Stochastic Fluid Models in Inventory Control Problems

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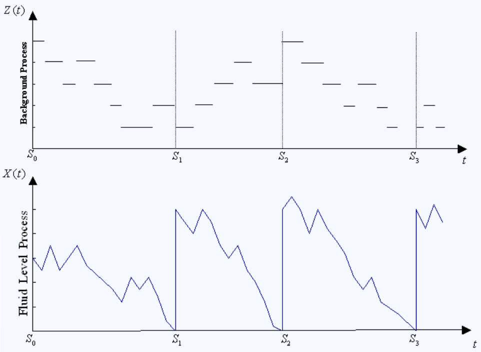
Outline

The Model

- A background process $\{Z(t), t \geq 0\}$. "Almost" an irreducible CTMC, finite state space S, generator Q.
- A fluid level process $\{X(t), t \geq 0\}$.
- When $Z(t) = i$, $X(t)$ changes at rate R_i.
- When the fluid level hits zero in state i, it instantaneously jumps to q, and the background process jumps to j with probability α_{ij}.



Outline

Stability Condition

The bivariate Markov process $\{(X(t), Z(t))\}$ has a limiting distribution

$$G_j(x) := \lim_{t \to \infty} P\{X(t) > x, Z(t) = j\}, \quad x \geq 0, t \geq 0, i \in S,$$

if the expected net production rate

$$\sum_{i \in S} \pi_i R_i < 0,$$

where π_i satisfies

$$\begin{cases} \pi Q = 0 \\ \pi e = 1. \end{cases}$$

Outline

Notation

- The row vector

$$G(x) := [G_1(x), ..., G_n(x)].$$

- Its derivative

$$G'(x) := \left[\frac{dG_1(x)}{dx}, ..., \frac{dG_n(x)}{dx}\right].$$

- Diagonal matrix

$$R := \begin{pmatrix} R_1 & & & \\ & R_2 & & \\ & & \ddots & \\ & & & R_n \end{pmatrix}.$$

Differential Equations

Theorem: Limiting Distribution

The limiting distribution $G(x)$ satisfies

$$\begin{aligned} G'(x)R &= G(x)Q + \beta, && \text{if } x < q, \\ G'(x)R &= G(x)Q, && \text{if } x > q, \end{aligned}$$

where the row vector $\beta := G'(0)RA$. The boundary conditions are given by

$$\begin{aligned} \lim_{x\to\infty} G(x) &= 0, \\ G_j(q^+) &= G_j(q^-), && \forall j : R_j \neq 0, \\ G_j'(0) &= 0, && \forall j : R_j > 0, \\ G(0)\, e &= 1. \end{aligned}$$

Proof.

Differential Equations

$$G'(x)R = G(x)Q, \qquad \text{if } x > q.$$

$$G'(x)R = G(x)Q + \beta, \qquad \text{if } x < q.$$

- $x > q$, standard fluid model,
- $x < q$, $\beta_j = \sum_{i:R_i<0} R_i G_i(0)\alpha_{ij}$

  the rate of transitions into state j due to the jump of the inventory level from 0 to q.

Proof. (continued)

Boundary Conditions

1. $\lim_{x \to \infty} G(x) = 0$.
   - $G(x)$ complementary probability at x.
2. $G_j(q^+) = G_j(q^-), \ \forall j : R_j \neq 0$
   - Continuity at q.
3. $G_j'(0) = 0, \ \forall j : R_j > 0$
   - $\frac{1}{G_j'(0)R_j}$: the expected time between two consecutive visits to the state $(0, j)$.
4. $G(0)e = 1$
   - $G_j(0) = \lim_{t \to \infty} P\{X(t) \in [0, \infty), Z(t) = j\}$.

Proof. (continued)

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Outline

Notation

Let (λ_i, ϕ_i) be an (eigenvalue, eigenvector) pair that solves

$$\phi_i(\lambda_i R - Q) = 0.$$

Fact: There is exactly one eigenvalue 0, and its corresponding eigenvector is π.

$$\Lambda := \begin{pmatrix} 0 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}, \quad \Phi := \begin{pmatrix} \pi \\ \phi_2 \\ \cdots \\ \phi_n \end{pmatrix}.$$

Solution to the Differential Equations

Theorem: Solving the differential equations by solving a linear system

The solution to the differential equations is given by

$$G(x) = \begin{cases} ce^{\Lambda x}\Phi + sx\pi + d & x < q, \\ ae^{\Lambda x}\Phi & x > q, \end{cases}$$

where a, c, d and s are the unique solution to the following system of equations:

$$\begin{aligned} c\Lambda\Phi R + s\pi R + dQ &= 0, \\ (ae^{\Lambda q}\Phi - ce^{\Lambda q}\Phi - sq\pi - d)I_{R\neq 0} &= 0, \\ a_i &= 0, \quad \forall i : \lambda_i \geq 0, \\ \sum_{i=0}^{m} c_i\lambda_i\phi_{ij} + s\pi_j &= 0, \quad \forall j : R_j > 0, \\ (c\Phi + d)e &= 1, \\ d\, e &= 1. \end{aligned}$$

Proof.

1. $c\Lambda\Phi R + s\pi R + dQ = 0$
   - $sx\pi + d$ is a particular solution to $G'(x)R = G(x)Q + \beta$.
2. $(ae^{\Lambda q}\Phi - ce^{\Lambda q}\Phi - sq\pi - d)I_{R\neq 0} = 0$
   - $G_j(q^+) = G_j(q^-), \quad \forall j : R_j \neq 0.$
3. $a_i = 0, \quad \forall i : \lambda_i \geq 0$
   - $\lim_{x\to\infty} G(x) = 0.$
4. $\sum_{i=0}^{m} c_i\lambda_i\phi_{ij} + s\pi_j = 0, \quad \forall j : R_j > 0$
   - $G_j'(0) = 0, \quad \forall j : R_j > 0.$
5. $(c\Phi + d)e = 1$
   - $G(0)e = 1.$
6. $d\,e = 1.$
   - Uniquely determine all the coefficients.

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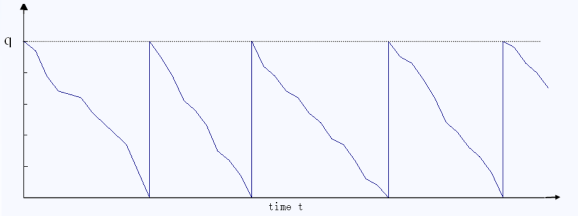
A Special Case: $R < 0$ *and* $A = I$

Uniform distribution in steady state

When the net input rate is non-positive, i.e., $R_i < 0, \forall i$, and $A = I$,

$$G(x) = (1 - x/q)\pi, \qquad x \in [0, q].$$

Remark: In steady state,
(1) $X \sim U[0, q]$; (2) X is independent of Z.



Outline

An Inventory Model

- $X(t)$ = inventory level at time t.
- $Z(t)$ = state of the environment at time t, a CTMC with generator matrix Q.
- R_i = rate at which inventory changes when $Z(t) = i$.
- The assumptions:
  - No back order.
  - Zero leadtimes.
  - Order when inventory reaches zero.
  - Order sizes independent of the environment state.

$\{(X(t), Z(t)), t \geq 0\}$: A special case of the fluid model with jumps where $A = I$.

Cost Model

- The costs:
  - h: holding cost of one unit of product per unit of time;
  - k: fixed set-up cost to place an order;
  - p_1: per unit purchasing cost;
  - p_2: per unit production cost.
- The goal: to compute the optimal q^* that minimizes the long-run total cost per unit of time.

Cost Rates Calculation

Total cost rate $c(q)$ as a function of q

$$c(q) = c_h(q) + c_o(q) + c_p(q),$$

where $c_h(q)$, $c_o(q)$ and $c_p(q)$ are the steady-state holding, ordering, and production cost rates.

Theorem: Costs as functions of q

$$c_h(q) = h\left[(c-a)\tilde{\Lambda}e^{\Lambda q}\Phi + \frac{s}{2}\pi q^2 + (d + c_1\pi)q - c\tilde{\Lambda}\Phi\right]e$$

$$c_o(q) = (k + p_1 q)(c\Lambda\Phi + s\pi)Re$$

$$c_p(q) = p_2(c\Phi + d)\tilde{R}e$$

Notation

$\tilde{\Lambda} = (0, \frac{1}{\lambda_2}, ..., \frac{1}{\lambda_n})$

r_i = production rate at state i

d_i = demand rate at state i

$\tilde{R} = \text{diag}(r_1, r_2, ...r_n)$

$R = \text{diag}(R_1, R_2, ...R_n)$

$R_i = r_i - d_i$

Proof.

1. $c_h(q) =$
   $h\left[(c-a)\tilde{\Lambda}e^{\Lambda q}\Phi + \frac{s}{2}\pi q^2 + (d + c_1\pi)q - c\tilde{\Lambda}\Phi\right] e$
   - $c_h(q) = h \sum_{j \in S} \int_{x=0}^{\infty} G_j(x)dx.$
2. $c_o(q) = (k + p_1 q)(c\Lambda\Phi + s\pi)Re$
   - $c_o(q) = (k + p_1 q) \sum_{j \in S} G_j'(0)R_j.$
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Optimality of EOQ with Stochastic Production and Demand

Theorem: Stochastic EOQ formula

Suppose $\Delta > 0$. The optimal order size q^* that minimizes the total cost rate $c(q)$ is given by:

$$q^* = \sqrt{2k\Delta/h},$$

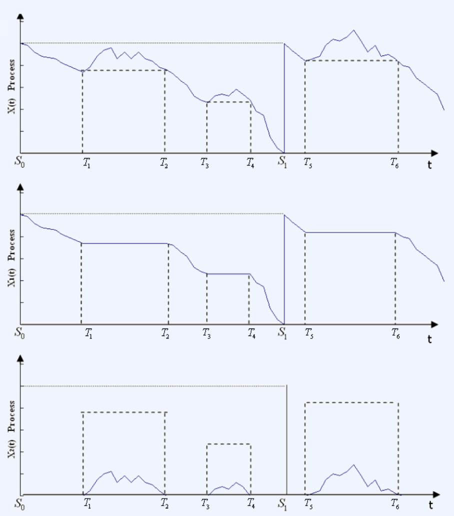
where Δ is the expected *"net" demand* rate

$$\Delta = -\sum_i \pi_i R_i.$$

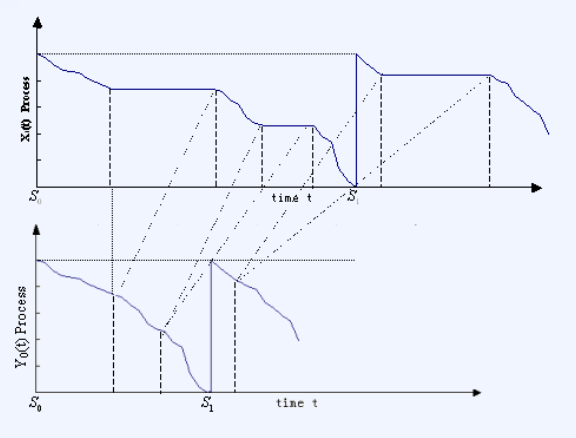
Proof.

1. Decompose $\{X(t)\}$ into $\{X_1(t)\}$ and $\{X_2(t)\}$.
2. In steady state $\{X_1(t)\}$ has uniform distribution.
   - Decompose $\{X_1(t)\}$ into $\{Y_0(t)\}$ and $\{Y_1(t)\}$.
     - $Y_0 \sim U[0, q]$ (The special case).
     - $Y_1 \sim U[0, q]$ (Construct a semi-Markov process).
3. $\{X_2(t)\}$ is independent of q.
4. $c(q) = \frac{hq}{2} + \frac{k\Delta}{q} + C$.
5. $q^* = \sqrt{\frac{2k\Delta}{h}}$.

Decompose $\{X(t)\}$ into $\{X_1(t)\}$ and $\{X_2(t)\}$



Correspondence of $\{X_1(t)\}$ and $\{Y_0(t)\}$



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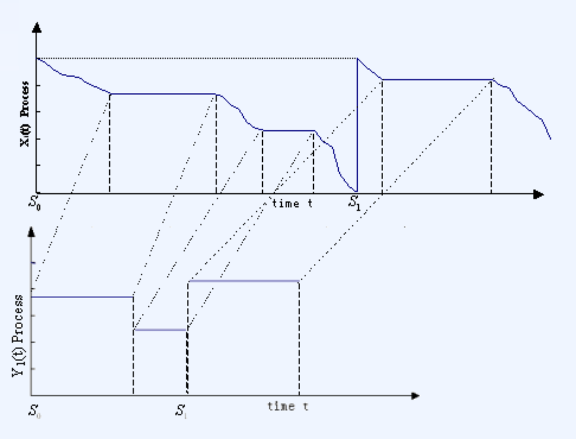
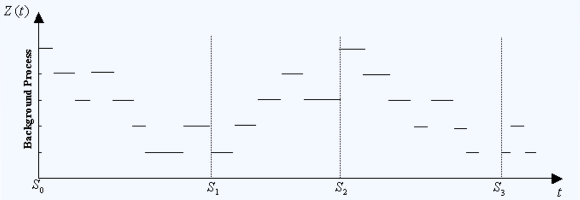


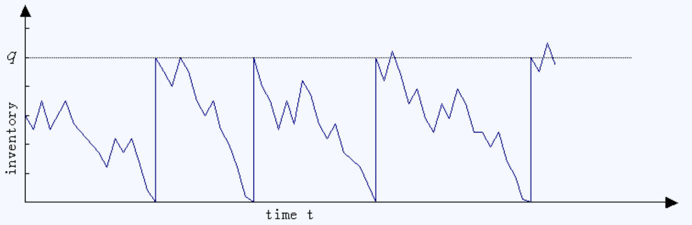
A Numerical Example

- n identical independent machines.
- Lifetime $exp(\mu)$, repair time $exp(\lambda)$.
- $Z(t)$ = number of working machines at time t.
- Background process $\{Z(t), t \geq 0\}$: CTMC.



Inventory Level Process $\{X(t), t \geq 0\}$

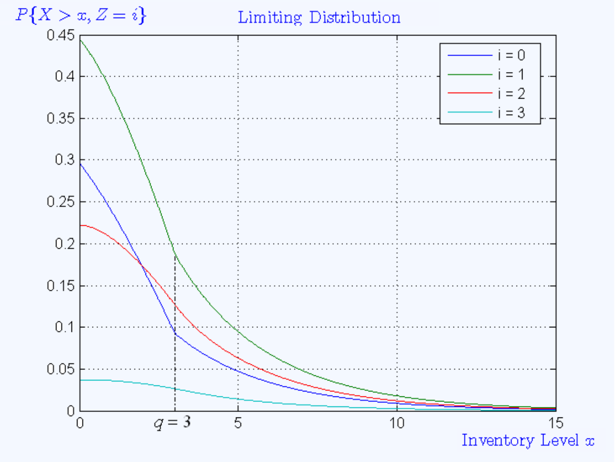
- When there are i working machines,
  - Production rate $i\,r$.
  - Demand rate d (constant).
  - $R_i = i\,r - d$, positive, or negative.
- When inventory level hits 0, an order of size q is placed and arrives instantaneously.



Limiting Distribution

Complementary Cumulative Distribution Function

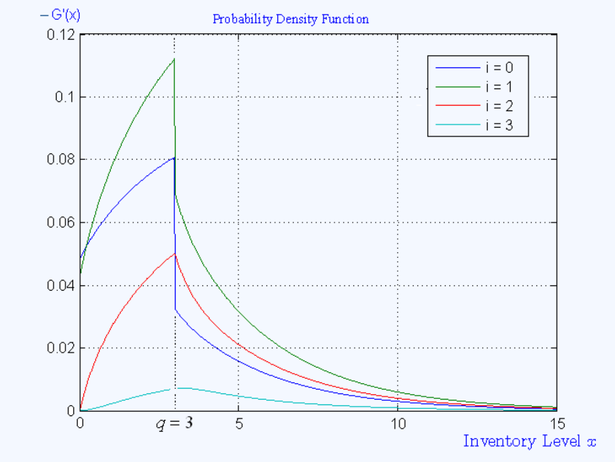
$n = 3, d = 3, r_i = 2.5\,i, \lambda = 1, \mu = 2, q = 3.$



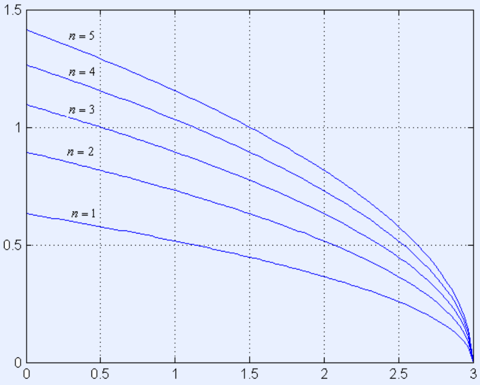
Limiting Distribution

Probability Density Function

$n = 3, d = 3, r_i = 2.5\, i, \lambda = 1, \mu = 2, q = 3.$



$d = n$, $r_i = i\,r$, $\lambda = 1$, $\mu = 2$, $h = 10$, $k = 2$, $p_1 = 8$ and $p_2 = 5$. Let r vary in $(0, 3)$.

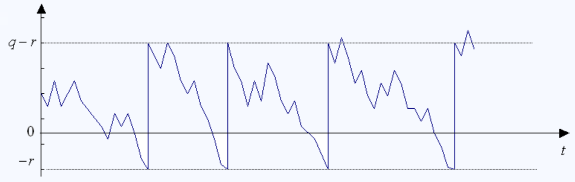


Optimal order size vs. production rate.

- q^* decreases with r .
- q^* reaches zero when r increases to 3.
- For a fixed r, q^* increases with n, but sublinearly.

An Extension: Backlogging Allowed

- Basic model: place an order as soon as inventory reaches 0.
- Extended model:
  - Don't place an order until backorders accumulate to r.
  - b: backlogging cost of one unit of product per unit of time;.



Stochastic EOQ with Backlogging

Theorem: Stochastic EOQ with backlogging

Suppose $A = I$, $\Delta > 0$. Then

$$q^* = \sqrt{\frac{2k(b+h)\Delta}{hb}}$$

$$r^* = \left(\frac{h}{b+h}\right) q^*.$$

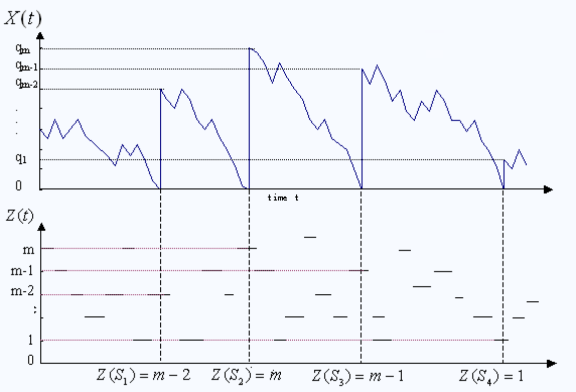
Remark: This is equivalent to the deterministic EOQ formula with backlogging.

Proof. sample path is the same as in basic model except for a shift in the y-axis.

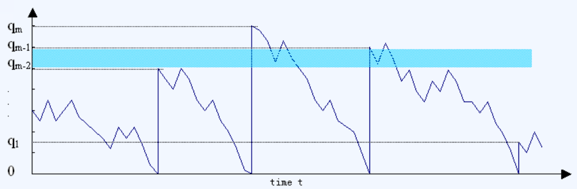
Outline

State-dependent Ordering Policy

- Basic model: one common order size q.
- Extended model: place an order of size q_i if the environment state is i at the time of ordering.



Method 1: Piecewise Function



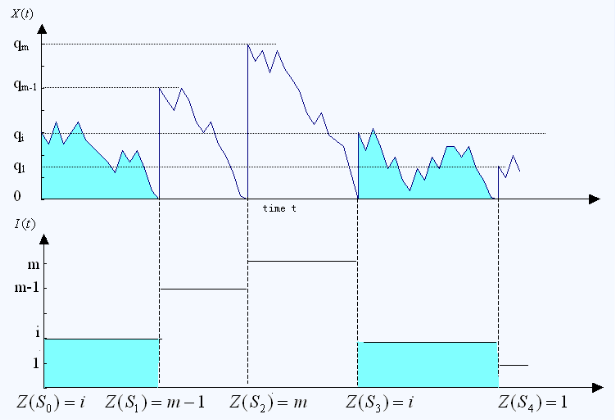
- $$G'(x)R = \begin{cases} G(x)Q, & \text{when } x > q_m \\ G(x)Q + \underline{\beta}^{(i)}, & \text{when } q_{i-1} < x < q_i, \forall i \in S \end{cases}$$

- $$\underline{\beta}^{(i)} = G'(0)RI^{(i)}$$

- $$I^{(i)} = \begin{array}{c} \\ 1 \\ \vdots \\ i-1 \\ i \\ \vdots \\ n \end{array} \begin{array}{c} \qquad\qquad\qquad i-1 \quad i \qquad\qquad \\ \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \end{array}.$$

Method 2: Sample Path Decomposition

the Inventory Level Process and the Cycle Type Process



Sample Path Decomposition Method

- $I(t)$: cycle type at time t.
- $\{(X(t), Z(t)), t \geq 0\}$ restrict to $I(t) = i$ is a special case of the basic fluid model with

$$A = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{ccccc} & & i & & \end{array} \\ \left(\begin{array}{ccccc} 0 & \dots & 1 & \dots & 0 \\ 0 & \dots & 1 & \dots & 0 \\ & & & & \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & \dots & 0 \end{array} \right) \end{array}.$$

- $p^{(i)} = \lim_{t \to \infty} P\{I(t) = i\}$
  steady-state probability that the system is in the i-th type cycle (SMP).

$$G(x) = \sum_{i:R_i<0} G^{(i)}(x) p^{(i)}$$

Advantages of Method 2

1. Complexity in matrix calculation.
   - Method 1: we need to solve m groups of equations simultaneously.
   - Method 2: solve m groups of equations separately.
2. Varying ranges of q_i's.
   - Method 1: q_i varies in $[q_{i-1}, q_{i+1}]$.
   - Method 2: no need to specify ranges.

Fluid Models in Inventory Control

Introduction

Limiting behavior

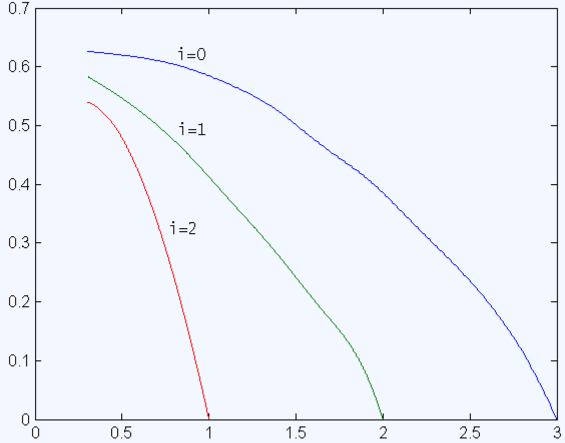
Optimal ordering policies

- Basic EOQ models
- State-dependent ordering policy
- Stochastic leadtimes

Future research

A Machine Shop Example

$n = 2$, $d = 2$, $r_i = i\, r$, $\lambda = 1$, $\mu = 2$, $h = 5$, $k = 0.5$, $p_1 = 5$ and $p_2 = 8$. Let r vary in $(0, 3)$.



Optimal order size vs. production rate.

Outline

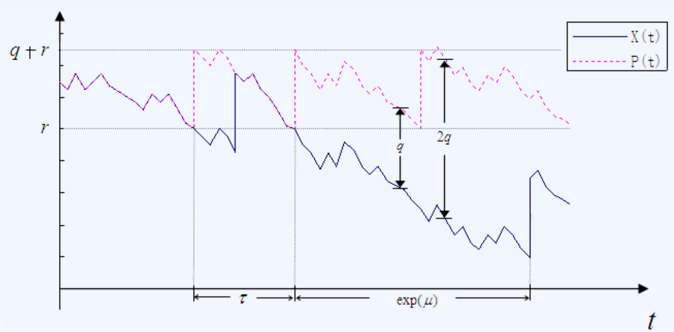
Stochastic Leadtimes

- Basic model: external orders arrive instantaneously.
- Extended model: leadtimes are iid exp(μ) random variables.
- $X(t)$: inventory level at time t;
- $Z(t)$: external environment state at time t;
- $O(t)$: number of outstanding orders at time t;
- $W(t) = (Z(t), O(t))$.
- $P(t)$: inventory position at time t, $P(t) = X(t) + q\,O(t)$.

Policy: External orders of size q are placed when the inventory position reaches r.
Observation: $\{(P(t), W(t)), t \geq 0\}$ is a fluid model with jumps.

Sample Path of

the Inventory Position Process and the Inventory Level Process



One External Supplier with Limited Number of Outstanding Orders

- Let $G^{(i)}(x) = \lim_{t\to\infty} P\{X(t) > x, O(t) = i\}$
- Then $G(x) = G^{(i)}(x - i\,q), \forall i \in \{0, 1, \ldots, N\}$
  - $G^{(0)'}(x)R = G^{(0)}(x)Q + \mu G^{(1)}(x)$
  - $G^{(i)'}(x)R = G^{(i)}(x)Q + \mu G^{(i+1)}(x) + G^{(i-1)'}(0)R$
    $i = 1, 2, \ldots, N-1$
  - $G^{(N)'}(x)R = G^{(N)}(x)Q + \mu G^{(N)}(x) + G^{(N-1)'}(0)R$

One External Supplier with Limited Number of Outstanding Orders

- $\bar{G}(x) := [G^{(0)}(x), G^{(1)}(x), \ldots, G^{(N)}(x)]$

- $$\bar{Q} := \begin{pmatrix} Q & & & \\ \mu I & Q - \mu I & & \\ & \ddots & \ddots & \\ & & \mu I & Q - \mu I \end{pmatrix}$$

- $$\bar{A} := \begin{pmatrix} 0 & I & & & \\ & 0 & I & & \\ & & \ddots & \ddots & \\ & & & 0 & I \\ & & & & I \end{pmatrix}$$

Infinite External Suppliers

- $G^{(0)'}(x)R = G^{(0)}(x)Q + \mu G^{(1)}(x)$
- $G^{(i)'}(x)R = G^{(i)}(x)Q + (i+1)\mu G^{(i+1)}(x) + G^{(i-1)'}(0)R$
  $i = 1, 2, \ldots$
- $$\bar{Q} := \begin{pmatrix} Q & & & & \cdots \\ \mu I & Q-\mu I & & & \cdots \\ & 2\mu I & Q-2\mu I & & \cdots \\ & & 3\mu I & Q-3\mu I & \cdots \\ & & & & \ddots \end{pmatrix}$$
- $$\bar{A} := \begin{pmatrix} 0 & I & & & \cdots \\ & 0 & I & & \cdots \\ & & 0 & I & \cdots \\ & & & & \ddots \end{pmatrix}$$

Future Research

State-Dependent Cost Rates

- Basic model: state-independent linear cost rates.
- Extended model: cost rates depend on the environmental process.

SMP Background Process

- Basic model: CTMC background process.
- Extended model: SMP background process.

Thank you!

Questions?

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