Moments of the maximum of the Gaussian random walk

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Model definition

Consider the partial sums

$$S_n = X_1 + \ldots + X_n$$

with

$$X_1, X_2, \ldots \text{ i.i.d.}, \quad X_i \sim N(-\beta, 1), \quad \beta > 0$$

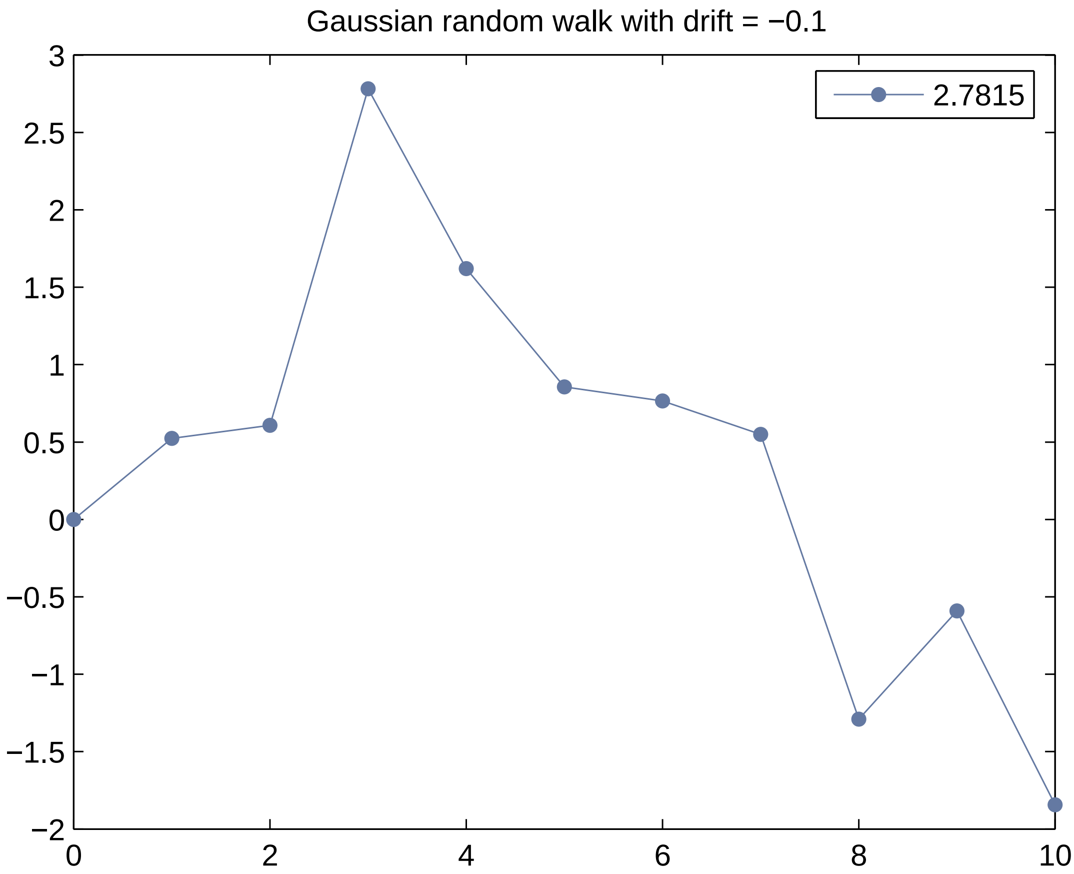
The **Gaussian random walk** is then defined as

$$\{S_n : n \geq 0\} \quad ; \quad S_0 = 0$$

We are interested in the all-time maximum

$$M_\beta = \max\{S_n : n \geq 0\}$$

We consider the moments $\mathbb{E}M_\beta^k$ and the role of the drift β


Gaussian random walk with drift = −0.1
2.7815
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
0
2
4
6
8
10

Motivation for studying $\mathbb{E}M_\beta^k$

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Queues in conventional heavy traffic

Scaled version of the queue behaves **approximately** as M_β

- *G/G/1 queue*: Kingman (1962,1965)

Queues in Halfin-Whitt scaling

In the limit, the queue behaves **exactly** as M_β

- *G/M/N queue*: Halfin-Whitt (1981)
- *G/D/N queue*: Jelenković-Mandelbaum-Momčilović (2004)
- *Call centers*: Borst-Mandelbaum-Reiman (2004)

Equidistant sampling of Brownian motion

- *Testing for drift*: Chernoff (1965)
- *Corrected diffusion approximations*: Siegmund (1979,1985)
- *Option pricing*: Broadie-Glasserman-Kou (1997)

Queues and Halfin-Whitt scaling

Example

$$W_{\lambda,n+1} = (W_{\lambda,n} + A_{\lambda,n} - s)^+$$

where $A_{\lambda,n}$ is Poisson distributed with mean $\lambda < s$. Let

$$W_\lambda = \lim_{n\to\infty} W_{\lambda,n}$$

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Square-root staffing

$$s = \lambda + \beta\sqrt{\lambda}$$

and

$$W = \lim_{\lambda\to\infty} \frac{1}{\sqrt{\lambda}} W_\lambda$$

gives

$$W \stackrel{d}{=} (W + N(-\beta, 1))^+ \stackrel{d}{=} \max\{S_n : n \geq 0\} =: M_\beta$$

Outline

1. Exact expression for $\mathbb{E}M_\beta$

2. Exact expressions for all moments $\mathbb{E}M_\beta^k$

3. Equidistant sampling of Brownian motion

"Despite the apparent simplicity of the problem, there does not seem to be an explicit expression even for $\mathbb{E}M_\beta$..., but it is possible to give quite sharp inequalities and asymptotic results for small β."

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Kingman showed that for $\beta \downarrow 0$

$$\mathbb{E}M_\beta = \frac{1}{2\beta} - c + \mathcal{O}(\beta)$$

where

$$c = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{\sqrt{n}(\sqrt{n}+\sqrt{n-1})^2}} \approx 0.58$$

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$$c = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{\sqrt{n}(\sqrt{n}+\sqrt{n-1})^2}} = -\frac{\zeta(\frac{1}{2})}{\sqrt{2\pi}}$$

Riemann zeta function

The Riemann zeta function ζ is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \mathrm{Re}\, s > 1$$

This definition is extended by analytic continuation to the entire complex plane except $s = 1$, where ζ has a simple pole.

Calculation of ζ is routine.

Theorem

For $0 < \beta < 2\sqrt{\pi}$ we have

$$\mathbb{E}M_\beta = \frac{1}{2\beta} + \frac{\zeta(\frac{1}{2})}{\sqrt{2\pi}} + \frac{1}{4}\beta + \frac{\beta^2}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \frac{\zeta(-\frac{1}{2} - r)}{r!(2r+1)(2r+2)} \left(\frac{-\beta^2}{2}\right)^r$$

Proof

For $M_\beta = \max\{S_n : n \geq 0\}$ we have from **Spitzer's identity**

$$J_1(\beta) \ := \ \mathbb{E}M_\beta \ = \ \sum_{n=1}^{\infty} \frac{1}{n}\mathbb{E}S_n^+$$

From $S_n \sim N(-\beta n, n)$ we get

$$J_1(\beta) = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{2\pi}} \int_{\beta\sqrt{n}}^{\infty} (\sqrt{n}x - \beta n) e^{-x^2/2} \mathrm{d}x$$

Upon changing variables according to $y = x/\sqrt{n}$ ...

Proof (cont'd)

... we can write $J_1(\beta)$ as

$$J_1(\beta) = \sum_{n=1}^{\infty} \frac{n^{1/2}}{\sqrt{2\pi}} \int_\beta^\infty (y-\beta) e^{-\frac{1}{2}ny^2} \mathrm{d}y$$

Differentiating twice w.r.t. β yields *

$$J_1^{(2)}(\beta) = \sum_{n=1}^{\infty} \frac{n^{1/2}}{\sqrt{2\pi}} e^{-\frac{1}{2}n\beta^2}$$

using

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\Big[\int_\beta^\infty f(y,\beta)\mathrm{d}y\Big] = -f(\beta,\beta) + \int_\beta^\infty \frac{\partial f}{\partial \beta}(y,\beta)\mathrm{d}y$$

* Idea of **Chang-Peres (1997)** in their analysis of $\mathbb{P}(M_\beta = 0)$

Lerch's transcendent

Lerch's transcendent Φ is defined as the analytic continuation of the series

$$\Phi(z, s, v) = \sum_{n=0}^{\infty} (v+n)^{-s} z^n$$

Note that $\zeta(s) = \Phi(1, s, 1)$.

Lemma

For $|\ln z| < 2\pi$, $s \neq 1, 2, 3, \ldots$, and $v \neq 0, -1, -2, \ldots$ we have

$$\Phi(z, s, v) = \frac{\Gamma(1-s)}{z^v} (\ln 1/z)^{s-1} + z^{-v} \sum_{r=0}^{\infty} \zeta(s-r, v) \frac{(\ln z)^r}{r!}$$

with $\zeta(s, v) := \Phi(1, s, v)$ the Hurwitz zeta function.

Proof Bateman §1.11(8)

Proof (cont'd)

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$$J_1^{(2)}(\beta) = \sum_{n=1}^{\infty} \frac{n^{1/2}}{\sqrt{2\pi}} e^{-\frac{1}{2}n\beta^2}$$
$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\beta^2} \Phi(z = e^{-\frac{1}{2}\beta^2}, s = -\tfrac{1}{2}, v = 1)$$

Using Bateman's result we get for $0 < \beta < 2\sqrt{\pi}$

$$J_1^{(2)}(\beta) - \frac{\Gamma(\frac{3}{2})}{\sqrt{2\pi}2^{3/2}} \frac{1}{\beta^3} = \frac{1}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \zeta(-r - \tfrac{1}{2}) \frac{(-\frac{1}{2}\beta^2)^r}{r!}$$

The rhs is a well-behaved function of β. Integrating twice yields

$$J_1(\beta) - \frac{1}{2\beta} = L_0 + L_1\beta + \frac{1}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \frac{\zeta(-r - \frac{1}{2})(-\frac{1}{2})^r \beta^{2r+2}}{r!(2r+1)(2r+2)}$$

where L_1 and L_0 are integration constants

Proof (cont'd)

These can be found using **Euler-Maclaurin summation**, among other things.

$$L_0 = \frac{\zeta(1/2)}{\sqrt{2\pi}} \quad , \quad L_1 = \frac{1}{4}$$

This in total gives

$$\mathbb{E}M_\beta = \frac{1}{2\beta} + \frac{\zeta(\frac{1}{2})}{\sqrt{2\pi}} + \frac{1}{4}\beta + \frac{\beta^2}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \frac{\zeta(-\frac{1}{2} - r)}{r!(2r+1)(2r+2)} \left(\frac{-\beta^2}{2}\right)^r$$

Cumulants

The k-th cumulant of a random variable A is defined as the k-th derivative of $\log \mathbb{E}e^{sA}$ evaluated at $s = 0$.

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Spitzer's identity leads to

$$\mathbb{E}(e^{sM_\beta}) = \exp\left\{ \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}(e^{sS_n^+} - 1) \right\}, \quad \operatorname{Re} s \leq 0$$

Thus

$$\log \mathbb{E}(e^{sM_\beta}) \;=\; \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} (sx + \tfrac{1}{2}s^2x^2 + \ldots) f_{S_n^+}(x) \mathrm{d}x$$

with $f_{S_n^+}$ the density function of S_n^+, and

$$\frac{\mathrm{d}^k}{(\mathrm{d}s)^k} \log \mathbb{E}(e^{sM_\beta})\Big|_{s=0} \;=\; \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}((S_n^+)^k) \;=:\; J_k(\beta), \quad k = 1, 2, \ldots$$

Recall

$$J_1(\beta) = \mathbb{E}M_\beta \quad ; \quad J_2(\beta) = \mathrm{Var}M_\beta \quad ; \quad J_3(\beta) = \mathbb{E}(M_\beta - \mathbb{E}M_\beta)^3$$

and all moments of M_β follow from the cumulants and vice versa.

Using $S_n \sim N(-\beta n, n)$, it follows that

$$J_k(\beta) = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{2\pi}} \int_{\beta\sqrt{n}}^{\infty} (\sqrt{n}x - \beta n)^k e^{-x^2/2} \mathrm{d}x$$

It obviously holds that

$$J_0(\beta) = \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}(S_n > 0)$$

From Spitzer's identity we then know that

$$J_0(\beta) = -\ln \mathbb{P}(M_\beta = 0)$$

Theorem

Assume $0 < \beta < 2\sqrt{\pi}$. There holds

$$J_0(\beta) \stackrel{\star}{=} -\ln\beta - \frac{\ln 2}{2} - \frac{\zeta(\frac{1}{2})}{\sqrt{2\pi}}\beta - \frac{1}{\sqrt{2\pi}}\sum_{r=1}^{\infty}\frac{\zeta(-r+\frac{1}{2})(-\frac{1}{2})^r\beta^{2r+1}}{r!(2r+1)}$$

and for $k = 1, 2, \ldots$

$$\begin{aligned} J_k(\beta) &= \frac{(k-1)!}{2^k}\beta^{-k} + \sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^j\Gamma(\frac{k-j+1}{2})}{\sqrt{2\pi}}\zeta(-\tfrac{1}{2}k - \tfrac{1}{2}j + 1)2^{\frac{k-j-1}{2}}\beta^j \\ &+ \frac{(-1)^{k+1}k!}{\sqrt{2\pi}}\sum_{r=0}^{\infty}\frac{\zeta(-k-r+\frac{1}{2})(-\frac{1}{2})^r\beta^{2r+k+1}}{r!(2r+1)\cdots(2r+k+1)} \end{aligned}$$

* Result of **Chang-Peres (1997)**

Proof

1. Use Spitzer's identity and normality of S_n to obtain

$$J_k(\beta) = \sum_{n=1}^{\infty} \frac{n^{k-1/2}}{\sqrt{2\pi}} \int_{\beta}^{\infty} (y-\beta)^k e^{-\frac{1}{2}ny^2} \mathrm{d}y$$

2. Differentiate $k+1$ times under the integral sign

$$J_k^{(k+1)}(\beta) = (-1)^{k+1} k! \sum_{n=1}^{\infty} \frac{n^{k-1/2}}{\sqrt{2\pi}} e^{-\frac{1}{2}n\beta^2}$$

3. Rewrite the expression in terms of **Lerch's transcendent**
4. Apply **Bateman's result**
5. Integrate $k+1$ times and find the $k+1$ integration constants using **Euler-Maclaurin summation**

Equidistant sampling of Brownian motion

Let $\{B_t : t \geq 0\}$ be a BM with $B_0 = 0$, drift $-\beta$ and variance 1, so that

$$B_t = -\beta t + W_t,$$

where $\{W_t : t \geq 0\}$ is a Wiener process. Let

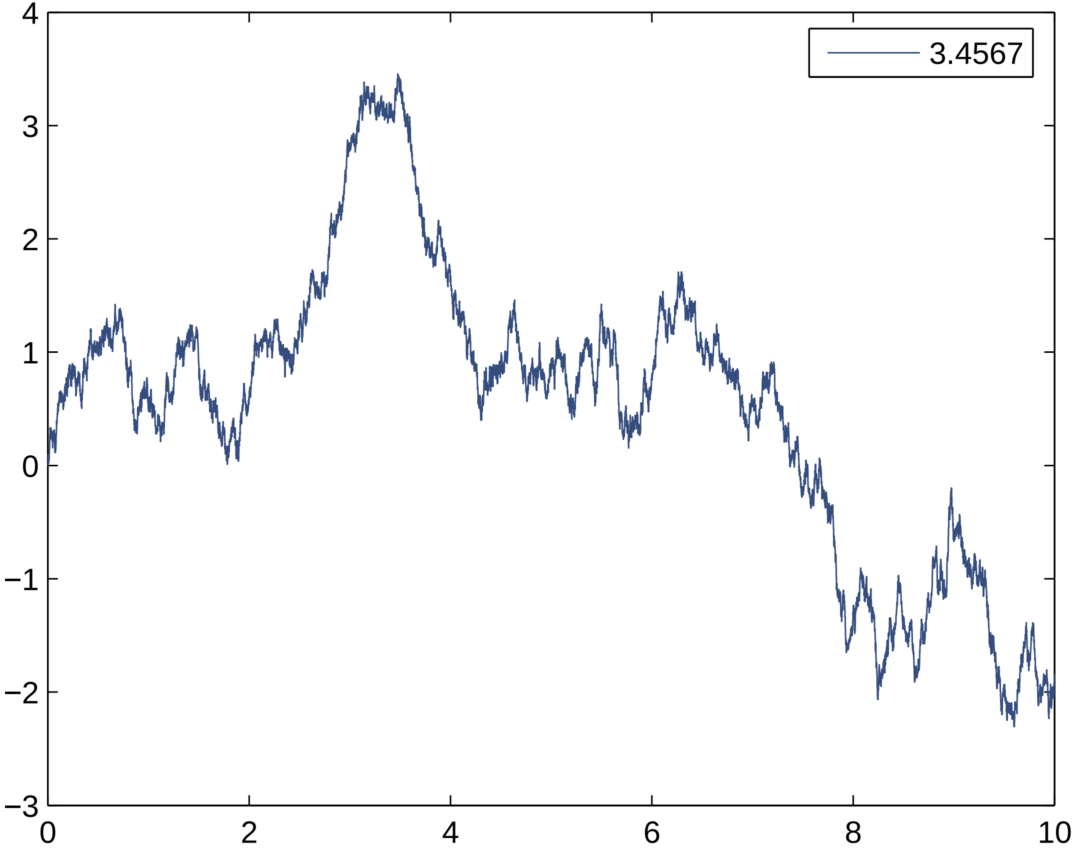
$$\tilde{M} = \max\{B_t : t \geq 0\}$$

It is well known that $\mathbb{P}(\tilde{M} \geq x) = e^{-2\beta x}$ and so the k-th cumulant of $\tilde{M}$ equals

$$\frac{(k-1)!}{2^k}\beta^{-k}$$

The GRW results from (equidistantly) sampling this BM, and by increasing the sampling frequency the GRW will converge to the BM.

Equidistant sampling of BM (drift=−0.1, variance=1).



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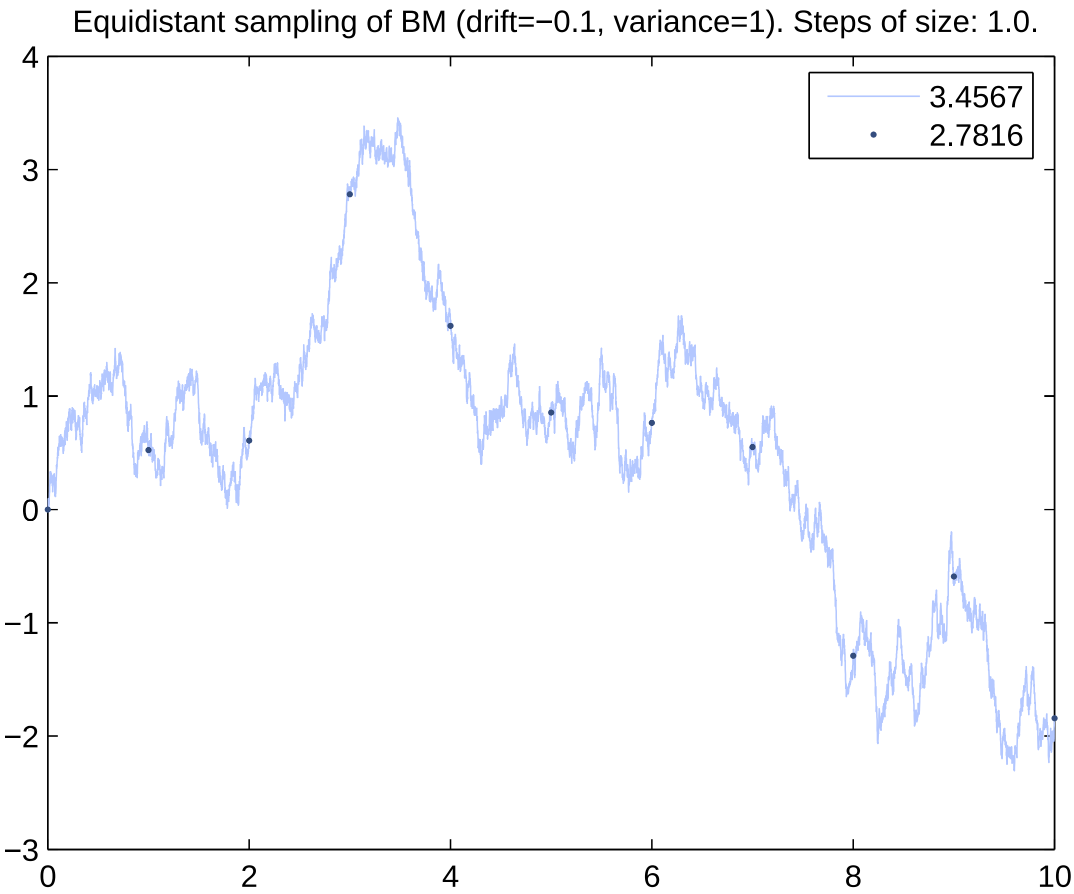
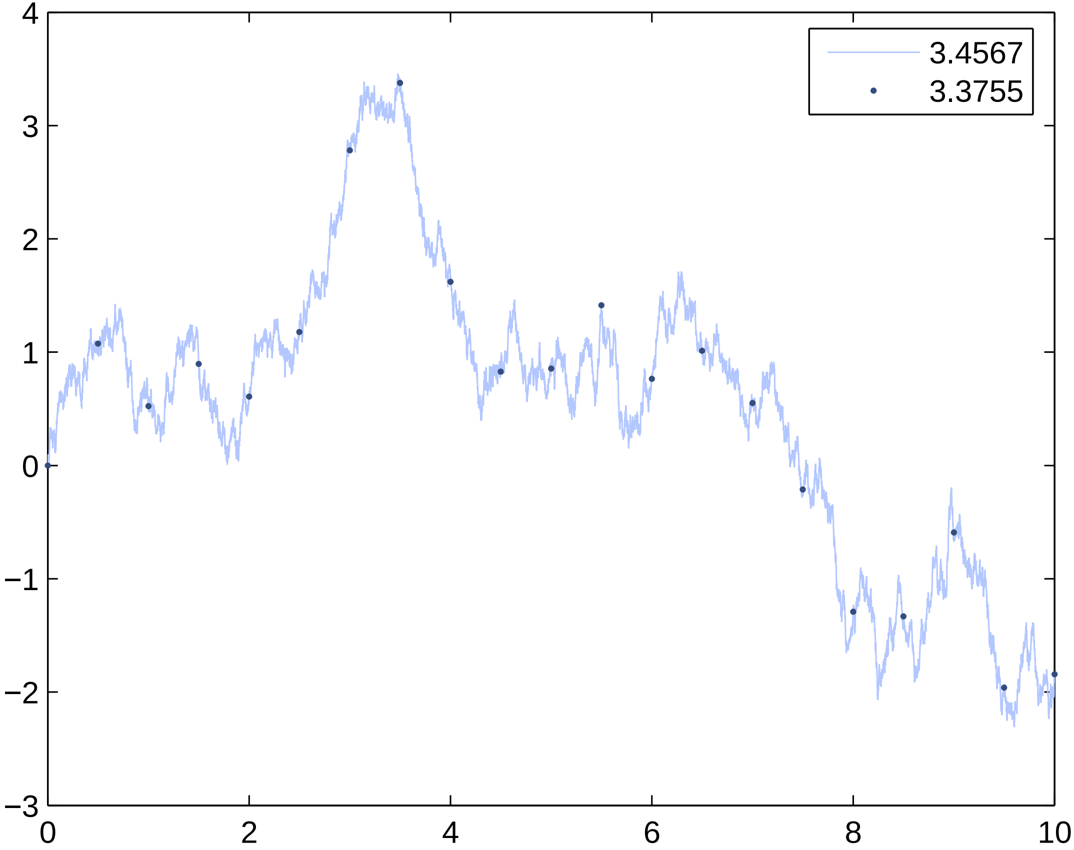
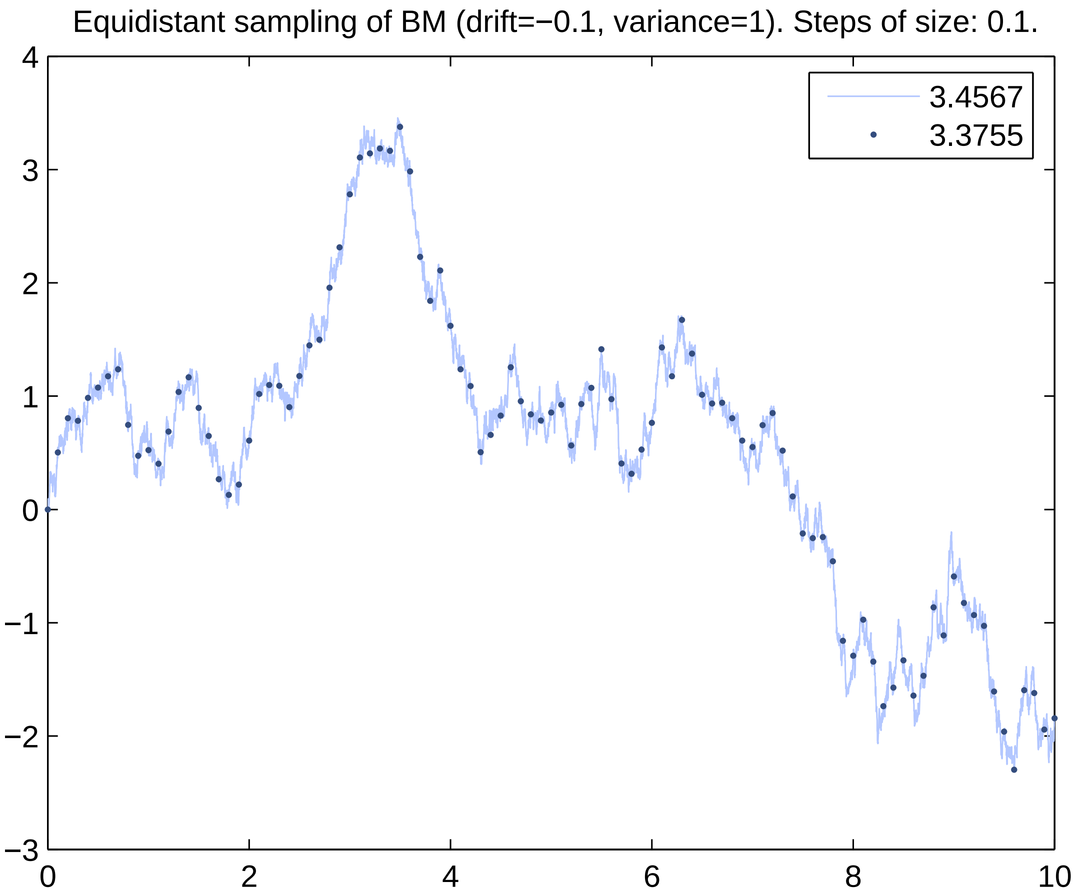
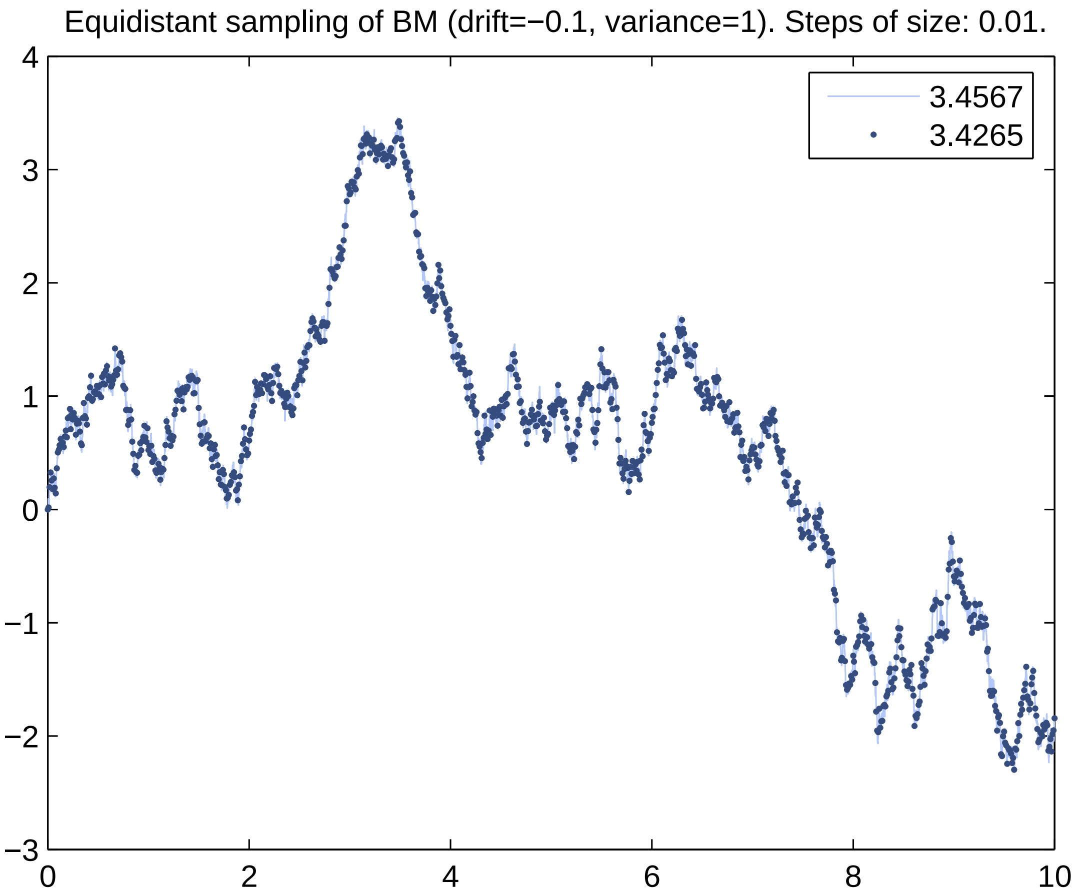


Equidistant sampling of BM (drift=−0.1, variance=1). Steps of size: 1.0.
3.4567
2.7816
4
3
2
1
0
−1
−2
−3
0
2
4
6
8
10

Equidistant sampling of BM (drift=−0.1, variance=1). Steps of size: 0.5.




Equidistant sampling of BM (drift=−0.1, variance=1). Steps of size: 0.1.
3.4567
3.3755
4
3
2
1
0
−1
−2
−3
0
2
4
6
8
10


Equidistant sampling of BM (drift=−0.1, variance=1). Steps of size: 0.01.
3.4567
3.4265
4
3
2
1
0
−1
−2
−3
0
2
4
6
8
10

Redefine the GRW as

$$\{S_n(\beta,\nu) : n = 0, 1, \ldots\},$$

where

$$S_n(\beta,\nu) = 0 \quad , \quad S_n(\beta,\nu) = X_{\nu,1} + \ldots + X_{\nu,n}$$

with

$$X_{\nu,1}, X_{\nu,2}, \ldots \ \text{i.i.d.} \quad , \quad X_i \sim N(-\beta/\nu, 1/\nu)$$

Let

$$M_{\nu,\beta} = \max\{S_n(\beta,\nu) : n = 0, 1, \ldots\}.$$

Earlier definition corresponds to $\nu = 1$ with $M_{1,\beta} =: M_\beta$.
Since

$$M_{\nu,\beta} \stackrel{d}{=} \nu^{-1/2} M_{\nu^{-1/2}\beta},$$

all characteristics of $M_{\nu,\beta}$ can be expressed in those of M_β.

Say we sample the BM at points

$$0, \frac{1}{\nu}, \frac{2}{\nu}, \frac{3}{\nu}, \dots$$

with ν some positive integer. From our results on $\mathbb{E}M_\beta$ we find that

$$\mathbb{E}\tilde{M} - \mathbb{E}M_{\nu,\beta} = -\frac{\zeta(\frac{1}{2})}{\sqrt{2\pi\nu}} + \mathcal{O}(1/\nu)$$

Similar result obtained by **Asmussen, Glynn and Pitman (1995).**

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Similar result obtained by **Asmussen, Glynn and Pitman (1995).** However, we can easily obtain

$$\mathbb{E}\tilde{M} - \mathbb{E}M_{\nu,\beta} = -\frac{\zeta(\frac{1}{2})}{\sqrt{2\pi\nu}} + \frac{\beta}{4\nu} + \mathcal{O}(\nu^{-3/2})$$

Moreover, our exact analysis of M_β leads to asymptotic expressions up to any order, for all cumulants of the maximum. E.g.

$$\mathrm{Var}\tilde{M} - \mathrm{Var}M_{\nu,\beta} = -\frac{1}{4\nu} - \frac{2\zeta(-\frac{1}{2})}{\sqrt{2\pi}}\frac{\beta}{\nu^{3/2}} + \mathcal{O}(\nu^{-2})$$

Other, related and future work

1. Exact expressions for $\mathbb{P}(M = 0)$, $\mathbb{E}M_\beta$, $\mathrm{Var}M_\beta$ (with A.J.E.M. Janssen)

2. Exact expressions for all cumulants $J_k(\beta)$, bounds, analytic continuation for all values of $\beta > 0$ (not only for $0 < \beta < 2\sqrt{\pi}$) (with A.J.E.M. Janssen)

3. Discrete queue and Halfin-Whitt scaling (with A.J.E.M. Janssen and Bert Zwart)

$$\lim_{\lambda\to\infty} \frac{1}{\sqrt{\lambda}} W_\lambda \stackrel{d}{=} \max\{S_n : n \geq 0\} = M_\beta$$