Multiresolution based Smoothing

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Contents

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Nonparametric Regression

Situation

- Data (x_i, y_i), $i = 0, \ldots, n$ with $x_i \in \mathbb{R}^k$, $k = 1, 2$ and $y_i \in \mathbb{R}$.
- Model $Y_i = f(x_i) + \sigma\,\varepsilon_i$, ε_i i.i.d. $\mathcal{N}(0, 1)$
- Wanted: Decomposition $y_i = f_n(x_i) + r_n(x_i)$

Smoothing Methods

- kernel estimator
- local polynomials
- smoothing splines
- wavelets
- ...

Different methods to balance **data closeness** and **simplicity**.

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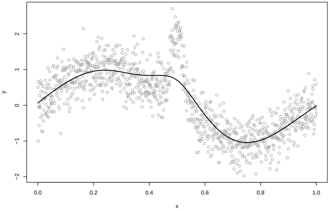

- Data closeness can best be evaluated considering the residuals of an approximation.
- A multiresolution analysis of the residuals can be used to decide whether an approximation is close enough to the data or not.

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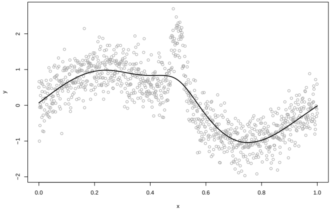

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1-dim. Multiresolution Conditions

White noise MR criterion (Davies and Kovac, 2001)

- residuals $r_n(x_i) = y_i - f_n(x_i)$
- MR coefficients

$$w_I(r) = \frac{1}{\sqrt{|I|}} \sum_{x_i \in I} r_n(x_i), \qquad |I| = \#\{x_i \,|\, x_i \in I\}$$

- MR constraints (modulus of continuity of the Brownian motion)

$$r_n \textit{ look like } \text{Gaussian white noise} \quad :\Leftrightarrow \quad |w_I| \le \sigma_n \sqrt{2.3 \log(n)} \quad \forall I$$

- $\sigma_n = \frac{1.48}{\sqrt{2}} \text{MED}(|y_2 - y_1|, \ldots, |y_n - y_{n-1}|)$ (e.g. $\sigma = 0.4$, $\sigma_n = 0.4056$)

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1-dim. Multiresolution Conditions

- Different sets of Intervals I can be used for the MR conditions.
- A dyadic scheme is usually sufficient.
- MRC on different scales with threshold $\sigma_n\sqrt{2.3\log(n)} \approx 1.62$

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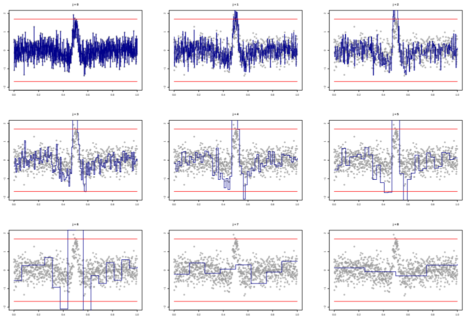
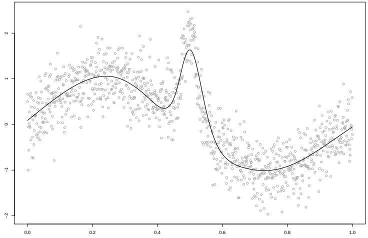


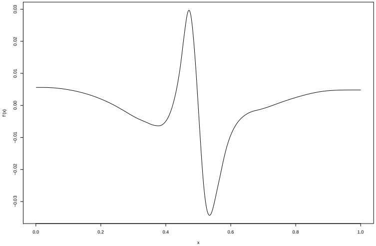
An Example: Locally Weighted Smoothing Splines

Iterative method to determine smoothing parameters λ_i automatically in

$$\min_g \sum_{i=1}^{n} \lambda_i \left(y_i - g(x_i)\right)^2 + \int_a^b \left(g''(x)\right)^2 dx.$$



Approximation



First derivative

Iteration Steps

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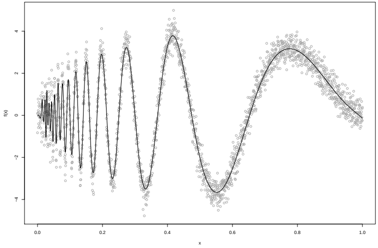

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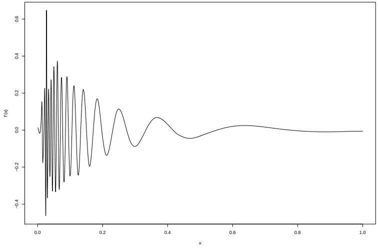

First derivative

Multiresolution based Smoothing

In general: MR conditions can be used to improve or automate smoothing methods. They work as:

- stopping criterion in iterative procedures
    - to determine locally defined smoothing parameters
        - kernel estimators, local polynomials, smoothing splines
        - inhomogeneos diffusion processes (cf. Stichtenoth)
    - for inverse problems (cf. Mildenberger, Weinert)
- linear side conditions in (linear) optimization methods
    - min $TV(f^{(k)})$ (cf. Kovac)
    - linear inverse problems

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An Example: min TV for Inverse Problems

- Model: $y = Kf + \varepsilon$ with ε_i i.i.d. $\mathcal{N}(0, \sigma^2)$
- Method: (find f with minimal total variation s.t. Kf satisfies the MRC)

$$\min_f TV(f^{(k)}) \quad \text{s.t.} \quad MKf \leq b.$$

- Example: K represents the Abel-Integration, $n = 400$, $k = 0$, $\sigma = 0.4$

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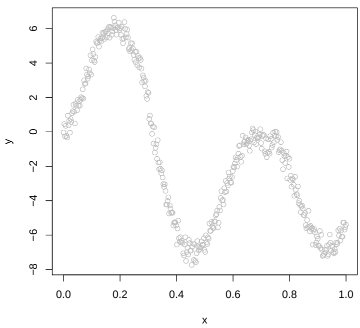

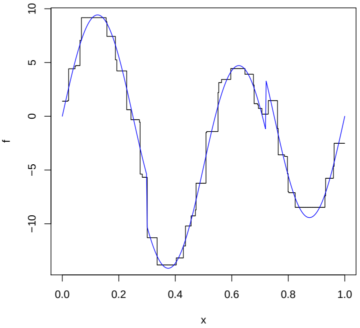

1-dim. Diffusion Process

$$\tilde{f}(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} y(s) \cdot \exp\left(-\frac{(x-s)^2}{4t}\right) ds$$

$\tilde{f}(x,t)$ satisfies the heat equation

$$\frac{\partial}{\partial t}\tilde{f}(x,t) - \frac{\partial^2}{\partial x^2}\tilde{f}(x,t) = 0$$

with the given data as starting points $\tilde{f}(x_i, 0) = y_i$.

$\tilde{f}_t$= diffusion process with homogeneous Neumann conditions stopped at t
t = smoothing parameter

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Local Smoothing Parameter

How can we realize a local smoothing parameter?

$\Longrightarrow$ inhomogeneous diffusion process

$$\frac{\partial}{\partial t}\tilde{f}(x,t) - \mathbf{a}(\mathbf{x}) \left(\frac{\partial^2}{\partial x^2}\tilde{f}(x,t) \right) = 0$$

$a(x)$ = thermal diffusivity at x, new smoothing parameter

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MRC to select local Smoothing Parameter automatically

goal:

find the smoothest among all the estimators $\left\{ \tilde{f}_a;\ a(x) \geq 0 \right\}$ which satisfies the white noise MR criterion

i.e. find the largest $a(x) \geq 0$, so that $\tilde{f}_a$ satisfies the MR criterion

$\Longrightarrow$ iterative algorithm

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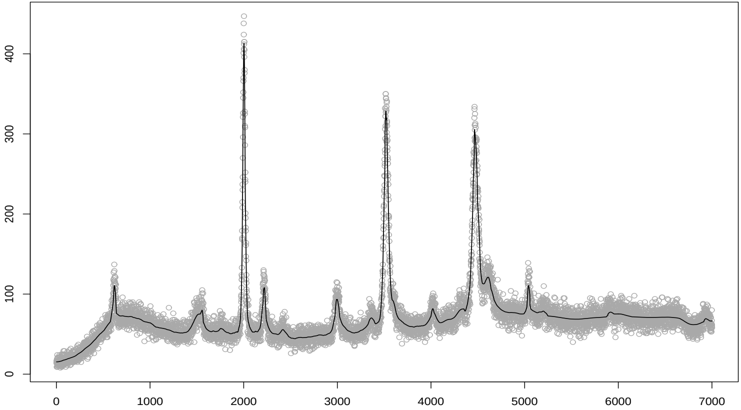
Diffusion Process with locally adapted a



computing time: 0.5 sec (kernel estimator: 163 sec)

2-dim. Diffusion Process

- inhomogeneous heat equation
  $\frac{\partial}{\partial t}\tilde{f}(\mathbf{x},t) - a(\mathbf{x})\,\Delta\tilde{f}(\mathbf{x},t) = 0$

- estimator $\tilde{f}_a$:
  inhomogeneous diffusion process stopped at $t = 1$ with
  - diffusivity $a(\mathbf{x})$
  - starting values $\tilde{f}(x_{ij}, 0) = y_{ij}$
  - homogeneous Neumann conditions $\nabla\tilde{f}(\mathbf{x},t) = 0$ for $\mathbf{x} \in \partial([0,1]x[0,1])$

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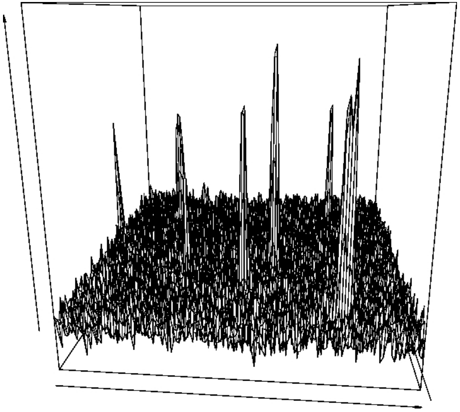
Noisy Dataset



256x256 pixel

2-dim. Multiresolution Conditions

- MR criterion for bivariate data:

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$$|w_P| \leq \sigma_n \sqrt{2.3 \log n} \quad \forall P \in \mathcal{P}$$

- $\sigma_n = \frac{1.48}{2}$ MED$(|y_{i+1,j+1} - y_{i+1,j} + y_{i,j+1} - y_{i,j}|)$
- the partition $\mathcal{P}$ should
  - include different scales
  - be fine enough to detect also small features
  - allow a fast computation of the MR coefficients

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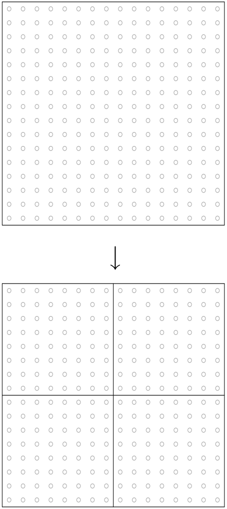
MRC: different Partitions $\mathcal{P}$

different partitions $\mathcal{P}$:

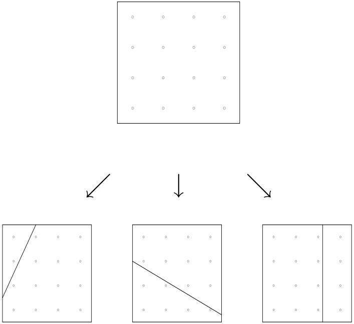
- dyadic squares
- wedges
- dyadic squares on 2 levels

Dyadic Squares and Wedges

dyadic squares

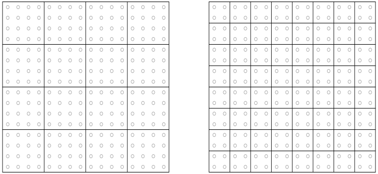


wedges

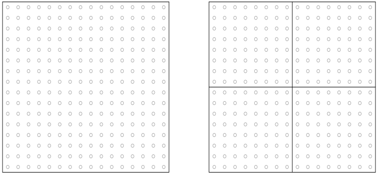


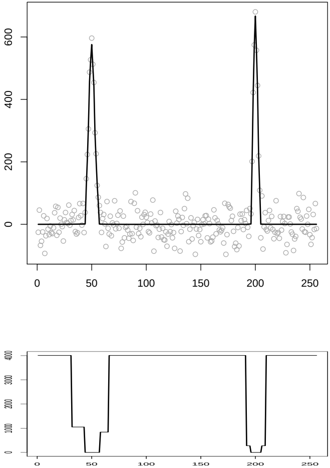
Dyadic Squares on 2 Levels

first level

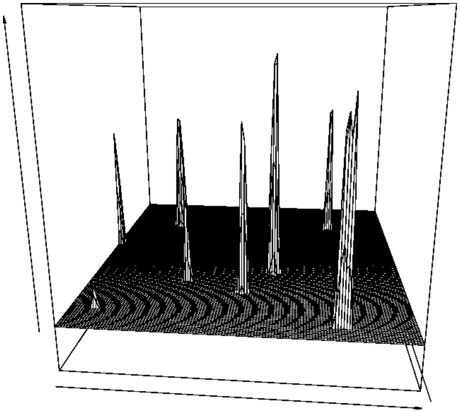


second level





$\tilde{f}_a$ for dyadic Squares on 2 Levels



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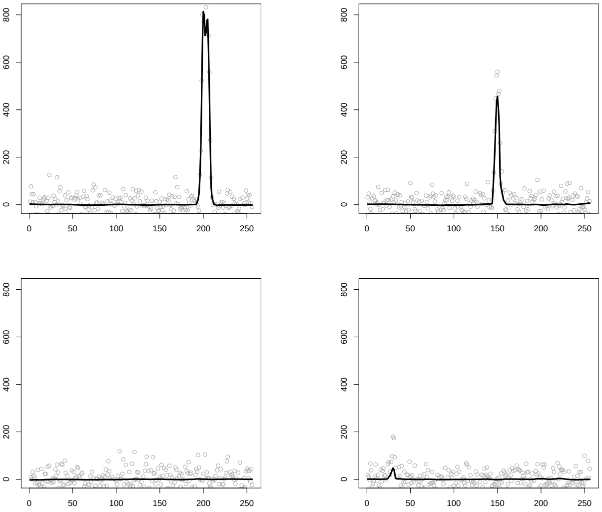

The End

Thank you for your attention!

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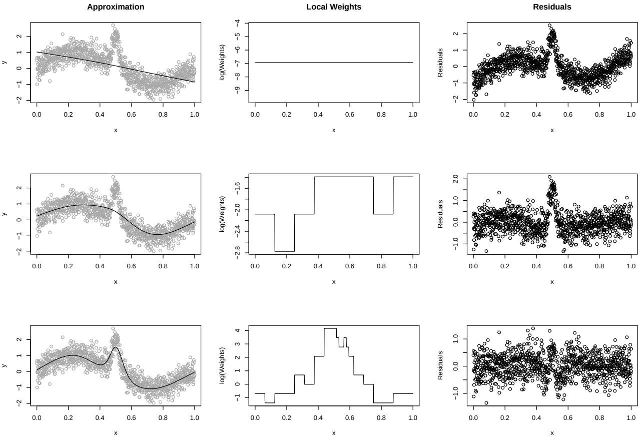
Iteration Steps



Return