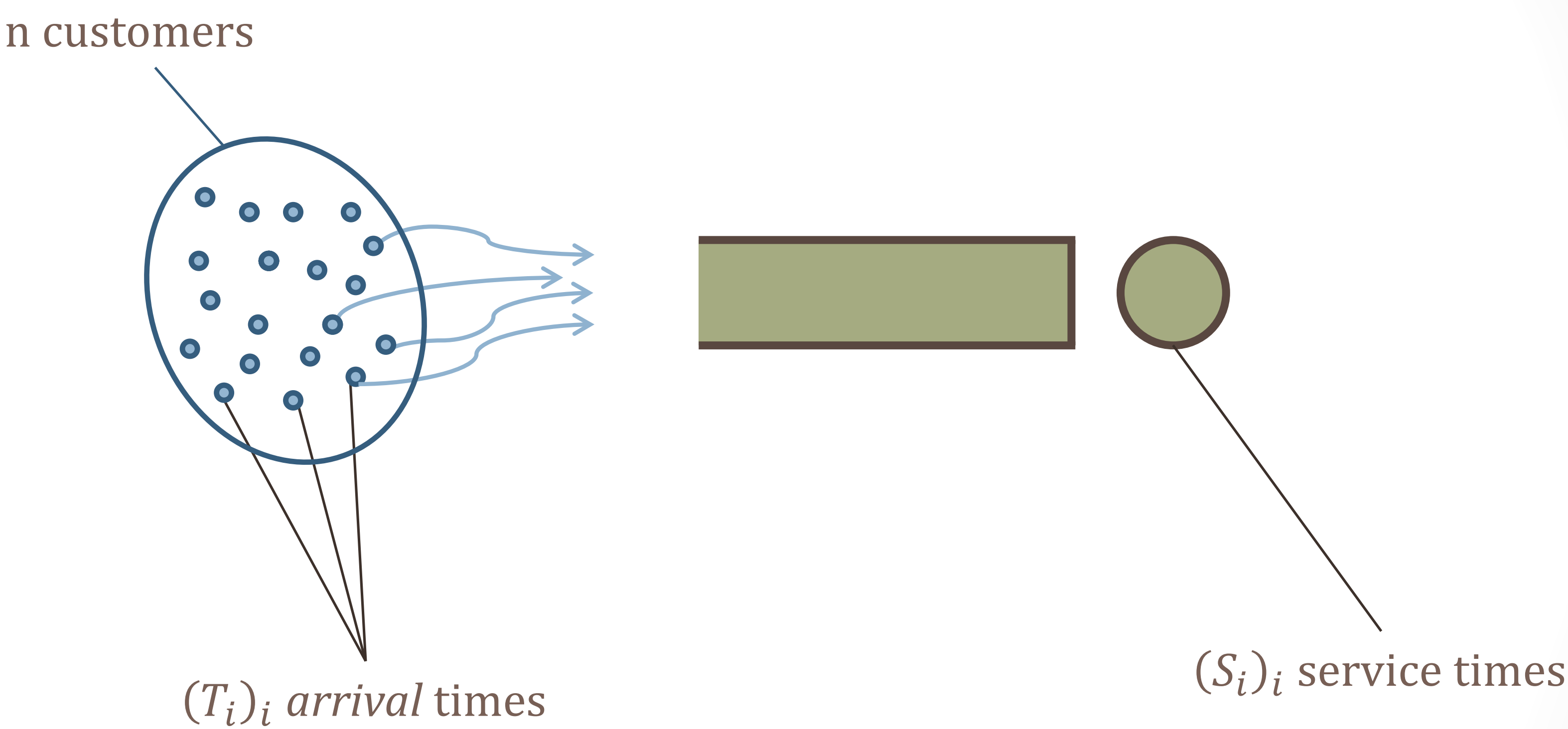
Critical queues exhibiting the depletion-of-points effect

Gianmarco Bet YEQT IX

12/11/2015

Joint work with: Remco van der Hofstad & Johan van Leeuwaarden

The model



Another arrival process

Π(t) Poisson process

$M_i \in \{1, .., n\}$ Uniform 'marks'

(what I mean by) Heavy-traffic

Overloaded if it grows linearly with time

$$Q_n(t) \approx tn$$

Critically loaded ..if it is not easy to guess how it behaves

Stable if it does not grow as a power of n

$$(Q_n(t))_n \text{ is tight}$$

Heavy-traffic (cont'd)

$$Q_n(t) \quad \cong \quad n\left(F_T(t) \quad - \quad \frac{t}{\mathbb{E}[S]}\right) \quad + \quad n^{\frac{1}{2}}..$$

$$\max_t f_T(t) \cdot \mathbb{E}[S] = 1$$

For rate λ exponentials arrivals

$$\lambda \cdot \mathbb{E}[S] = 1$$

"Critical window"

$$\lambda \cdot \mathbb{E}\left[S(1+\beta n^{-1/3})\right] = 1 + \beta n^{-1/3}$$

Main result

Scale service times as S_i/n. Then:

Theorem (Scaling limit for exponential arrivals)

Assume ***heavy-traffic*** *and* $\mathbb{E}[S^2] < \infty$, $Q_n(0) = n^{1/3}q$

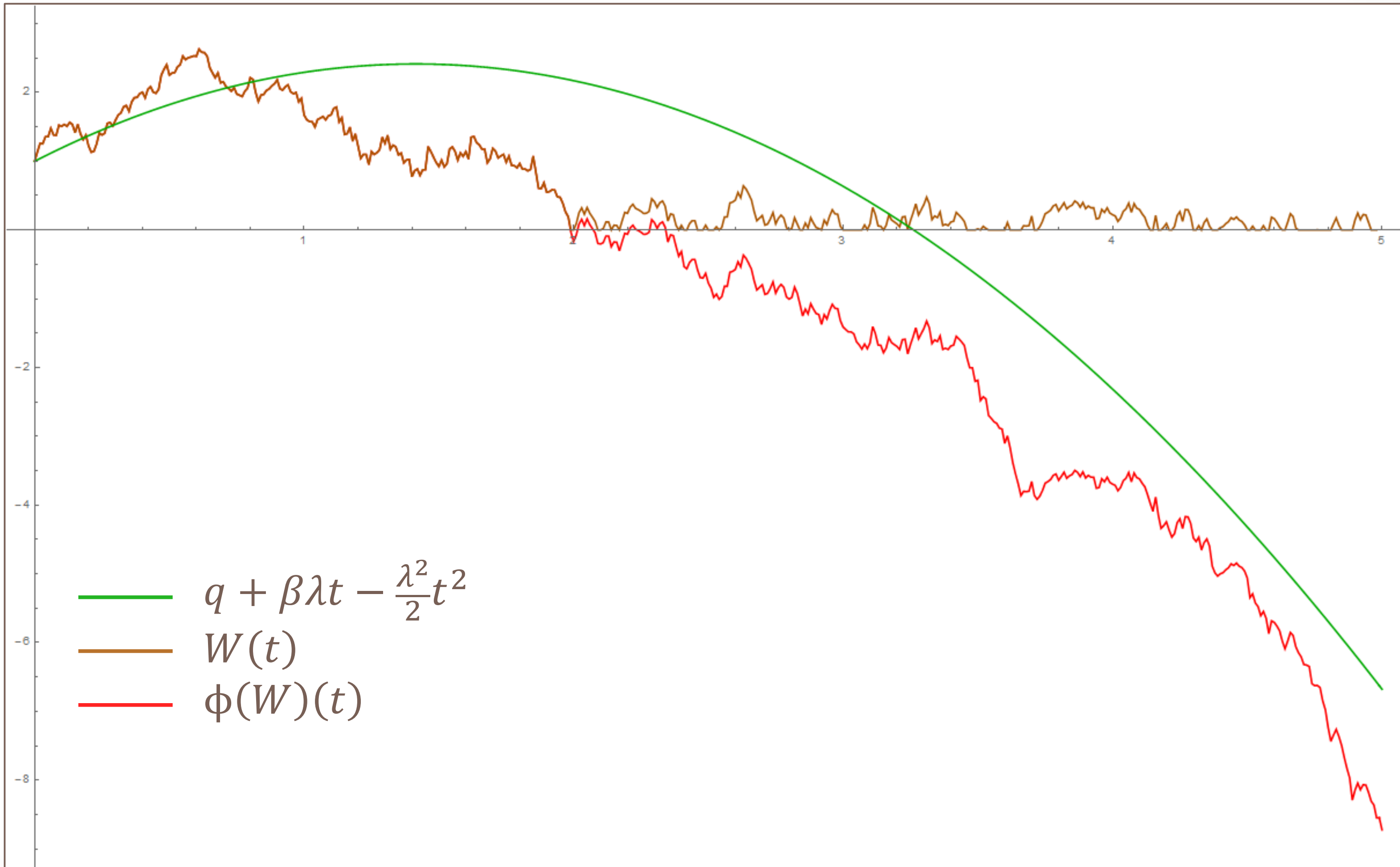
$$n^{-1/3}Q_n\left(tn^{-1/3}\right) \overset{d}{\to} \phi(W)(t),$$

where $\phi(W)$ *is the reflected version of*

$$W(t) = q + \beta\lambda t - \frac{\lambda^2}{2}t^2 + \sigma B(t),$$

and $B(t)$ *is a standard Brownian motion.*

A sample path



Idea of the proof

Define $Q_n(t)$ as $\quad Q_n(t) = \phi(W_n)(t)$

where $W_n(t)$ is $\quad W_n(t) = A_n(t) - S_n(t)$

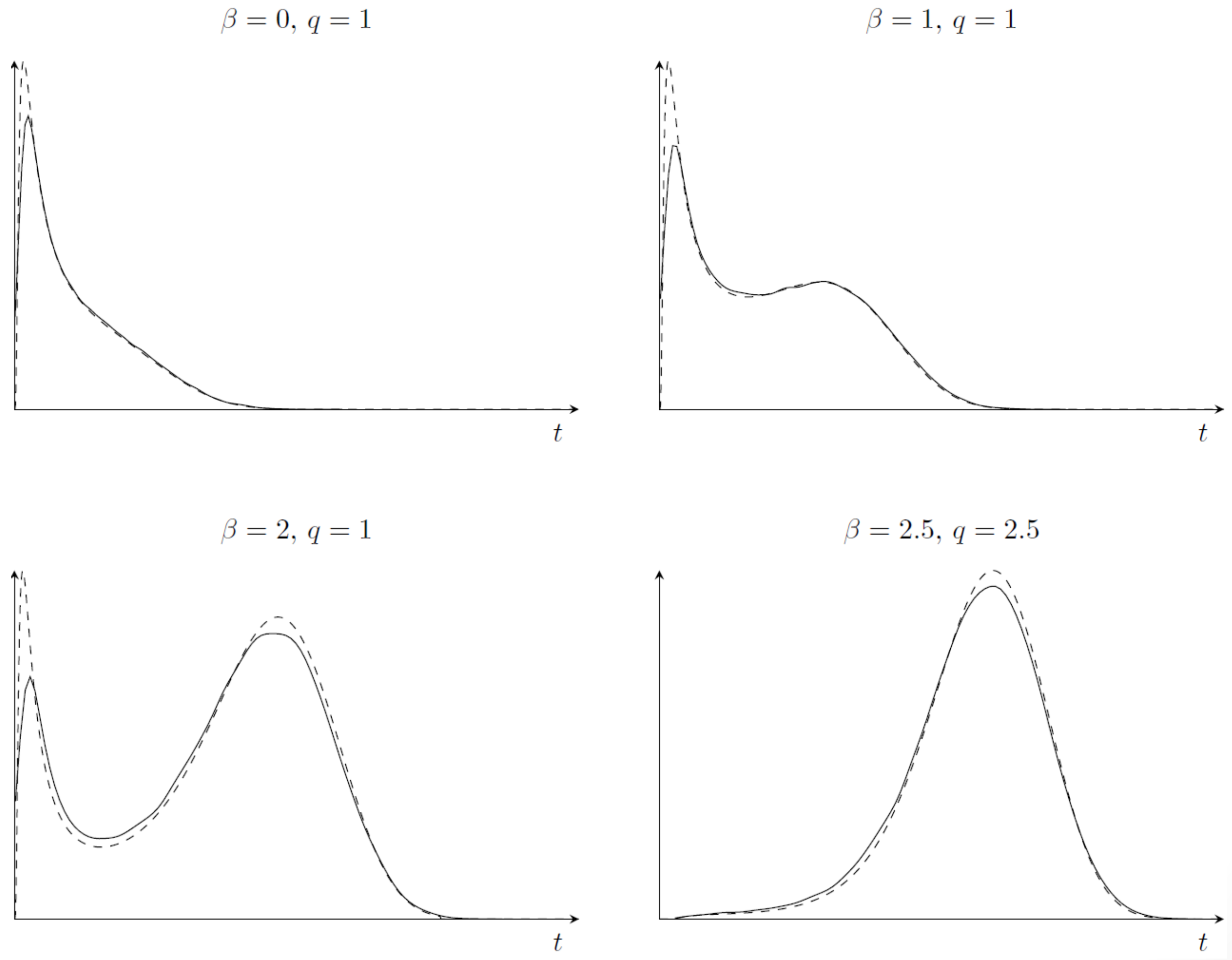
$$\begin{cases} A_n(t) = \sum_{i=1}^{n} \mathbb{1}_{\{T_i \le t\}} \\ S_n(t) = \max\{k : \sum_i^k S_i \le tn\} \end{cases}$$

rescaling

$$\begin{aligned} n^{-1/3} W_n\left(tn^{-1/3}\right) = {} & n^{-1/3}\left(A_n(tn^{-1/3}) \ - F_T(tn^{-1/3})n\right) \\ & -n^{-1/3}\left(S_n(tn^{-1/3}) - 1/\mathbb{E}[S]tn^{2/3}\right) \\ & +n^{2/3}\left(F_T(tn^{-1/3}) - 1/\mathbb{E}[S]tn^{-1/3}\right) \end{aligned}$$

where $\quad F_T(t) = \mathbb{P}(T \le t)$

Busy period density



Infinite second moment

New assumption is *power-law service distribution*

$$\mathbb{P}(S_i > x) \sim cx^{-\alpha} \qquad \alpha \in (1,2)$$

$$\begin{cases} \mathbb{E}[S^p] = \infty & p \geq \alpha \\ \mathbb{E}[S^p] < \infty & p < \alpha \end{cases}$$

Infinite second moment

Theorem (Scaling limit for power-law service times)

*Assume **heavy-traffic** and **power-law service distribution**.*

$$n^{-1/(2\alpha-1)} Q_n\left(t n^{(1-\alpha)/(2\alpha-1)}\right) \xrightarrow{d} \phi(W)(t),$$

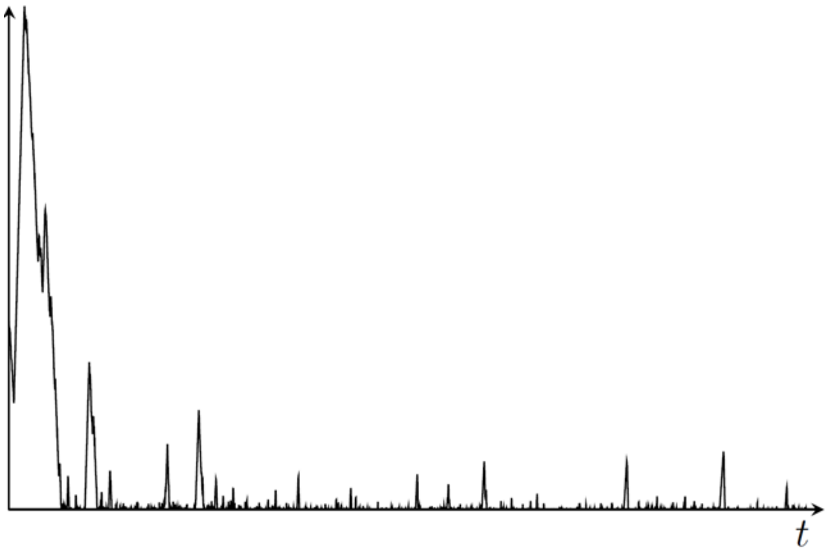
where $\phi(W)$ is the reflected version of

$$W(t) = -\frac{\lambda^2}{2} t^2 + \sigma \boldsymbol{S}(t),$$

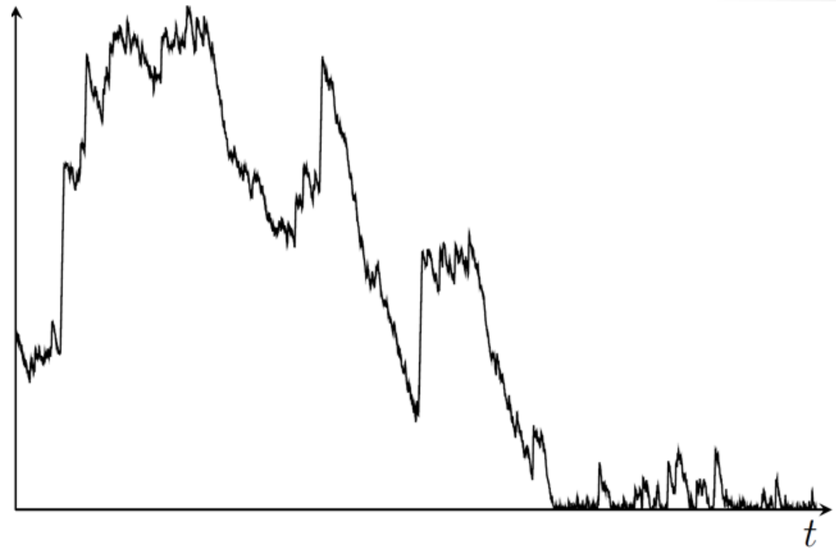
and $\boldsymbol{S}(t)$ is an α-stable process.

Sample paths

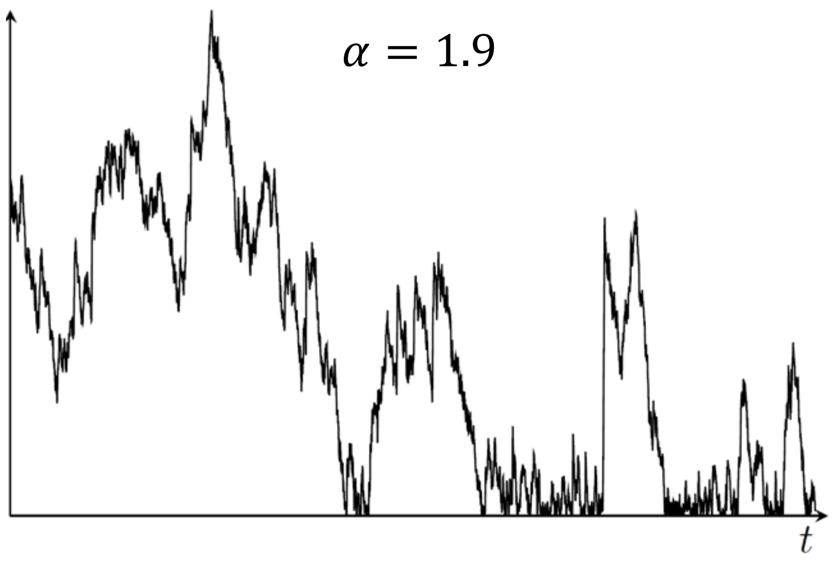
$\alpha = 1.1$



$\alpha = 1.5$



$\alpha = 1.9$



Open problems

- Exact results on $\phi(q + \beta\lambda t - \frac{\lambda^2}{2}t^2 + \sigma B(t))$?
- Exact relationship between q and first busy period?
- Scaling limit for $\text{argmax}_t f_T(t) > 0$?

Thank you!