Dynamics of Large-Scale Spiking Neural Networks

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November, 11 2015

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Stochastic Models

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Stochastic Models

- Communication Networks
  Large Distributed Systems
  Resource Allocation :
  Bandwidth, memory, cores, . . .

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Stochastic Models

- **Communication Networks**
  Large Distributed Systems
  Resource Allocation :
  Bandwidth, memory, cores, . . .

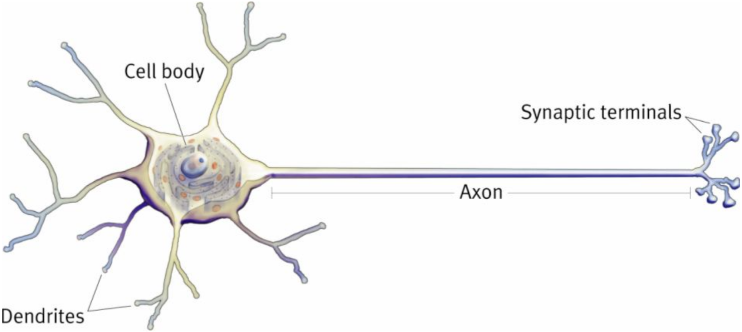
- **Biological Systems**
  Protein Production in Cells,
  Polymerisation Processes of Proteins
  Resource Allocation :
  Ribosomes, Polymerases, ATP, . . .

joint work with Jonathan Touboul

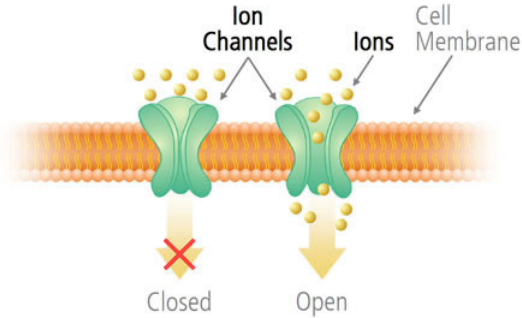
INRIA and Collège de France

Introduction

Neuronal Cell



Ion Channels



Dynamics

Simple Classical Model

Evolution of membrane potential of a neuron :

- **increases with external inputs of other neurons**

Dynamics

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- **decays at some rate**

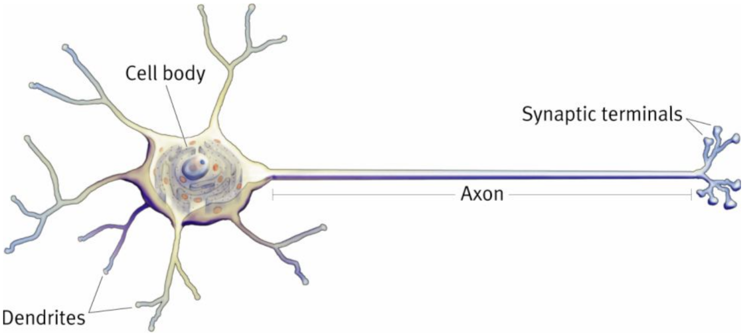
Dynamics

Simple Classical Model

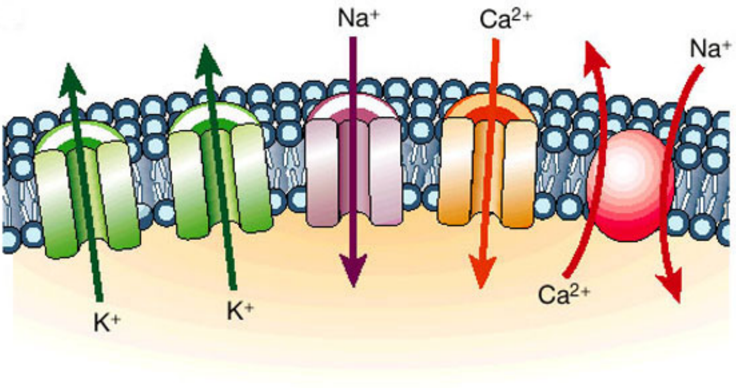
Evolution of membrane potential of a neuron :

- **increases with external inputs of other neurons**
- **decays at some rate**
- **if fixed threshold is hit :**
  - **input sent to neighboring neurons**
  - **returns to some fixed value**

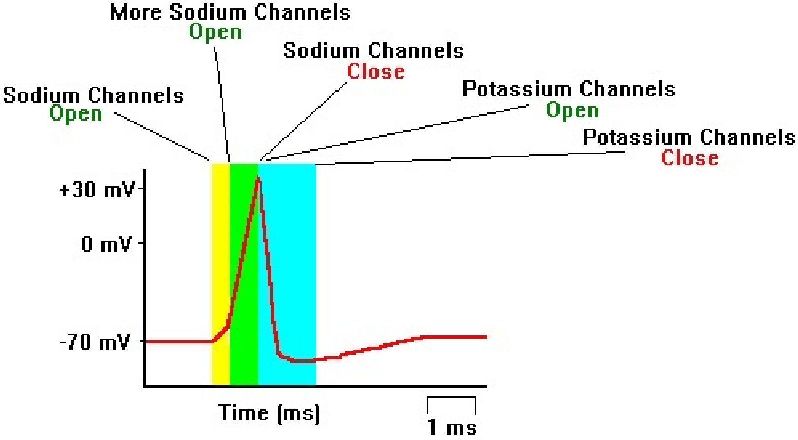
Neuronal Cell



Ion Channels Dynamics



Evolution of Potential



Mathematical Models

Empirical model : Hodgkin-Huxley (1952)

$$\dot{V} = g_{Na} m^3 h (V - V_{Na}) + g_K n^4 (V_K - V) + g_L (V_L - V) + I_e$$

$$\dot{m} = \alpha_m(V)(1 - m) - \beta_m(V) m$$
$$\dot{h} = \alpha_h(V)(1 - m) - \beta_h(V) m$$
$$\dot{n} = \alpha_n(V)(1 - m) - \beta_n(V) m$$

Other Mathematical Models

Empirical models

- FitzHugh-Nagumo
- Wilson-Cowan
- Morris-Lecar
- ...

A classsical Model

Leaky integrate and fire model (LIFM)

Simple ODE for Potential $X_i(t)$ of neuron i

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if $X_i(t) < V_F$

$$dX_i(t) = -X_i(t)dt + \sum_{j \neq i} W_{ji}(t)dD_j(t) + dI_i^e(t),$$

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- $D_j(t)$ nb of spikes of j up to t
- $W_{ji}(\cdot)$ spike from j to i
- I_i^e external input to i

A classsical Model

Leaky integrate and fire model (LIFM)
Simple ODE for Potential $X_i(t)$ of neuron i

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$$dX_i(t) = -X_i(t)dt + \sum_{j \neq i} W_{ji}(t)dD_j(t) + dI_i^e(t),$$

if $X_i(t-) = V_F$ then $dD_i(t) = 1, \quad X_i(t) = V_R$

- $D_j(t)$ nb of spikes of j up to t
- $W_{ji}(\cdot)$ spike from j to i
- I_i^e external input to i

Stochastic Versions of LIFM

Where is the randomness ?

- I_i^e external input to i is Gaussian

$$dI_i^e(t) = dB_i(t)$$

  a common assumption in literature

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- Instants of spikes

  Instead of deterministic threshold V_R

  Randomness in firing instants

  Chichilnisky (2001), Pillow et al. (2005)

  Non-Linear Poisson Model

Literature

- Brunel (2000)
- Càceres, Carrillo and Perthame (2010)
- Rao, Pakdaman (2011)
- Delarue, Inglis, Rubenthaler, Tanré (2012)
- ...

A stochastic LIFM

Main Assumptions

- external input may be zero
- randomness of firing instants
  a neuron i in state $X_i(t)$ fires at rate $b(X_i(t))$

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$$x \mapsto b(x) \nearrow$$

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Model with threshold :

$b(x) = 0$ if $x < V_F$ and $+\infty$ if $x \geq V_F$

A stochastic LIFM

Main Assumptions

- external input may be zero
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  a neuron i in state $X_i(t)$ fires at rate $b(X_i(t))$

  $(D_i(t))$ is a Poisson process with intensity

$$\left(\int_0^t b(X_i(u))\,du, \quad t \geq 0\right)$$

  $x \mapsto b(x) \nearrow$

  Model with threshold :

  $b(x) = 0$ if $x < V_F$ and $+\infty$ if $x \geq V_F$

Outline of the talk

- **Stability Properties of Finite Networks**
- **Mean-Field Limits**
- **Invariant Distributions**

Finite Networks

Stochastic Model

N **nodes**

$$dX_i(t) = -X_i(t-)dt + \sum_{j \neq i} W_{ji}(t-)\mathcal{N}^j_{b(X_j(t-))}(dt)$$
$$- X_i(t-)\mathcal{N}^i_{b(X_i(t-))}(dt)$$

- (W_{ij}) **i.i.d. random variables**
- $\mathcal{N}_{b(y)}(dt)$ **Poisson process with rate** $b(y)$
- $y \mapsto b(y) \nearrow$

Stochastic Model

$$\mathrm{d}X_i(t) = -X_i(t-)\mathrm{d}t + \sum_{j \neq i} W_{ji}(t-)\mathcal{N}^{j}_{b(X_j(t-))}(\mathrm{d}t) - X_i(t-)\mathcal{N}^{i}_{b(X_i(t-))}(\mathrm{d}t)$$

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$$\mathcal{N}_{b(y)} \overset{\text{dist.}}{=} \mathcal{N}_{b(0)} + \mathcal{N}_{b(y)-b(0)}$$

sum of two ind. Poisson processes.

Stochastic Model

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sum of two ind. Poisson processes.

If $b(0)>0$

Proc. $\mathcal{N}_{b(0)}$ can be seen as external input

Firing instants

Initial state $(x_i, 1 \leq i \leq N)$,

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Initial state $(x_i, 1 \leq i \leq N)$,
first firing instant is

$$t_1 = \min(\tau_i, 1 \leq i \leq N),$$

where (τ_i) are independent with distribution

$$\mathbb{P}(\tau_i > t) = \exp\left(-\int_{x_i e^{-t}}^{x_i} \frac{b(s)}{s}\,ds\right)$$

Stability of Finite Networks

Theorem

- If $\mathbf{b(0) > 0}$
  $(\mathsf{X_i}(t), 1 \leq \mathsf{i} \leq \mathsf{N})$ **Ergodic Markov process**
  with non-trivial invariant distribution

Stability of Finite Networks

Theorem

- If $b(0) > 0$
  $(X_i(t), 1 \leq i \leq N)$ Ergodic Markov process
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- If
  $$\int_0^1 \frac{b(s)}{s}\, ds < +\infty$$
  no spike occurs after a finite (random) time

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  $$\lim_{t \to +\infty} (X_i(t), 1 \leq i \leq N) = 0$$
  δ_0 is the unique invariant distribution

The case of constant firing rate

Theorem If $b(\cdot) \equiv \lambda$

- X_i state of node i at equilibrium then

$$\mathbb{E}\left(e^{-\xi X_i}\right) = \int_0^{+\infty} e^{-\lambda(N-1)\int_0^x (1-\widetilde{W}(\xi e^{-u}))du} \lambda e^{-\lambda x} dx$$

where $\widetilde{W}(\xi) = \mathbb{E}(\exp(-\xi W))$

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where $\widetilde{W}(\xi) = \mathbb{E}(\exp(-\xi W))$

Proof

Backward coupling arguments

Representation as a functional of Poisson processes with rate λ and i.i.d. (W_{ij})

Constant rate : Mean-Field Asymptotics

- $\mathbf{b(\cdot) \equiv \lambda}$
- $\mathbf{W_{ij} = V_{ij}/N}$ **with** $\mathbf{(V_{ij})}$ **i.i.d.**

Constant rate : Mean-Field Asymptotics

- $b(\cdot) \equiv \lambda$
- $W_{ij} = V_{ij}/N$ with (V_{ij}) i.i.d.

Proposition At equilibrium

- X_i^N converges in dist. to X^∞ with density

$$\frac{1}{\mathbb{E}(V)}\left(1-\frac{u}{\lambda\mathbb{E}(V)}\right)^{\lambda-1} \quad \text{for } u \in [0, \lambda\mathbb{E}(V))$$

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- Invariant distribution :
  a bounded support in the limit $N \to +\infty$

Large Networks

Mean-Field Asymptotics

Large Networks

Assumptions

- $x \mapsto b(x) \nearrow C^1$-function

$$b'(x) \leq \gamma b(x) + c$$

- $W_{ij} = V_{ij}/N$ with (V_{ij}) i.i.d. with bounded support and

$$3\mathbb{E}(V)\gamma < 1$$

Large Networks : Asymptotics

$$\mathrm{d}X_i^N(t) = -X_i^N(t)\mathrm{d}t - X_i^N(t)\mathcal{N}^i_{b(X_i^N(t))}(\mathrm{d}t) + \frac{1}{N}\sum_{j\neq i} V_{ji}(t)\mathcal{N}^j_{b(X_j^N(t))}(\mathrm{d}t)$$

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Mean-Field Approximation

$$\frac{1}{N}\sum_{j=1}^{N} V_{ji}(t)\mathcal{N}^j_{b(X_j^N(t))}(dt) \sim \mathbb{E}(V)\mathbb{E}(b(X_1^N(t)))\,dt$$

Large Networks : Asymptotics

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McKean-Vlasov Process

Large Networks : main steps

- **Analysis of McKean-Vlasov process**
- **Proof of the Mean-Field Convergence**

Literature : Mean-Field Results

Brownian setting

- **Delarue, Inglis, Rubenthaler, Tanré (2012), . . .**
- **Touboul (2014)**

Literature : Mean-Field Results

Poissonnian Setting

$$\mathrm{d}X_i(t) = -X_i(t)\mathrm{d}t$$
$$+\sum_{j\neq i} W_{ji}(t)\mathcal{N}^{j}_{b(X_j(t))}(\mathrm{d}t) - X_i(t)\mathcal{N}^{i}_{b(X_i(t))}(\mathrm{d}t)$$

Literature : Mean-Field Results

Poissonnian Setting

- Fournier, Löcherbach (2014)
  de Masi, Galves, Löcherbach, Presutti (2014)

Literature : Mean-Field Results

Poissonnian Setting

- **Fournier, Löcherbach (2014)**
  de Masi, Galves, Löcherbach, Presutti (2014)
  with self-centering decay term

$$\mathrm{d}X_i(t) = -\left(X_i(t) - \frac{1}{N}\sum_{j=1}^{N} X_j(t)\right)\mathrm{d}t$$

$$+\sum_{j\neq i} W_{ji}(t)\mathcal{N}^{j}_{b(X_j(t))}(\mathrm{d}t) - X_i(t)\mathcal{N}^{i}_{b(X_i(t))}(\mathrm{d}t)$$

McKean-Vlasov process

Theorem

There exists a unique càdlàg process $(Z(t))$ such that

$$dZ(t) = -Z(t)dt + \mathbb{E}(V)\mathbb{E}(b(Z(t)))dt - Z(t-)\mathcal{N}_{b(Z(t-))}(dt)$$

and $Z(0) = x > 0$

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and $Z(0) = x > 0$

Proof
Iteration scheme + stochastic calculus

Mean-Field Results

Empirical Distribution

$$\langle \Lambda_N(t), \phi \rangle = \frac{1}{N} \sum_{i=1}^{N} \phi(X_i^N(t)).$$

Mean-Field Results

Empirical Distribution

$$\langle \Lambda_N(t), \phi \rangle = \frac{1}{N} \sum_{i=1}^{N} \phi(X_i^N(t)).$$

Fraction of nb of nodes with potential $\leq x$

$$\langle \Lambda_N(t), \mathbb{1}_{[0,x]} \rangle$$

Mean-Field Results

Empirical Distribution

$$\langle \Lambda_N(t), \phi \rangle = \frac{1}{N} \sum_{i=1}^{N} \phi(X_i^N(t)).$$

Mean-Field result :

If $(Z(t))$ is McKean-Vlasov process

$$\lim_{N \to +\infty} \left(\langle \Lambda_N(t), \phi \rangle, t \geq 0 \right) = \left(\mathbb{E}(\phi(Z(t))), t \geq 0 \right)$$

Mean-Field Results

Empirical Distribution

$$\langle \Lambda_N(t), \phi \rangle = \frac{1}{N} \sum_{i=1}^{N} \phi(X_i^N(t)).$$

Mean-Field result :

If $(Z(t))$ is McKean-Vlasov process

$$\lim_{N \to +\infty} \left(\langle \Lambda_N(t), \phi \rangle, t \geq 0 \right) = \left(\mathbb{E}(\phi(Z(t))), t \geq 0 \right)$$

or, for $i \geq 1$

$$\lim_{N \to +\infty} \left(X_i^N(t), t \geq 0 \right) = \left(Z(t), t \geq 0 \right)$$

Mean-Field Results

Technical Difficulties

$$
\begin{aligned}
dX_i^N(t) = -X_i^N(t)dt - X_i^N(t)\mathcal{N}^i_{b(X_i^N(t))}(dt) \\
+ \frac{1}{N}\sum_{j\neq i} V_{ji}(t)\mathcal{N}^j_{b(X_j^N(t))}(dt)
\end{aligned}
$$

$b(\cdot)$ NOT Lipschitz

Mean-Field Results

Technical Difficulties

$$\mathrm{d}X_i^N(t) = -X_i^N(t)\mathrm{d}t - X_i^N(t)\mathcal{N}^i_{b(X_i^N(t))}(\mathrm{d}t) + \frac{1}{N}\sum_{j\neq i} V_{ji}(t)\mathcal{N}^j_{b(X_j^N(t))}(\mathrm{d}t)$$

$b(\cdot)$ NOT Lipschitz $\Rightarrow$ pb in interaction term

Mean-Field without Lipschitz

Non-Lipschitz setting

- Scheutzow (1987)
- Benachour, Roynette, Talay and Vallois (1998)
- Malrieu (2003),
  Cattiaux, Guillin, Malrieu (2006)
- Bolley, Cañizo and Carillo (2011)
- ...

Proof of Mean-Field in our case

- Stochastic calculus with Poisson processes

- **Estimations of the scaled global firing rate**

$$\langle \Lambda_N(t), b \rangle = \frac{1}{N} \sum_{i=1}^{N} b(X_i(t)),$$

Proof of Mean-Field in our case

- Stochastic calculus with Poisson processes

- **Estimations of the scaled global firing rate**

$$\langle \Lambda_N(t), b^p \rangle = \frac{1}{N}\sum_{i=1}^{N} b^p(X_i(t)), \quad p \leq 3$$

Proof of Mean-Field in our case

- Stochastic calculus with Poisson processes

- Estimations of the scaled global firing rate

$$\langle \Lambda_N(t), b^p \rangle = \frac{1}{N} \sum_{i=1}^{N} b^p(X_i(t)), \quad p \leq 3$$

- Compactness/Uniqueness

Invariant States

of McKean Vlasov Processes

McKean-Vlasov process : Equilibrium

Stochastic Differential Equation

$$\mathbf{dZ(t) = -Z(t)dt + \mathbb{E}(V)\mathbb{E}(b(Z(t)))dt \\ - Z(t-)\mathcal{N}_{b(Z(t-))}(dt)}$$

McKean-Vlasov process : Equilibrium

Stochastic Differential Equation

$$\mathbf{dZ(t)} = -\mathbf{Z(t)}\mathrm{d}\mathbf{t} + \mathbb{E}(\mathbf{V})\mathbb{E}(\mathbf{b(Z(t))})\mathrm{d}\mathbf{t} - \mathbf{Z(t-)}\mathcal{N}_{\mathbf{b(Z(t-))}}(\mathrm{d}\mathbf{t})$$

if at equilibrium

$$\mathbb{E}(\mathbf{b(Z(t))}) = \mathbb{E}(\mathbf{b(Z(0))}) \stackrel{\text{def.}}{=} \alpha, \quad \forall \mathbf{t} \geq \mathbf{0}$$

McKean-Vlasov process : Equilibrium

Stochastic Differential Equation

$$\mathbf{dZ(t) = -Z(t)dt + \mathbb{E}(V)\mathbb{E}(b(Z(t)))dt - Z(t-)\mathcal{N}_{b(Z(t-))}(dt)}$$

if at equilibrium

$$\mathbb{E}(b(Z(t))) = \mathbb{E}(b(Z(0))) \stackrel{\text{def.}}{=} \alpha, \quad \forall t \geq 0$$

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McKean-Vlasov process : Equilibrium

$$dZ(t) = -Z(t)\mathrm{d}t + \alpha\mathbb{E}(V)\mathrm{d}t - Z(t-)\mathcal{N}_{b(Z(t-))}(\mathrm{d}t)$$

McKean-Vlasov process : Equilibrium

$$dZ(t) = -Z(t)dt + \alpha\mathbb{E}(V)dt - Z(t-)\mathcal{N}_{b(Z(t-))}(dt)$$

Starting from 0, evolves as

$$\dot{z} = \alpha\mathbb{E}(V) - z$$

until it jumps back to 0 at rate $b(z)$

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Remember that $\alpha = E(b(Z(0)))$

McKean-Vlasov process : Equilibrium

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until it jumps back to 0 at rate $b(z)$

Remember that $\alpha = E(b(Z(0)))$

$b(0) = 0 \Rightarrow \delta_0$ is invariant

McKean-Vlasov process : Equilibrium

$$\dot{\mathbf{z}}_\alpha = \alpha\mathbb{E}(\mathbf{V}) - \mathbf{z}_\alpha \quad \textbf{with } \mathbf{z}(0) = 0$$

McKean-Vlasov process : Equilibrium

Instant of first spike τ

$$\mathbb{P}(\tau \geq t) = \exp\left(- \int_0^t b(z_\alpha(u))\, du \right)$$

McKean-Vlasov process : Equilibrium

Instant of first spike $\boldsymbol{\tau}$

$$\mathbb{P}(\boldsymbol{\tau} \geq \mathbf{t}) = \mathbf{exp}\left(- \int_0^{\mathbf{t}} \mathbf{b}(\mathbf{z}_\alpha(\mathbf{u}))\, \mathbf{du} \right)$$

Invariant distribution $\boldsymbol{\pi}$ **of** $\mathbf{(Z(t))}$

$$\boldsymbol{\pi}_\alpha(\mathbf{f}) = \frac{\mathbf{1}}{\mathbb{E}(\boldsymbol{\tau})} \mathbb{E}\left(\int_0^{\boldsymbol{\tau}} \mathbf{f}(\mathbf{z}_\alpha(\mathbf{u}))\, \mathbf{du} \right)$$

McKean-Vlasov process : Equilibrium

Instant of first spike τ

$$\mathbb{P}(\tau \geq t) = \exp\left(-\int_0^t b(z_\alpha(u))\,du\right)$$

Invariant distribution π **of** $(Z(t))$

$$\pi_\alpha(f) = \frac{1}{\mathbb{E}(\tau)}\mathbb{E}\left(\int_0^\tau f(z_\alpha(u))\,du\right)$$

$\alpha = E_{\pi_\alpha}(b(Z(0))) \Rightarrow$ **fixed point equation**

$$\alpha = \int b(u)\,\pi_\alpha(du)$$

McKean-Vlasov process : Equilibrium

Proposition If

$$\mathbf{C}(\beta) = \int_0^1 \frac{1}{1-\mathbf{x}} \exp\left(-\int_0^{\mathbf{x}} \frac{\mathbf{b}(\beta \mathbb{E}(\mathbf{V})\mathbf{y})}{1-\mathbf{y}}\,\mathrm{d}\mathbf{y} \right) \mathrm{d}\mathbf{x}$$

if there exists $\beta > 0$ satisfying fixed point equation

$$\beta \mathbf{C}(\beta) = 1$$

McKean-Vlasov process : Equilibrium

Proposition If

$$C(\beta) = \int_0^1 \frac{1}{1-x} \exp\left(- \int_0^x \frac{b(\beta \mathbb{E}(V) y)}{1-y} \, dy \right) dx$$

if there exists $\beta > 0$ satisfying fixed point equation

$$\beta C(\beta) = 1$$

then distribution with density

$$u \mapsto \frac{1}{C(\beta)(\beta \mathbb{E}(V) - u)} \exp\left(- \int_0^u \frac{b(s)}{\beta \mathbb{E}(V) - s} \, ds \right)$$

on $[0, \beta \mathbb{E}(V))$ is invariant

Examples

$$\mathbf{b(x) = \lambda x + \delta}$$

If $\delta > 0$

- **unique non-trivial invariant distribution**

Examples

$$\mathbf{b}(\mathbf{x}) = \lambda \mathbf{x}$$

Examples

$\mathbf{b(x) = \lambda x}$

If $\boldsymbol{\lambda \mathbb{E}(V) < 1}$

- $\boldsymbol{\delta_0}$ **is the unique invariant distribution**
- **it is stable**

Examples

$b(x) = \lambda x$

If $\lambda \mathbb{E}(V) < 1$

- δ_0 is the unique invariant distribution
- it is stable

If $\lambda \mathbb{E}(V) > 1$

- $\exists$ non trivial invariant distribution
- inv. dist. δ_0 is not stable

The linear case : a summary

If $b(x) = \lambda x$ and $\overline{W} = \lambda \mathbb{E}(V) > 1$

- For $N \geq 1$

$$\lim_{t \to +\infty} (X_i^N(t), 1 \leq i \leq N) = 0$$

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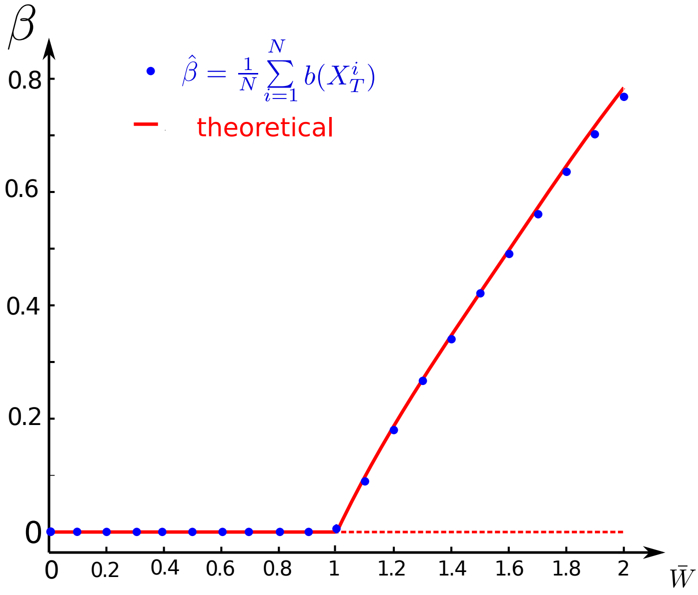
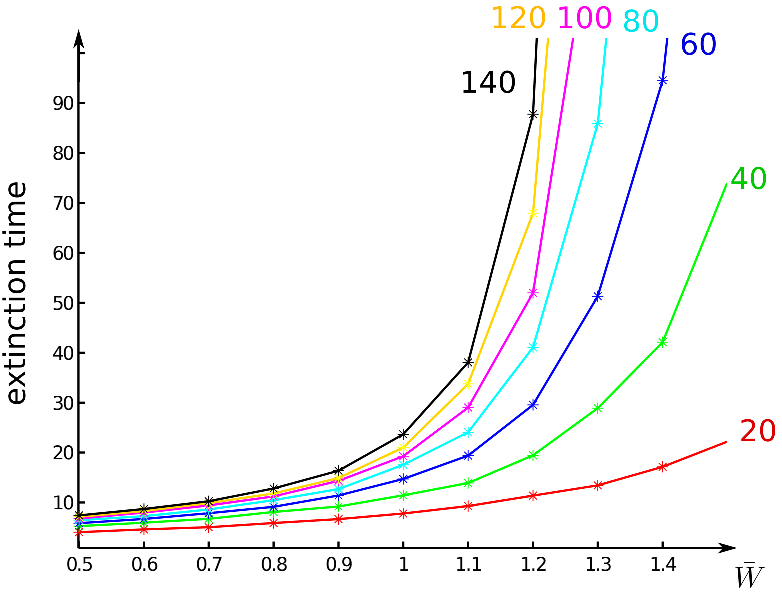
$\Rightarrow$ suggests a non-trivial quasi-stationary distribution before "death"

The linear case : mean firing rate



The linear case : Extinction Time



Examples

$\mathbf{b(x) = \lambda x^{\alpha}}$ **with** $\alpha > 1$

Proposition

There exists some $\rho_c > 0$ **such that**

- **If** $\overline{W} = \lambda \mathbb{E}(V) < \rho_c$
  δ_0 **is the unique invariant distribution**

Examples

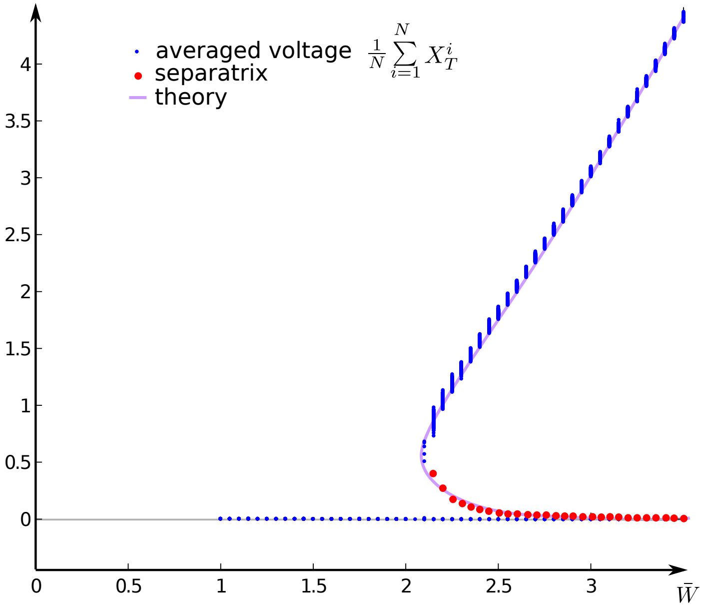
$b(x) = \lambda x^{\alpha}$ with $\alpha > 1$

Proposition

There exists some $\rho_c > 0$ such that

- If $\overline{W} = \lambda \mathbb{E}(V) < \rho_c$
  δ_0 is the unique invariant distribution

- If $\overline{W} > \rho_c$
  There exist at least TWO non-trivial invariant distributions



averaged voltage $\frac{1}{N}\sum_{i=1}^{N} X_T^i$
separatrix
theory
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5
3
$\bar{W}$

Conclusion

Future work

- When $b(0) = 0,$ for a fixed N
  Phase transition for instant of last spike
  as $N \to +\infty$?

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  Phase transition for instant of last spike
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- Stability properties of
  non-trivial invariant distributions

The End