Equilibrium fluctuations for one-dimensional conservative systems

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(INRIA Lille)
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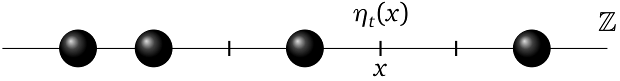
CEMPI CENTRE EUROPÉEN POUR LES MATHÉMATIQUES, LA PHYSIQUE ET LEURS INTERACTIONS

Inria INVENTEURS DU MONDE NUMÉRIQUE

I. Particle systems in 1D



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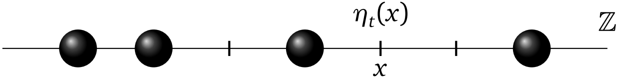

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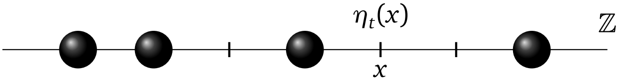

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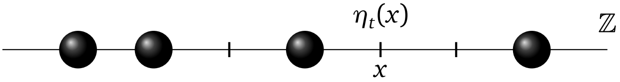

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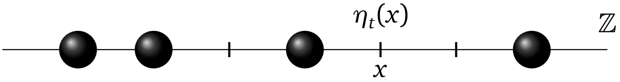

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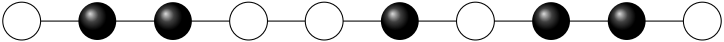
- **Markov process with jumps:**

$$\boxed{\mu(t) = \text{probability law of } \big\{\eta_t(x)\,;\, x \in \mathbb{Z}\big\}}$$

Example: porous media



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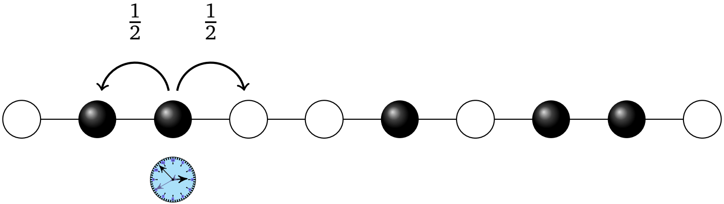

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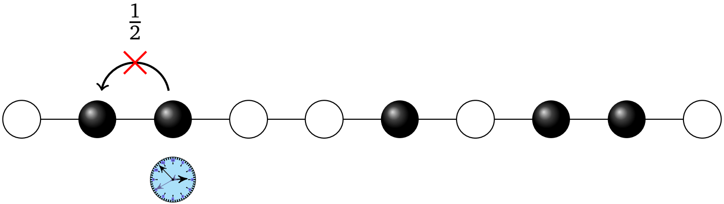

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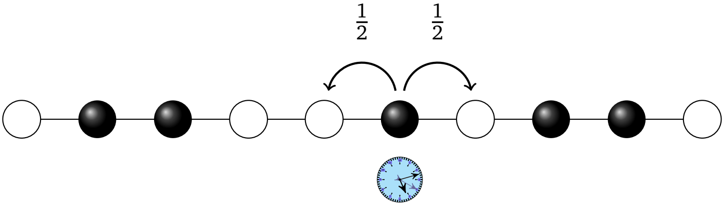

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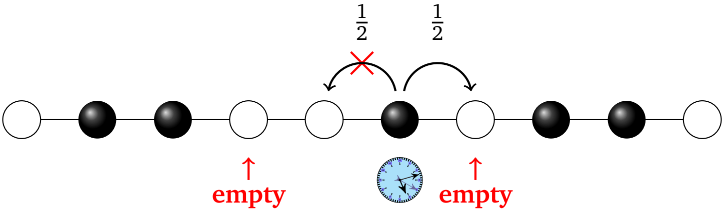

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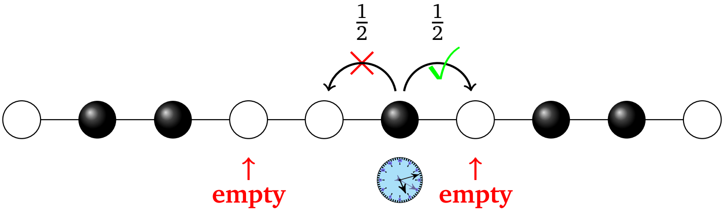

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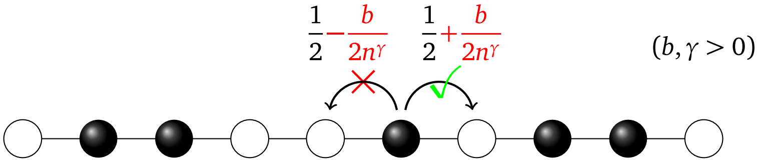

$\boxed{\textbf{Weakly asymmetric}}$

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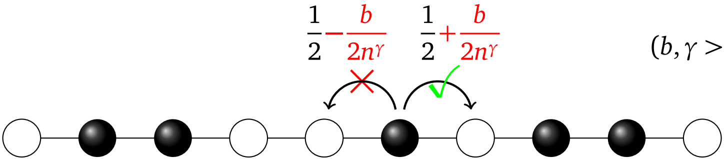

$(b, \gamma > 0)$

- **Rate** to exchange $\eta(x)$ and $\eta(x+1)$

$$\boxed{r(\eta) = \eta(x)\big(1-\eta(x+1)\big)\big(\eta(x-1)+\eta(x+2)\big)\left(\frac{1}{2}+\frac{b}{2n^\gamma}\right)}$$

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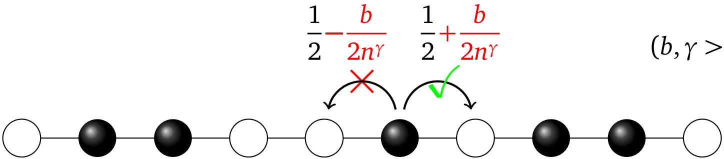

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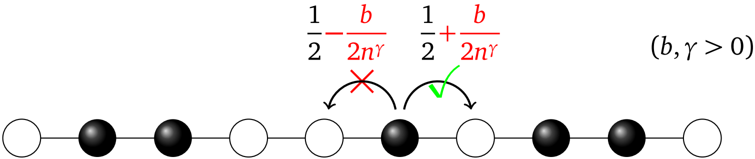

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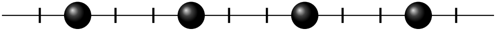
- **Time-invariant measures** = Bernoulli ν_ρ with $\rho \in (0,1)$
- **Generator** of the Markov process:

$$\frac{d\mu}{dt} = \mu\mathcal{L}, \qquad \text{and} \qquad \mathcal{L} = \mathcal{A} + \mathcal{S} \quad \text{in } \mathbb{L}^2(\nu_\rho).$$

Specificities of the dynamics

- **Blocked configuration:**

No possible jump!



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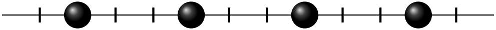

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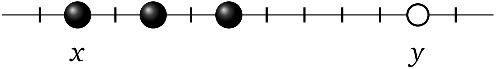

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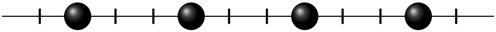

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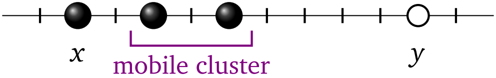

$\boxed{\textit{sequence of moves to bring } x \to y \text{ ??}}$

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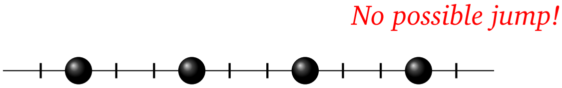

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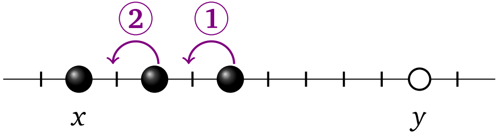

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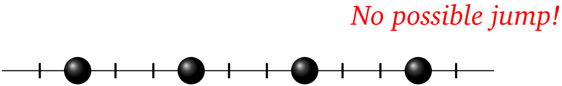

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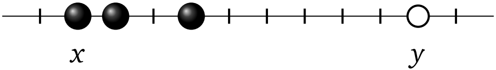

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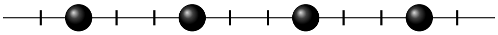

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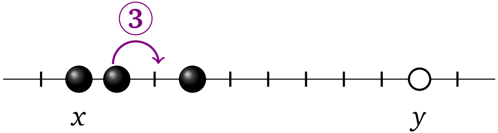

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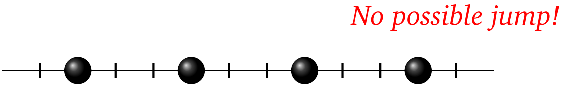

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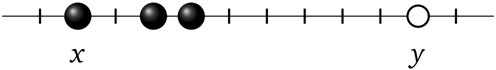

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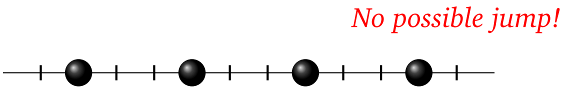

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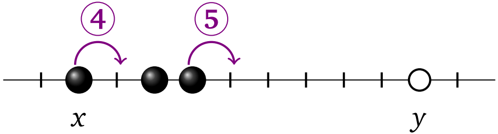

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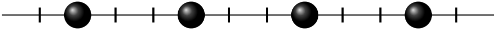

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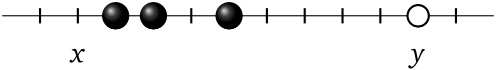

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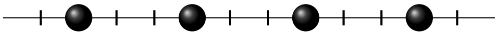

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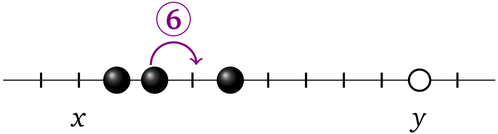

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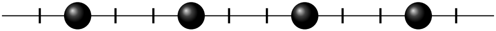

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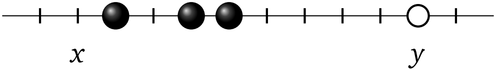
. . . etc.



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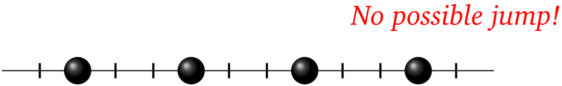

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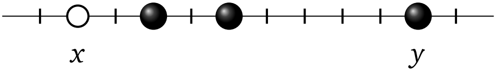
  ▷ Final configuration:



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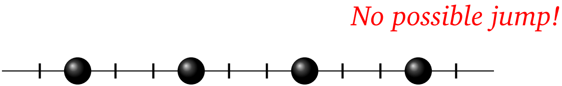

- **Practical tool:** **mobile cluster = pair at distance $\leqslant 2$**

For any x, y that do not belong to the cluster, there exists an allowed path that transports the cluster to the vicinity of x, y and uses it to exchange $\eta(x), \eta(y)$.

Weight of blocked configurations

For $x \in \mathbb{Z}$ and $\ell \in \mathbb{N}$,

$$\Lambda_\ell(x) := \left\{x+1, \ldots, x+\ell\right\} = \text{ box of size } \ell.$$

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Exponentially small weight

$$\nu_\rho\left(\mathcal{B}_\ell(x)\right) \leqslant (1-\rho^2)^{\ell/2}$$

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- **Density fluctuation field:**

$$\boxed{\mathcal{Y}_t^n(\varphi) := \frac{1}{\sqrt{n}} \sum_{x\in\mathbb{Z}} \varphi\Big(\frac{x}{n}\Big)\Big(\eta_{tn^2}(x) - \rho\Big)} \qquad \varphi \in \mathcal{S}(\mathbb{R})$$

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  - ▷ For each **fixed time**

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  - ▷ **Limiting process for $t \in [0,T]$ and $n \to \infty$??**

Previous result when $b = 0$

Ornstein-Uhlenbeck process (OU)

$\left\{\mathcal{Y}_t^n(\cdot)\ ;\ t \in [0,T]\right\}$ converges to the stationary solution of

$$\boxed{d\mathcal{Y}_t = D(\rho)\,\Delta\mathcal{Y}_t\,dt + \sqrt{2\chi(\rho)D(\rho)}\,\nabla(d\mathcal{B}_t)}$$

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is the diffusion coefficient of the **porous media equation**

$$\partial_t \rho(t,u) = \tfrac{1}{2}\Delta\big(\rho^2\big)(t,u), \qquad (t,u) \in \mathbb{R}_+ \times \mathbb{R}.$$

[Gonçalves-Landim-Toninelli 2008]

II. Boltzmann-Gibbs Principles (BG)

- **Remember:**

$$\mathcal{Y}_t^n(\varphi) = \frac{1}{\sqrt{n}} \sum_{x\in\mathbb{Z}} \varphi\Big(\frac{x}{n}\Big)\big(\eta_{tn^2}(x) - \rho\big)$$

$$\Rightarrow \quad \boxed{\mathcal{M}_t^n(\varphi) = \mathcal{Y}_t^n(\varphi) - \mathcal{Y}_0^n(\varphi) - \int_0^t n^2 \mathcal{L}\big(\mathcal{Y}_s^n(\varphi)\big)\, ds}$$

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- **Density current:**
  - ▷ **Conservation law**:

$$\mathcal{L}\left(\eta(x)\right) = -\nabla\big(j_{x,x+1}\big) = j_{x-1,x}(\eta) - j_{x,x+1}(\eta)$$

II. Boltzmann-Gibbs Principles (BG)

- **Remember:**

$$\mathcal{Y}_t^n(\varphi) = \frac{1}{\sqrt{n}} \sum_{x\in\mathbb{Z}} \varphi\Big(\frac{x}{n}\Big)\big(\eta_{tn^2}(x) - \rho\big)$$

$$\Rightarrow \quad \boxed{\mathcal{M}_t^n(\varphi) = \mathcal{Y}_t^n(\varphi) - \mathcal{Y}_0^n(\varphi) - \int_0^t n^2 \mathcal{L}\big(\mathcal{Y}_s^n(\varphi)\big)\, ds}$$

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  ▷ **Integral part**:

$$\int_0^t n^2 \mathcal{L}\big(\mathcal{Y}_s^n(\varphi)\big)\, ds = \frac{n}{\sqrt{n}} \int_0^t \sum_{x\in\mathbb{Z}} \varphi'\Big(\frac{x}{n}\Big)\, j_{x,x+1}(\eta_{sn^2})\, ds$$

- **Decomposition of the current:**

$$\boxed{j_{x,x+1}(\eta) = \underbrace{\nabla(h_x)}_{\text{gradient}} + \frac{b}{2n^\gamma}\left\{\underbrace{\eta(x)\eta(x+1)}_{\text{polynomial}} + \underbrace{\eta(x)\eta(x+1)\eta(x-1) + \dots}_{\text{degree 3}}\right.}$$

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1. GRADIENT PART $\nabla(h_x)$: FIRST-ORDER BG

- **Second integration by part:**

$$\frac{1}{\sqrt{n}}\int_0^t \sum_{x\in\mathbb{Z}} \varphi''\Big(\frac{x}{n}\Big)\underbrace{\{\eta_{sn^2}(x)-\rho\}}_{\text{close the equation}} ds = \int_0^t \mathcal{Y}_s^n(\varphi'')\, ds$$

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Not true for $\gamma = \frac{1}{2}$!!

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Limiting process for $\gamma = \frac{1}{2}$

Stochastic Burgers Equation

$\left\{\mathcal{Y}_t^n(\cdot)\,;\; t \in [0,T]\right\}$ converges to the *unique energy solution* of

$$d\mathcal{Y}_t = D(\rho)\,\Delta\mathcal{Y}_t\,dt - 2b\nabla\left(\mathcal{Y}_t^2\right)dt + \sqrt{2\chi(\rho)D(\rho)}\,\nabla(d\mathcal{B}_t).$$

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  - ▷ **Exclusion processes** with slow bonds ($\longrightarrow$ *Talk of Patricia Gonçalves*)

Thank you for your attention!