)

Jim Pitman (1999), Coalescent with multiple collisions, Annals of Probability

Serik Sagitov (1999), The general coalescent with asynchronous mergers of ancestral lines, Annals of applied Probability

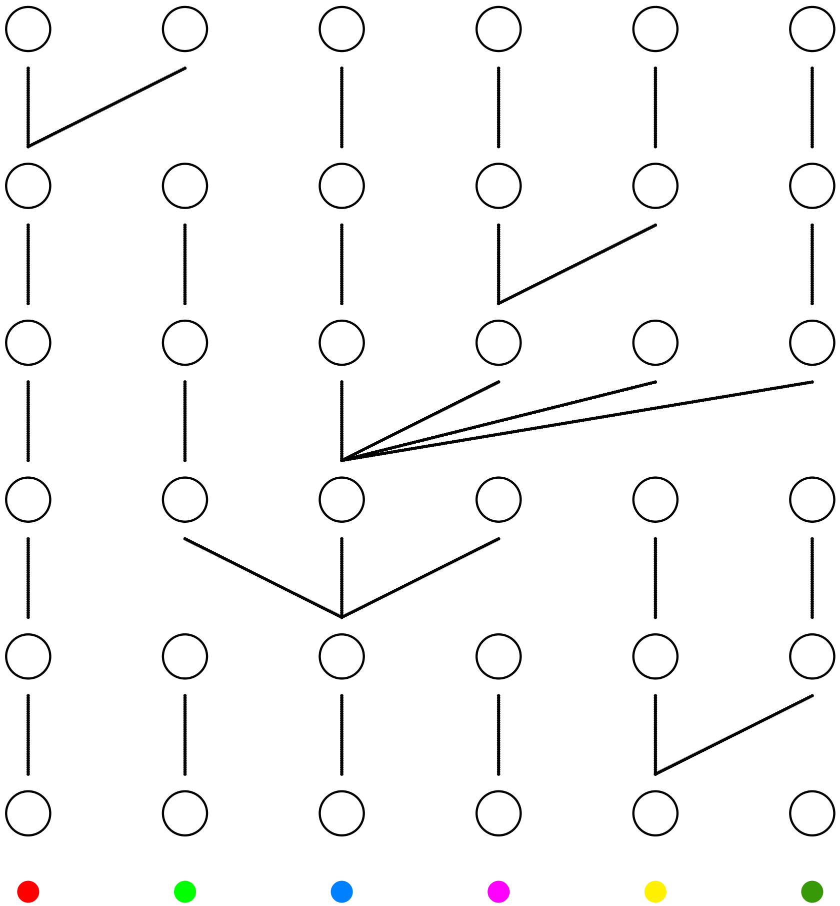
There exists a finite measure Λ on $[0,1]$ with

$$\lambda_{n,k} := \int_0^1 \Lambda(\mathrm{d}x)\, x^{k-2}(1-x)^{n-k}.$$

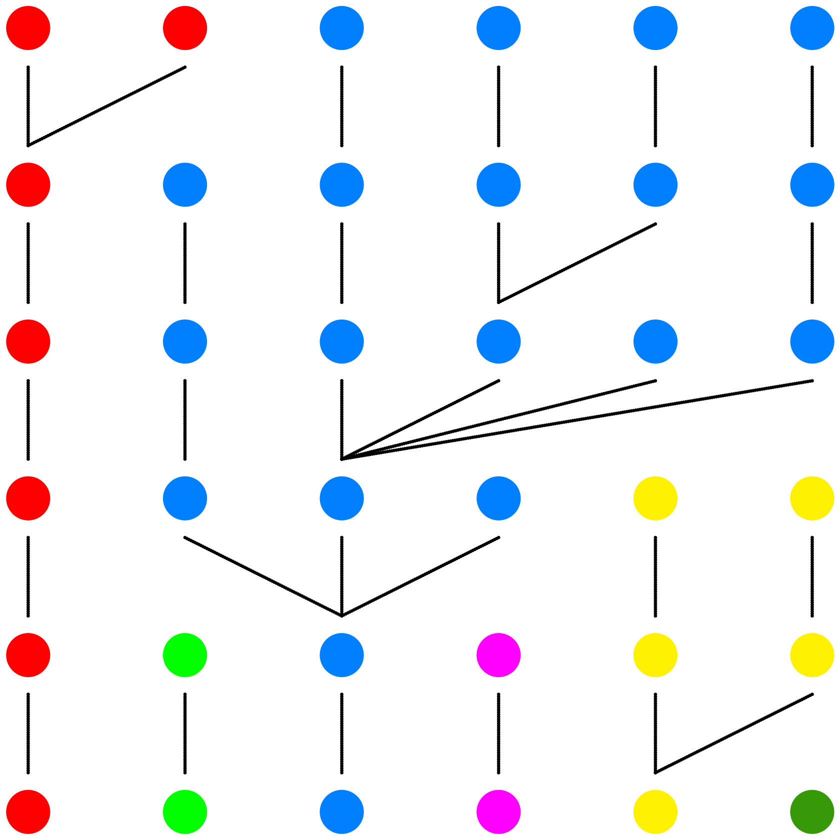
Particular instances of Λ-Cannings.

- $\Lambda = \delta_0$; **Moran model** = (locally) Voter model on the complete graph
- $\Lambda = \delta_1$ (locally) one individual takes over the whole population

Λ-Cannings dynamics



Λ-Cannings dynamics



From particle model to continuum limits

$X_t^{N,\Lambda}$:= empirical type distribution at time t

Jean Bertoin & Jean-Francois Le Gall (2003) Stochastic flows associated to stochastic processes, PTRF

Jochen Blath & Matthias Birkner (2009) Rescaled stable generalised Fleming-Viot processes: ... EJP

Measure-valued process. $(N \to \infty)$. X^Λ is a strong Markov process with values in $\mathcal{M}(K \times G)$ whose generator

$$\Omega := \Omega^{\Lambda}_{\text{resample}} + \Omega^{a(\cdot,\cdot)}_{\text{migration}}$$

acts on fcts of the form

$$\mu \mapsto \prod_{i=1}^{n} \langle \mu_{x_i}, \psi_i \rangle = \langle \otimes_{i=1}^{n} \mu_{x_i}, \prod_{i=1}^{n} \psi_i \rangle,$$

where we abbreviate

$$\mu_x := \mu\big(\cdot \times \{x\} \big), \qquad x \in G.$$

when we sample from a given $x \in G$

with local Λ-Cannings resampling and migration

$$\Omega^{\Lambda}_{\text{resample}} \prod_{i=1}^{n} \langle \cdot_{x_i}, \psi_i \rangle (\mu)$$

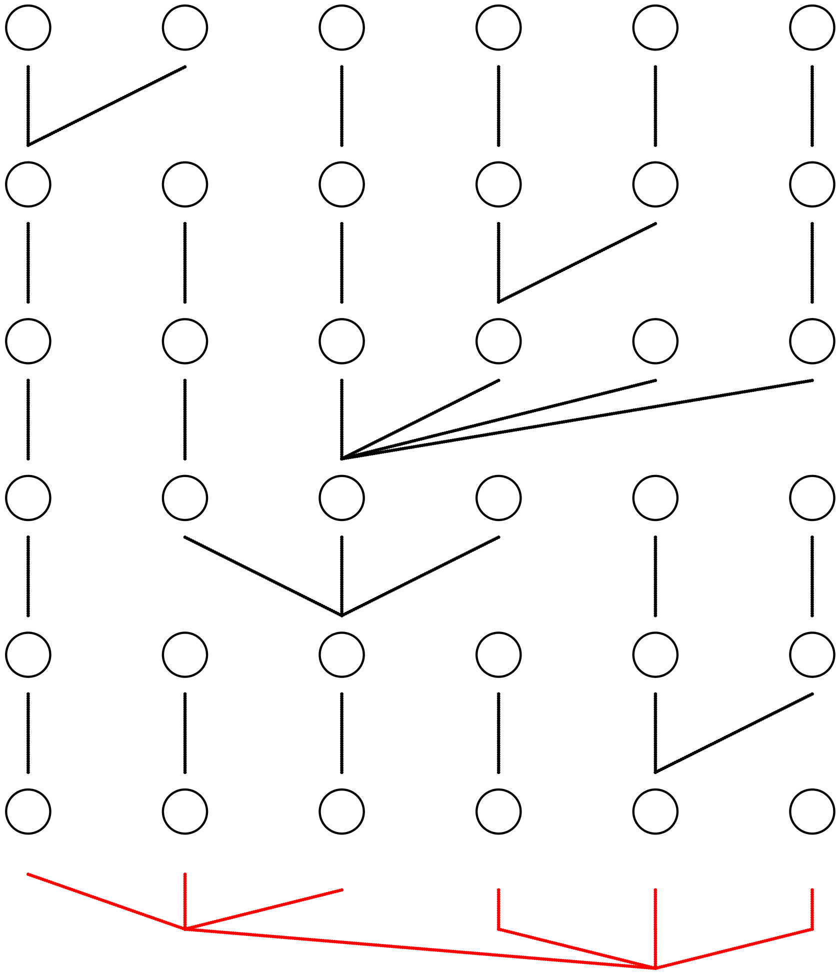
$$= \sum_{x \in \{x_1,\dots,x_n\}} \sum_{\substack{J_x \subseteq \{j \,:\, x_j = x\} \\ \# J_x \geq 2}} \lambda_{\#\{j \in \{1,\dots,n\}:\, x_j = x\}, \# J_x} \cdot \prod_{i \in \{1,\dots,n\} \setminus J_x} \langle \mu_{x_i}, \psi_i \rangle \cdot$$

$$\cdot \Big(\langle \mu_x, \prod_{j \in J_x} \psi_j \rangle - \prod_{j \in J_x} \langle \mu_x, \psi_j \rangle \Big).$$

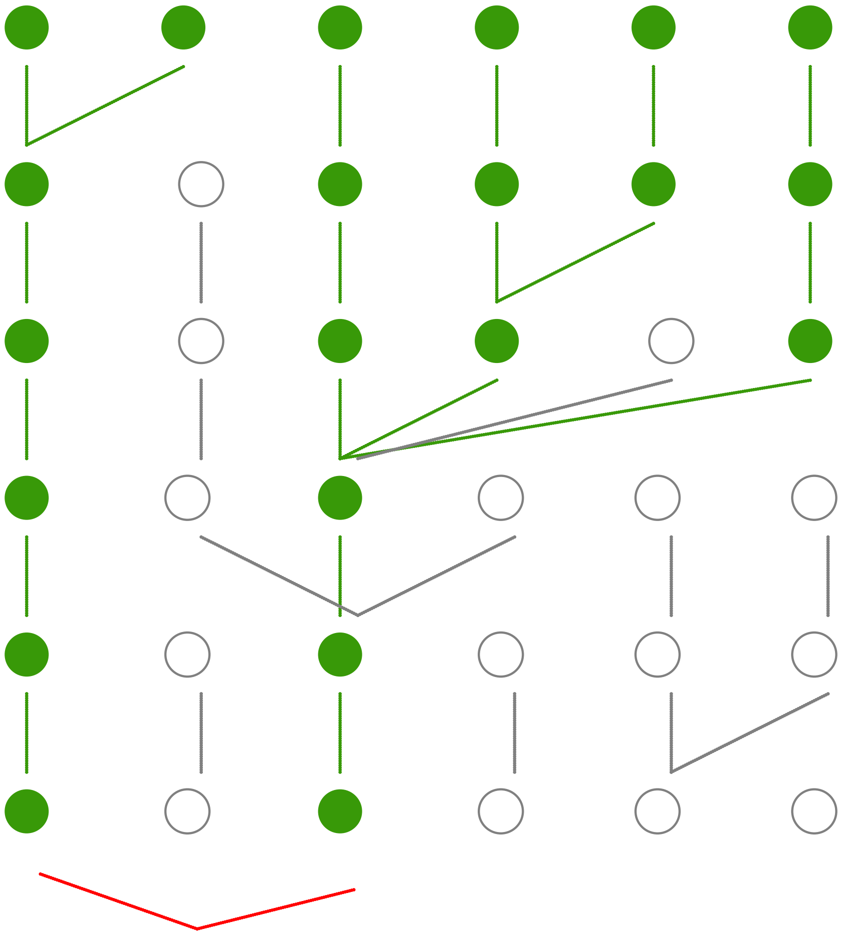
$$\Omega^{a(\cdot,\cdot)}_{\text{migration}} \prod_{i=1}^{n} \langle \cdot_{x_i}, \psi_i \rangle (\mu)$$

$$= \sum_{x \in G} \sum_{i=1}^{n} \big(a(x, x_i) - \delta(x, x_i) \big) \Big\langle \otimes_{l=1}^{i-1} \mu_{x_l} \otimes \mu_x \otimes_{l=i+1}^{n} \mu_{x_l}, \prod_{j=1}^{n} \psi_j \Big\rangle.$$

Tracing back ancestry



Tracing back ancestry



(locally)

Λ-coalescent (in backward picture)

given n ancestral lines,

k of them merge at rate $\lambda_{n,k}$

Genealogies evolve as spatial Λ-coalescent

Spatial Λ-coalescent is a strong Markov process which takes values in the set of partitions of all individuals where each partition block is assigned a site in G such that any "locally finite" subpopulation/-partition behaves as follows:

- **Migration.** **Partition elements change** their **position** according to a rate 1 random walk with transition probabilities $\bar{a}(x,y) := a(y,x)$.
- **Λ-coalescence.** Each **local** partition performs a **Λ-coalescent**.

Constructions of the Λ-coalescent.

- **finite G.** Limic & Sturm (2006), Spatial Λ-coalescent, EJP
- **countable G.** Donnelly & Kurtz's (1999) Particle representations for measure-valued population models, Annals of Probab.

Heuristics for the spatial coalescent with delay

Greven, Limic & W. (2005) Representation theorems for interacting Moran models, interacting Fisher-Wright diffusions, EJP

Limic & Sturm (2006) The spatial Λ-coalescent, EJP

1. **Not feeling the finiteness yet.** Two coalescing particles on G_N move **until one of them hits the boundary** which takes time $\ll T_N(1)$.

2. **Loosing a feeling for the geographic structure.** If the have not coalescent by that time, they (independently) **wrap around the torus**. For any $\varepsilon > 0$, by time of order $T_N(\varepsilon)$ they have forgotten their initial distances.

3. **Coalescing with (even more) delay on a complete graph.** From now on the geographic structure does not play a role anymore, and it takes the 2 particles an *exponential time with mean* $\#G_N$ to meet. Now they either coalesce or depart again. For the actual coalescence, they need a

   geometric number of trails with success probability $\frac{\lambda_{2,2}}{2+\lambda_{2,2}}$

   In a finite sample, at that stage you find **never more than 2 particles at the same location**.

Measure-valued finite system scheme: global perspective

$G := \mathbb{Z}^d$, $d \geq 3$, $G_N := [-N, N]^d \cap \mathbb{Z}^d$, $T_N(t) := t \cdot \# G_N$

For $\mu := \{\mu_x;\, x \in \mathbb{Z}^d\} \subset (\mathcal{M}_1(K))^{\mathbb{Z}^d}$, consider the **average measures**

$$\widehat{\theta}^N(\mu) := \frac{1}{\# G_N} \sum_{x \in G_N} \mu_x \in \mathcal{M}_1(K).$$

Let $\mu := (\mu_t)_{t \geq 0}$ be the measure-valued Λ-FV process and $(V_t^{\bar{\lambda} \cdot \delta_0})_{t \geq 0}$ the **non-spatial** measure-valued $\bar{\lambda}\delta_0$ **-FV diffusion**, where

$\bar{\lambda} := 2 \cdot \left(\rho + \frac{2}{\lambda_{2,2}}\right)^{-1}$ $\qquad$ $\rho :=$ escape probability on $\mathbb{Z}^d$.

$=$ probab. that 2 individuals do not merge due to delayed coalescence on $\mathbb{Z}^d$

Theorem. **(Greven, Klimovsky & W.)** Assume that $\hat{a}(x, y)$ is transient and irreducible, and that the initial family $\{\mu_x(0); x \in \mathbb{Z}^d\}$ is i.i.d. with mean measure $\theta \in \mathcal{M}_1$. Then

$$\mathcal{L}\left[\left(\widehat{\theta}^N(\mu^N_{T_N(t)})\right)_{t \geq 0}\right] \underset{N \to \infty}{\Longrightarrow} \mathcal{L}^\theta\left[\left(V_t^{\bar{\lambda} \cdot \delta_0}\right)_{t \geq 0}\right].$$

Measure-valued finite system scheme: local perspective

$G := \mathbb{Z}^d$, $d \geq 3$, $a_N(x,y) := \sum_{z:\, z=y \text{ mod } G_N} a(0,z)$

Fix a finite window $W \subset \mathbb{Z}^d$. Then $W \subset G_N$ for large enough $N \in \mathbb{N}$.

- **Restriction of a measure to the window W.** $\mu^W := \mu(\cdot \times (\cdot \cap W))$
- **Empirical measure.** $\widehat{\Sigma}^N(\mu) := \frac{1}{\# G_N} \sum_{x \in G_N} \delta_{(\sigma_x)_* \mu}$.

Theorem. **(Greven, Klimovsky & W.)** If $\hat{a}(x,y)$ is transient and irreducible, and the initial family $\{\mu_x(0); x \in \mathbb{Z}^d\}$ is i.i.d. with mean measure $\theta \in \mathcal{M}_1$, then there exists $\nu_\theta \in \mathcal{M}_1((\mathcal{M}_1(K))^{\mathbb{Z}^d})$ such that

$$\mathcal{L}\left[\mu_t\right] \underset{t \to \infty}{\Longrightarrow} \nu_\theta.$$

Moreover, for all $t > 0$,

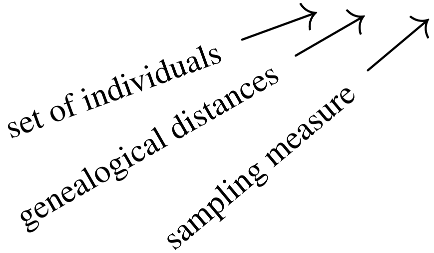
$$\mathcal{L}\Big[\Big(\widehat{\Sigma}^N\big(\mu^{N,W}_{T_N(t)+s}\big)\Big)_{s \geq 0}\Big|\widehat{\theta}^N\big(\mu^{N,W}_{T_N(t)}\big) = \theta'\Big] \underset{N \to \infty}{\Longrightarrow} \mathcal{L}_{\nu_{\theta'}}\big[\big(\mu^W_s\big)_{s \geq 0}\big].$$

Encoding Genealogies

Encoding genealogies in the non-spatial case ...

We aim to describe the **genealogical tree** of the **whole population** while making ancestral lines of **all possible samples explicit**.

We **encode** our genealogies by (U, r, μ)



and **evaluate samples** via **test functions** of the form

$$\Phi^{n,\phi}(U, r, \mu) := \int_{U^n} \mu^{\otimes n}(\mathrm{d}\underline{u})\, \phi\big((r(u_i, u_j))_{1 \le i < j \le n}\big).$$

Such test functions are referred to as **polynomials**.

The state space: more formal

$$\mathbb{U} := \{\text{isometry classes of ultra metric probability spaces}\}.$$

Misha Gromov (2000), Metric structures for Riemannian and non-Riemannian spaces; Chapter $3\frac{1}{2}$

Andreas Greven, Peter Pfaffelhuber & W. (2009), Convergence of random metric measure spaces: PTRF

Wolfgang Löhr (2013), Equivalence of Gromov-Prohorov- and Gromov's $\underline{\square}_\lambda$ -metric on the space of mm-spaces, ECP

We equip $\mathbb{U}$ with the **Gromov-weak topology** which means convergence in the sense of **convergence of all polynomials** (with continuous bounded test functions).

The Λ-coalescent tree

We can associate a realization of a Λ-coalescent as a metric space: indeed, first equip $\mathbb{N}$ with the genealogical distance $r_{\text{genealogy}}$, and then consider its completion $(\bar{\mathbb{N}}, r_{\text{genealogy}})$.

Question. Can we assign $(\bar{\mathbb{N}}, r_{\text{genealogy}})$ a probability measure such that any finite sample of size n is distributed like an n-Λ-coalescent.

Greven, Pfaffelhuber & W. (2009), Convergence of random metric measure spaces: The Λ-coalescent tree, PTRF

Theorem. The family

$$\left\{\left(\bar{\mathbb{N}}, r_{\text{genealogy}}, \tfrac{1}{n}\sum_{i=1}^{n}\delta_i\right);\ n \in \mathbb{N}\right\}$$

is tight if and only if the **dust-free property** holds, i.e., iff $\int_0^1 \Lambda(\mathrm{d}x)\frac{1}{x} = \infty$.

The limit point is unique and referred to as **Λ-coalescent tree**.

... and in the spatial case

We aim to describe the **genealogical tree** of the **whole population** while making ancestral lines and **locations** of **all possible samples explicit**.

We **encode** our genealogies by (U, r, μ)

set of individuals → U
genealogical distances → r
measure on $U \times G$ → μ

and **evaluate samples** via **test functions** of the form

$$\Phi^{(x_1,\dots,x_n),\phi}(U,r,\mu) := \int_{U^n} \otimes_{i=1}^n \mu_{x_i}(\mathrm{d}\underline{u})\big((\phi \circ \underline{\underline{r}})\big)(\underline{u}).$$

with

$$\underline{\underline{r}} : \underline{u} \mapsto \big(r(u_i, u_j)\big)_{1 \le i < j \le n}$$

The state space including types: more formal

$$\mathbb{U}^G := \{\text{isometry classes of ultra metric spaces such that } \mu_x(U) = 1,\ \forall x \in G\}.$$

Depperschmidt, Greven & Pfaffelhuber (2011) Marked metric measure spaces, ECP

Andreas Greven, Rongfeng Sun & W.; Continuum space limit of the genealogies of interacting Fleming-Viot processes on $\mathbb{Z}$, manuscript

We equip $\mathbb{U}^G$ with the **marked Gromov-weak $\#$ topology** which means convergence in the sense of **convergence of all polynomials** (with continuous bounded test functions).

The spatial Λ-coalescent tree

Start with a realization of a spatial Λ-coalescent starting with infinitely many singleton blocks at each location, and read off genealogical distances. As the migration is transient on $\mathbb{Z}^d$, $d \geq 3$, distances might be infinity due to avoidance of blocks. Consider the **transformation map** $\mathfrak{t}: \mathbb{U}^G \to \mathbb{U}^G$ defined as

$$\mathfrak{t}\Big(\big(U, r(\cdot,\cdot), \mu\big)\Big) := \big(U, 1 - e^{-r(\cdot,\cdot)}, \mu\big).$$

Question. Let $\mathcal{I}$ denote the set of all individuals collected in the blocks. Can we assign to $(\bar{\mathcal{I}}, r_{\text{genealogy}})$ a measure μ on $\bar{\mathcal{I}} \times G$ such that μ_x is probability measure and such that individuals sampled at prescribed locations $x_1, ..., x_n$ span the corresponding finite spatial Λ-coalescent tree.

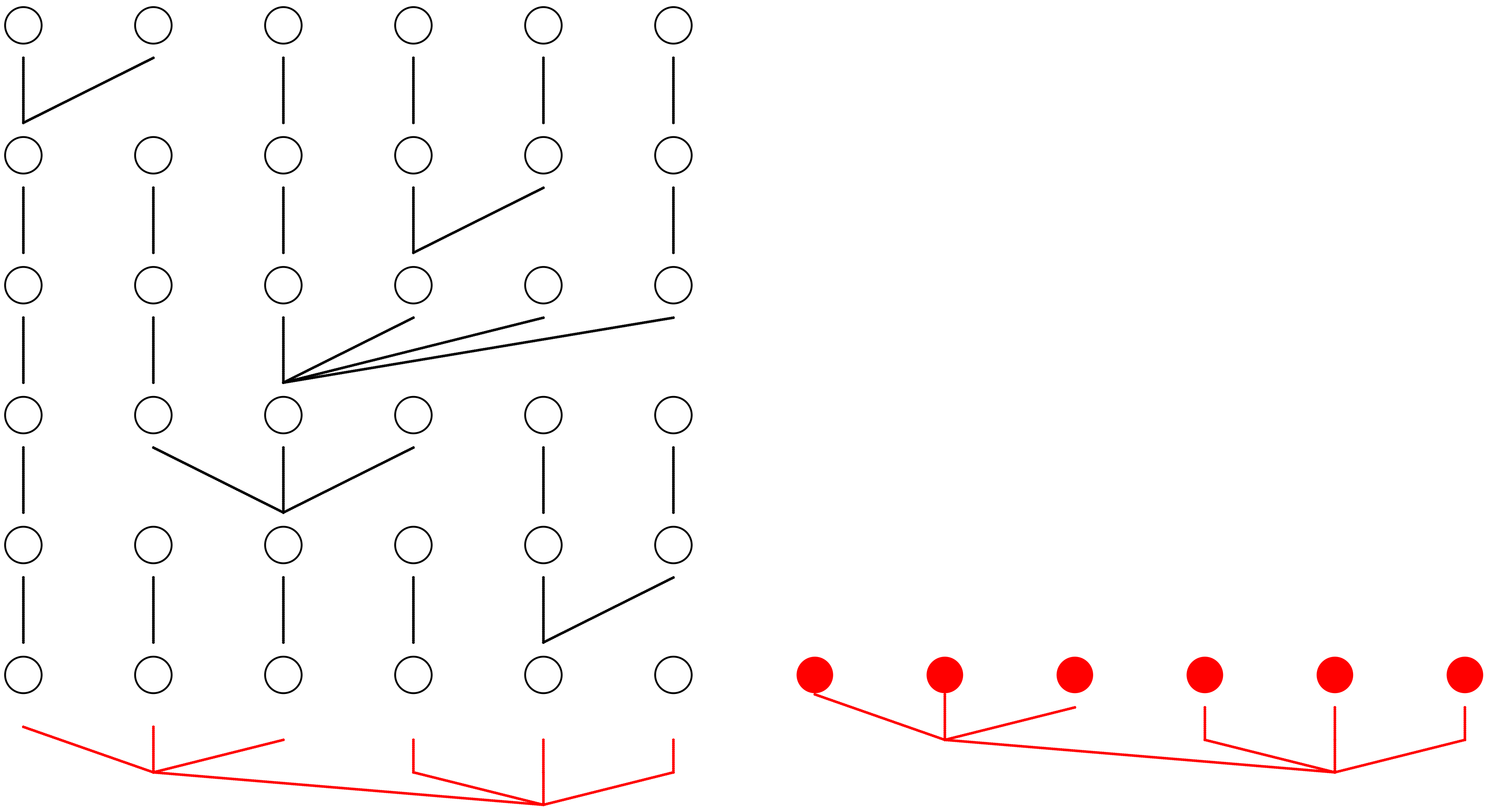
Under the **dust-free property**, we have a positive answer in the sense that

$$\mathfrak{t}\Big(\big(\bar{\mathcal{I}}, r_{\text{genealogy}}\big), \tfrac{1}{n}\sum_{x\in G} \sum_{\text{first } n \text{ individuals } \iota \text{ at } x} \delta_\iota\Big)\Big) \textit{ converges Gromov-}\#\textit{-weakly}$$

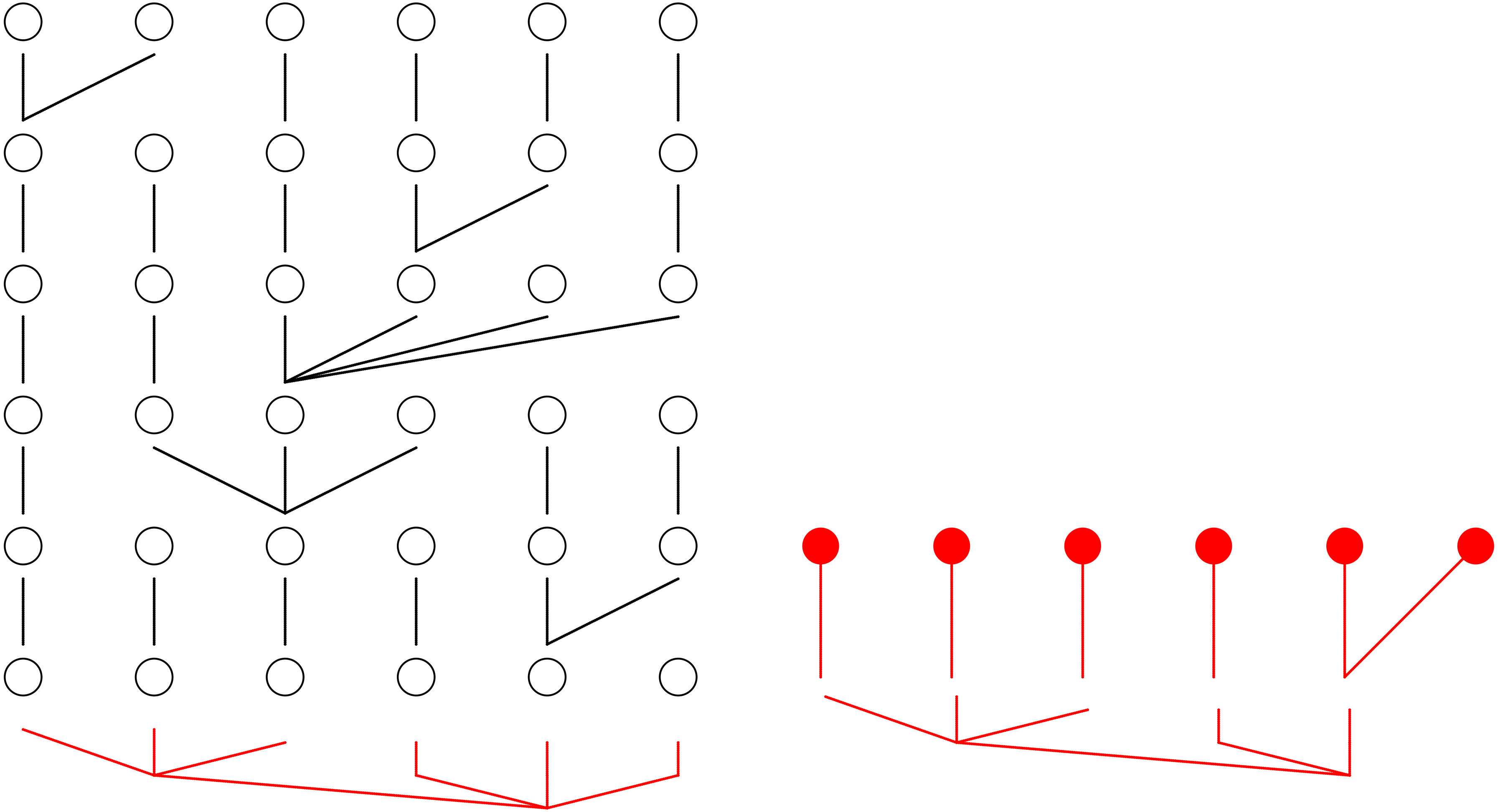
The limit is referred to as **spatial Λ-coalescent tree**.

Evolving Fleming-Viot genealogies

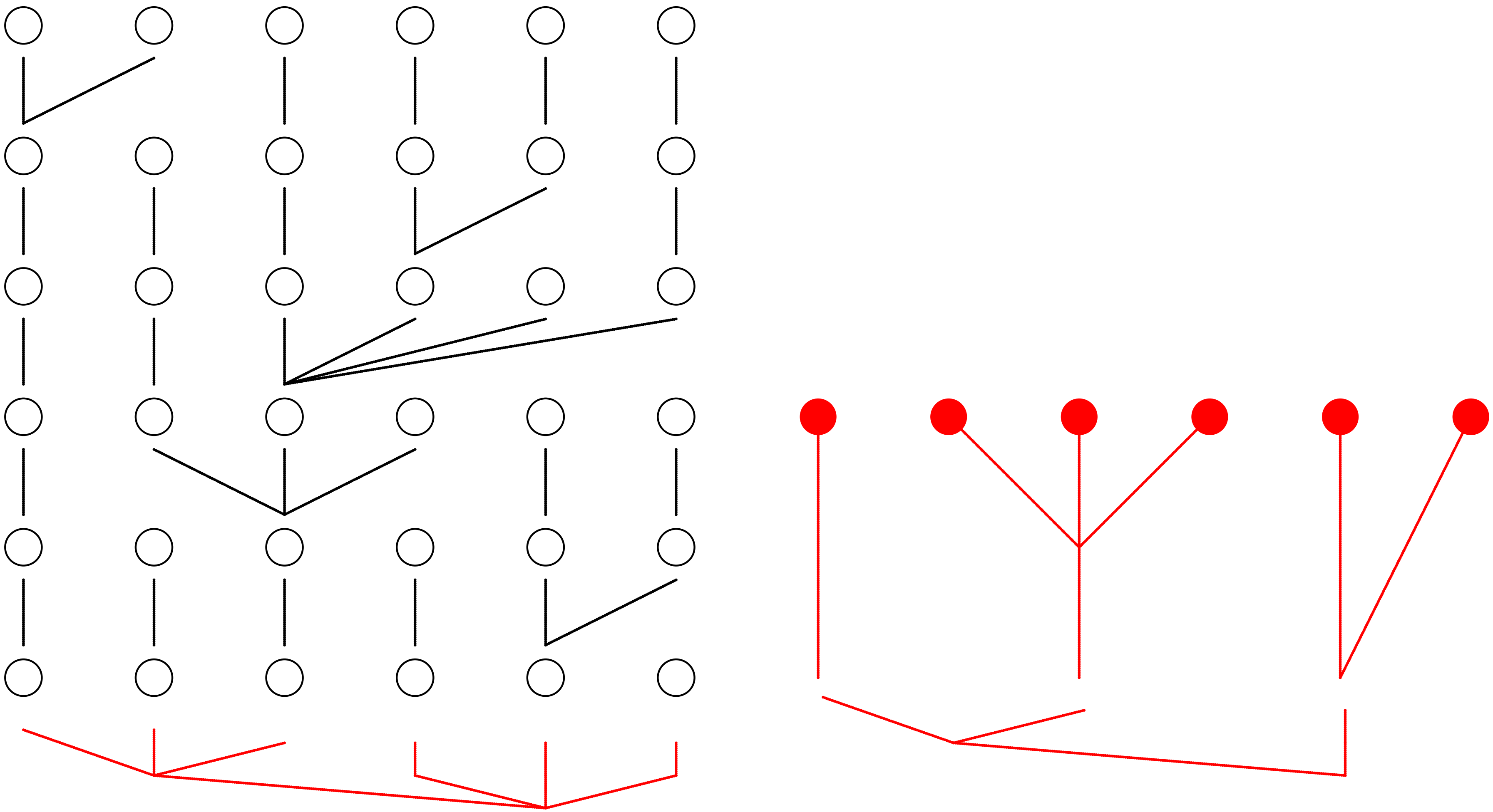
Evolving genealogies



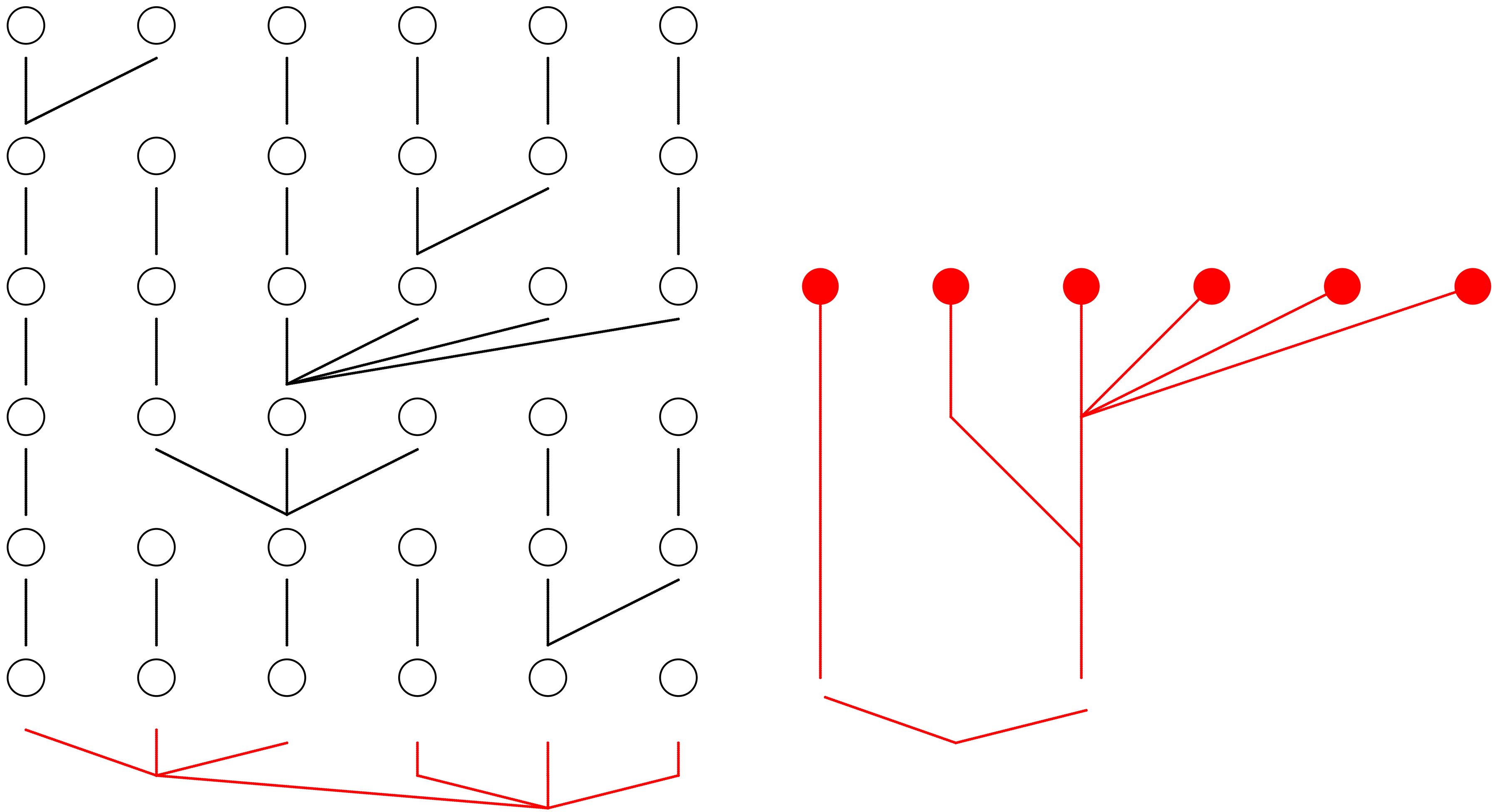
Evolving genealogies



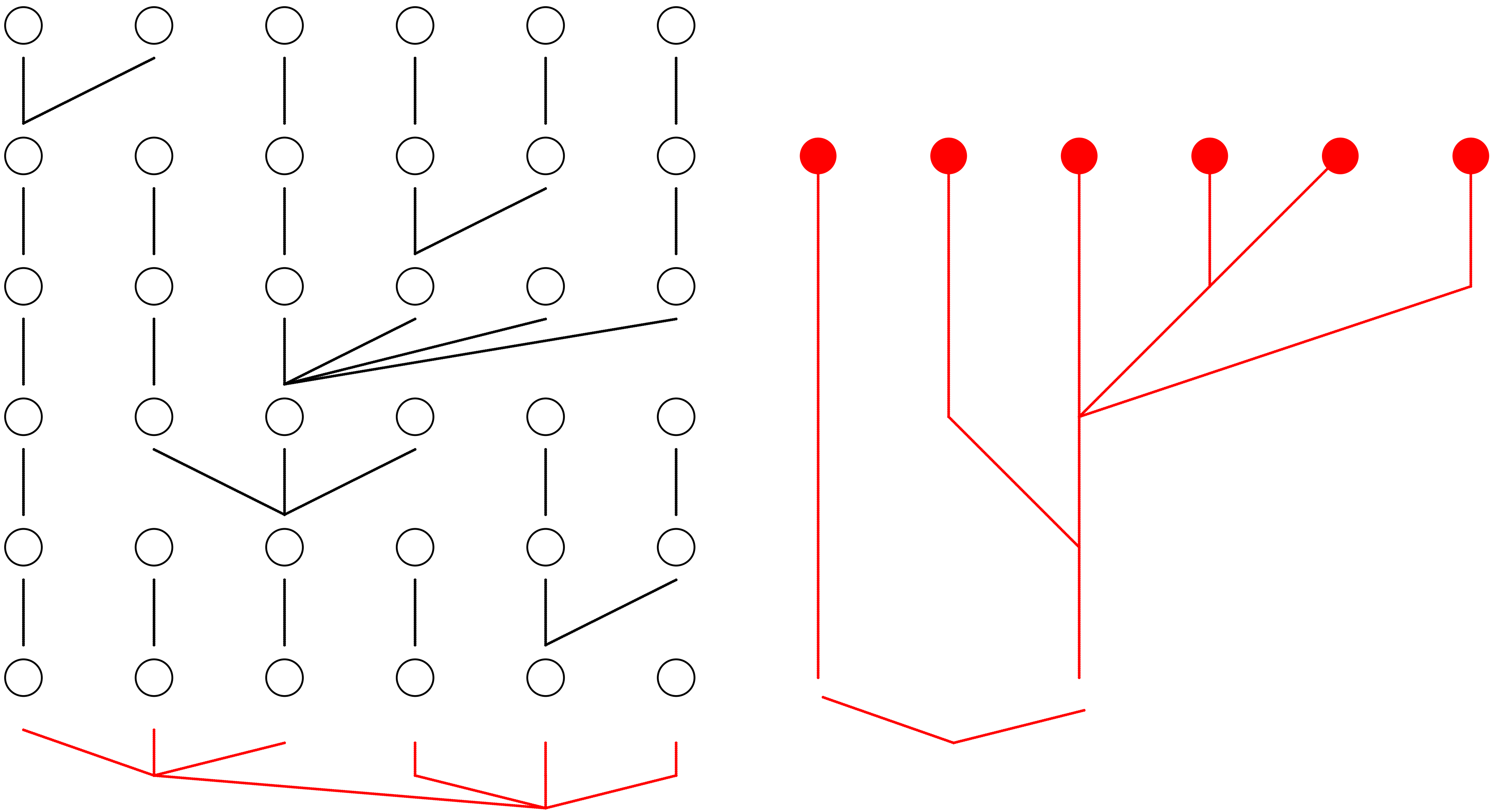
Evolving genealogies



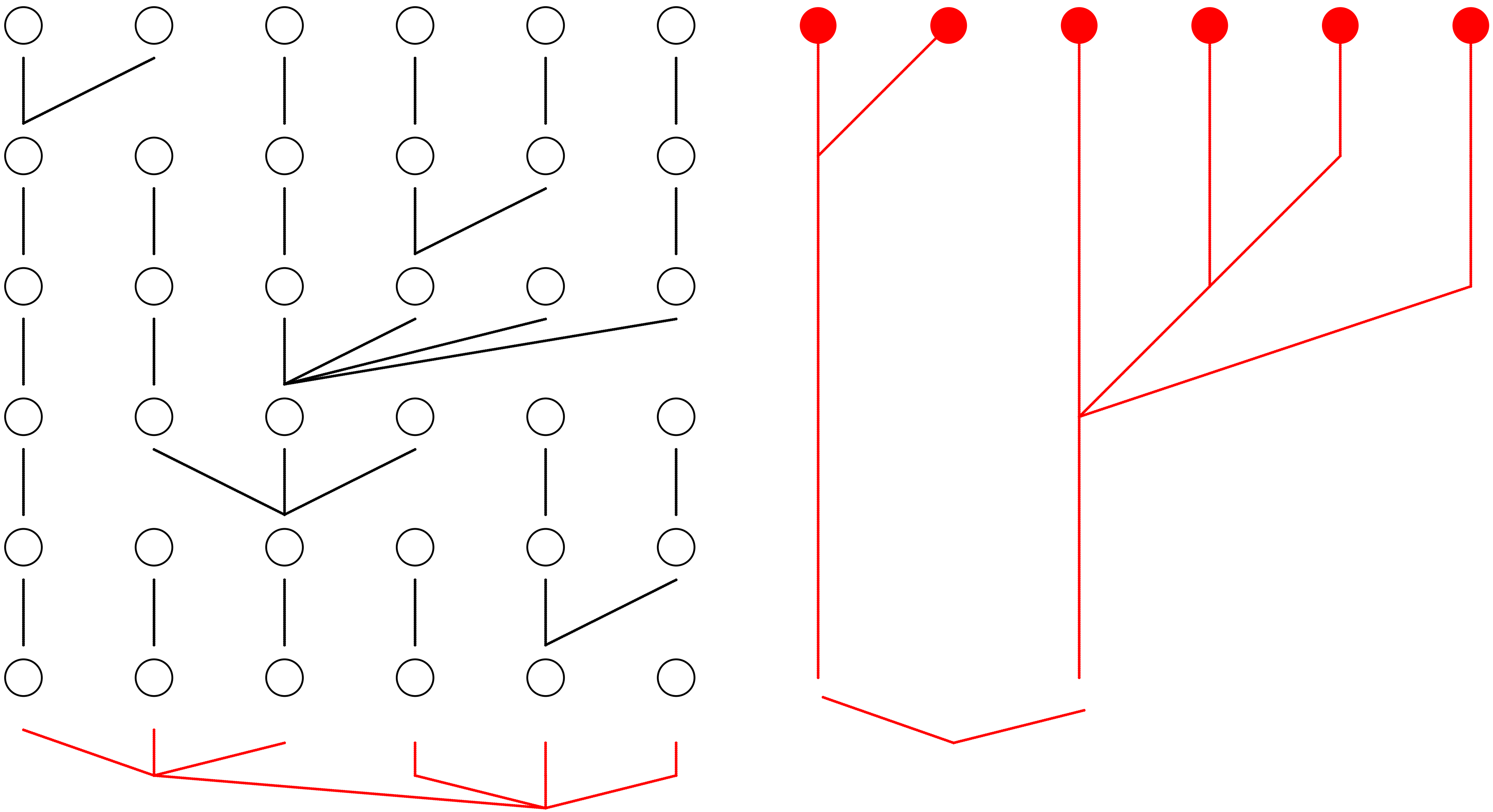
Evolving genealogies



Evolving genealogies



Evolving genealogies



Tree-valued spatial Λ-Fleming-Viot

$\mathcal{U}$ is the $\mathbb{U}^G$-valued strong Markov process whose generator

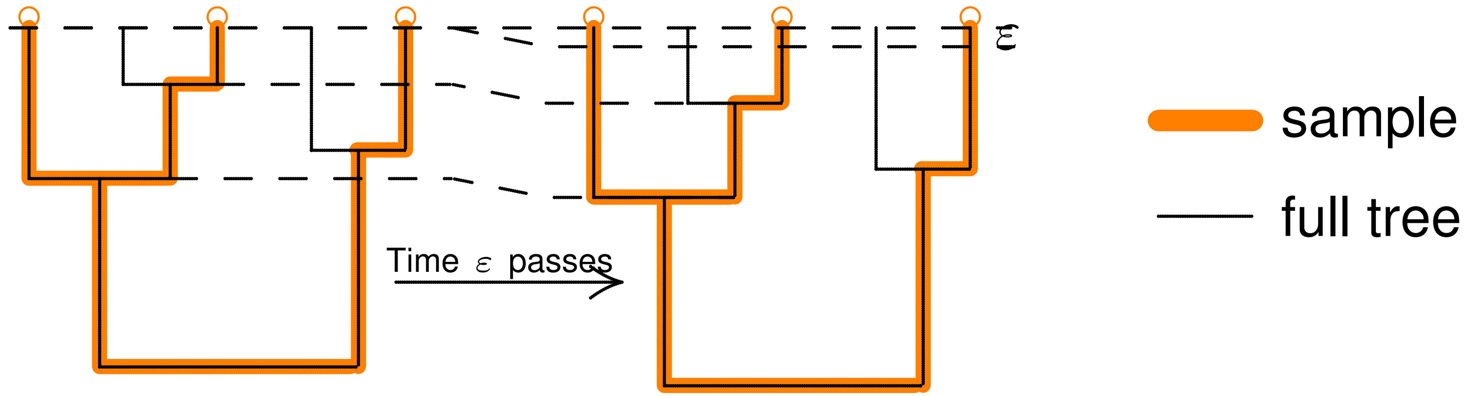
$$\Omega_{\text{tree }\Lambda\text{-FV}} = \Omega^{\Lambda}_{\text{resample}} + \Omega^{a(\cdot,\cdot)}_{\text{migration}} + \Omega_{\text{growth}}$$

acts on functions of the form

$$\Phi^{(x_1,\ldots,x_n),\phi}\big(U,r,\mu\big) := \int_{U^n} \otimes_{i=1}^{n} \mu_{x_i}(\mathrm{d}\underline{u})\big((\phi \circ \underline{\underline{r}})\big)\big(\underline{u}\big).$$

where $n \in \mathbb{N}$ and $x_1, ..., x_n \in G$.

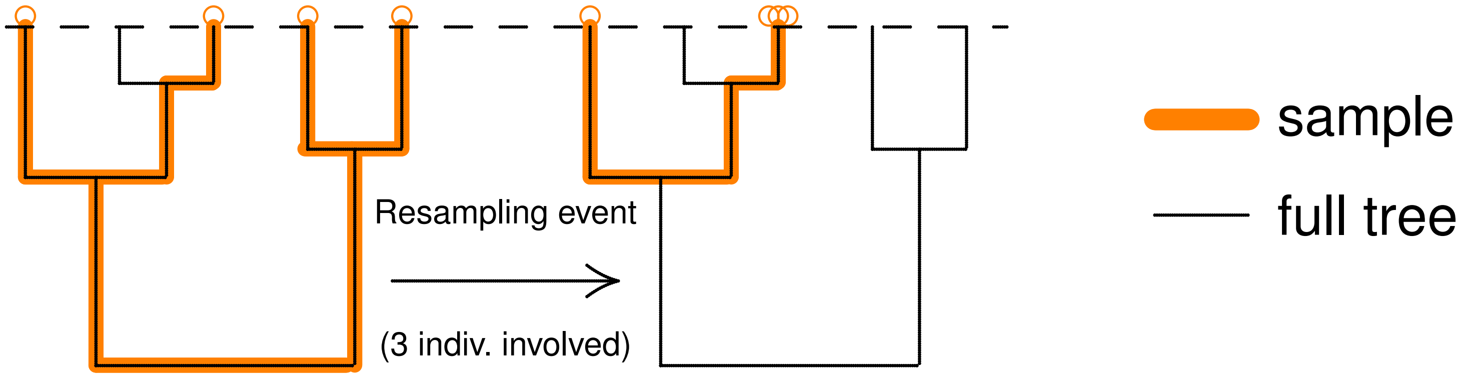
Tree growth



$$\Omega_{\text{growth}} \Phi^{(x_1,\dots,x_n),\phi}(U,r,\mu)$$

$$= 2\int_{U^n} \otimes_{i=1}^n \mu_{x_i}(\mathrm{d}\underline{u}) \sum_{1\le i<j\le n} \frac{\partial\phi}{\partial r_{i,j}}\Big(\big(r(u_i,u_j)\big)_{1\le i<j\le n}\Big).$$

Λ-Resampling



$$\Omega^{\Lambda}_{\text{resampling}}\Phi^{(x_1,\dots,x_n),\phi}(U,r,\mu)$$

$$= \sum_{x\in\{x_1,\dots,x_n\}} \sum_{\substack{J_x \subseteq \{j\,:\,x_j = x\} \\ \#J_x \geq 2}} \lambda_{\#\{j\in\{1,\dots,n\}:\,x_j=x\},\#J_x} \cdot \int \otimes_{i=1}^{n}\mu_{x_i}(\mathrm{d}\underline{u})\, \{\phi(\underline{\underline{r}}^{J_x}(\underline{u})) - \phi(\underline{\underline{r}}(\underline{u}))\},$$

where for each $n \in \mathbb{N}$, $J \subseteq \{1,\dots,n\}$, and $1 \leq i < j \leq n$,

$$r^{J}_{i,j} := \begin{cases} r_{i,j}, & \text{if } i,j \notin J, \\ r_{i\wedge \min J,\, i\vee \min J}, & \text{if } i \notin J, j \in J \\ r_{j\wedge \min J,\, j\vee \min J}, & \text{if } i \in J, j \notin J, \\ 0, & \text{if } i,j \in J. \end{cases}$$

Well-posed martingale problem

Consider the operator

$$\Omega_{\text{tree } \Lambda\text{-FV}} = \Omega^{\Lambda}_{\text{resample}} + \Omega^{a(\cdot,\cdot)}_{\text{migration}} + \Omega_{\text{growth}}$$

acting on the space

$\Pi^G :=$ (spatial) polynomials with differentiable, bounded test functions.

Theorem. **(Greven, Klimovsky & W.)** Let $\mathbf{P_0}$ be a probability measure on $\mathbb{U}^G$. The $(\mathbf{P_0}, \Omega_{\text{tree } \Lambda\text{-FV}}, \Pi^G)$ **-martingale problem is well-posed** provided that the **dust-free property** holds.

The dual process

Tree-valued spatial Λ-coalescent. $\mathcal{C} = (C_t, \underline{\underline{r}}_t)_{t\geq 0}$

- **Migration and coalescence.** $(C_t)_{t\geq 0}$ is the spatial Λ-coalescent.
- **Distance growth.** At time t, for all $1 \leq i < j$ the information on the genealogical distance $r'_\cdot(i,j)$ grows with constant speed 2 as long as the are not merged into the same partition element.

We **encode** the **states of** $(C_t)_{t\geq 0}$ by

$$\Big(\{(\pi, L_\pi) : \pi \in \mathcal{P}\}\Big)$$

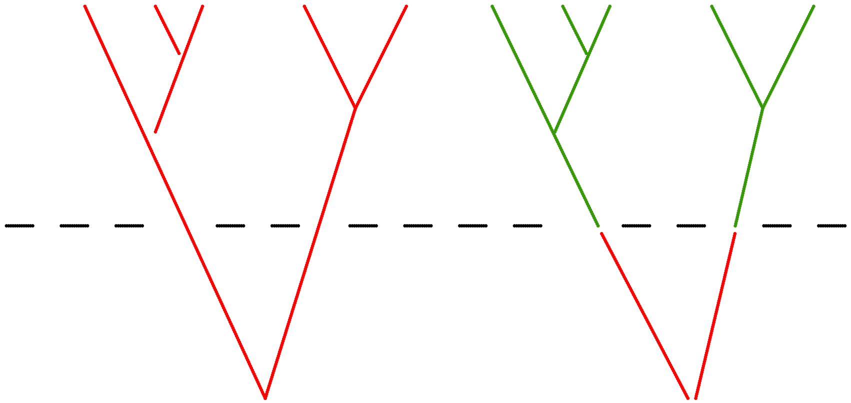
blocks

blocks' locations

partition of $\{1,\dots,n\}$

We usually start in $C_0 := \{(\{i\}, x_i);\ i = 1, \dots, n\}$.

Duality in the tree-valued setting



tree-valued spatial Λ-FV

$(U, r^{\uparrow}, \mu)_t$

tree-valued spatial Λ-coalescent

$(\{(\pi, L_\pi);\ \pi \in \mathcal{P}\}, r^{\downarrow})_t$

Greven, Pfaffelhuber & W. (2013) Tree-valued Fleming-Viot diffusion, PTRF

Theorem. (Greven, Klimovsky & W.)

$$\mathbf{E}^{(U_0, r_0, \mu_0)}\Big[\int \bigotimes_{l=1}^{n} (\mu_t)_{x_l}(\mathrm{d}\underline{u})\,\phi\big((r_t^{\uparrow}(u_i, u_j))_{1 \le i < j \le n}\big)\Big]$$

$$= \mathbf{E}^{(C_0, 0)}\Big[\int \bigotimes_{\varpi \in \mathcal{P}_t} (\mu_0)_{(L_t)_\varpi}(\mathrm{d}v_\varpi)\,\phi\big((r_t^{\downarrow}(i,j) + r_0^{\uparrow}(v_{\varpi(i)}, v_{\varpi(j)}))_{1 \le i < j \le n}\big)\Big]$$

Longterm behavior

Recall $r \mapsto 1 - e^{-r}$ and the corresponding map $\mathfrak{t} : \mathbb{U}^G \to \mathbb{U}^G$ (transforming distances accordingly).

As a consequence of the duality,

$$\mathfrak{t}(\mathcal{U}_t) \underset{t\to\infty}{\Longrightarrow} \mathfrak{t}\big(\text{ spatial } \Lambda\text{-coalescent tree}\big).$$

The tree-valued finite system scheme

Observing spatial genealogies

Encoding genealogies via metric measure spaces fits very well with concept of **sampling from the population**.

For observing the **spatial genealogies** as marked metric measure spaces we have different choices:

- **Global point of view (G finite necessary).** From a *macroscopic* point of view, we observe a finite sample from the whole population and on the macroscopic time scale.
- **Local point of view.** From a *microscopic* point of view, we take a finite sample from fixed locations, and observe an old population on the microscopic time scale.

The finite system scheme from a global point

$G_N := [-N,N]^d \cap \mathbb{Z}^d$, $d \geq 3$, $a_N(x,y) := \sum_{z:\, z=y \text{ mod } G_N} a(0,z)$

- **Macroscopic time scale.** $T_N(t) := t\#G_N$
- **Global average map.** $\mathfrak{q}_N : (U,r,\mu) \mapsto (U, \frac{1}{\#G_N} r, \frac{1}{\#G_N}\mu(\cdot \times G_N))$.
- **Limiting non-spatial dynamics.** $\mathcal{V}^{\bar{\lambda}\delta_0}$ is tree-valued $\bar{\lambda} \cdot \delta_0$-Fleming-Viot diffusion with

  $\bar{\lambda} := 2 \cdot \left(\rho + \frac{2}{\lambda_{2,2}}\right)^{-1}$ $\qquad \rho :=$ escape probability on $\mathbb{Z}^d$.

  $=$ probab. that 2 individuals do not merge due to delayed coalescence on $\mathbb{Z}^d$

Theorem **(Greven, Klimovsky and W.)** If for the initial states $\mathcal{U}_0$ of tree-valued Λ-FV dynamics $\mathfrak{q}_N(\mathcal{U}_0) \underset{N\to\infty}{\longrightarrow} \mathcal{V}_0^{\bar{\lambda}\delta_0}$ and $\hat{a}(x,y)$ is transient and irreducible, then

$$\left(\mathfrak{q}_N(\mathcal{U}^N_{T_N(t)})\right)_{t\geq 0} \underset{N\to\infty}{\Longrightarrow} \left(\mathcal{V}_t^{\bar{\lambda}\cdot\delta_0}\right)_{t\geq 0}.$$

The finite system scheme from a local point

Fix a finite window $W \subset \mathbb{Z}^d$. Then $W \subset G_N$ for large enough $N \in \mathbb{N}$.

- **Restriction of genealogies to windows.** $(U,r,\mu)^W := (U,r,\mu\big|_W)$
- **Empirical genealogy.** $\widehat{\Sigma}((U,r,\mu)) := \frac{1}{\#G_N}\sum_{x \in G_N} \delta_{(U,r,(\sigma_x^N)_*\mu)}$.

Theorem **(Greven, Klimovsky and W.)** Under the same assumptions as stated for the global finite system scheme, for all $t \geq 0$,

$$\Big(\widehat{\Sigma}^N\big(\mathcal{U}^{N,W}_{T_N(t)+s}\big)\Big)_{s\geq 0} \underset{N\to\infty}{\Longrightarrow} \mathcal{L}^{\text{spatial } \bar{\lambda}\cdot\delta_0 \text{-coal tree}}\Big[\big(\mathcal{U}_s\big)_{s\geq 0}\Big]$$

Two different strategies of proof

- **The straight forward approach.** Give the look-down construction for the spatial Λ-coalescent tree, use
  - estimates for **how fast the coalescent comes down** from

    Greven, Limic & W. (2005) Representation theorems for interacting Moran models, interacting Fisher-Wright diffusions, EJP
    Limic & Sturm (2006) The spatial Λ-coalescent, EJP
  - a criterion for the **compact containment condition** from

    Greven, Pfaffelhuber & W. (2013) Tree-valued Fleming-Viot diffusion, PTRF
  - **general techniques** for finite system schemes

    Cox & Greven (1994); Finite system scheme: an abstract theorem and a new example; CRM
- **More conceptional.** Prove that our convergence results towards a tree-valued strong Markov processes hold provided that
  - the one-dimensional (tree-valued) distributions converge,
  - they hold for the corresponding measure-valued processes,
  - any limit process can be shown to be a strong Markov process.

Many thanks