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Polynuclear growth on a flat substrate and edge scaling of GOE eigenvalues

Patrik Lino Ferrari
in collaboration with
Michael Prähofer and Herbert Spohn

Technische Universität München

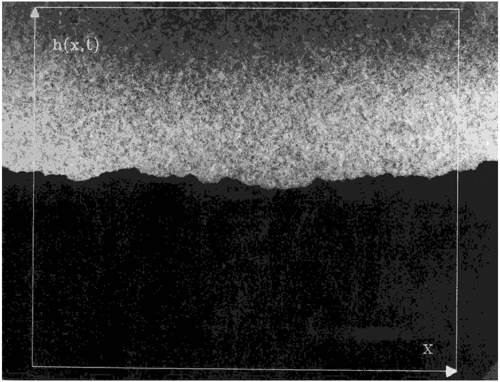
http://www−m5.ma.tum.de/pers/ferrari/

Outline

- Growth in $1+1$ dimension
- The polynuclear growth (PNG) model: flat growth has GOE Tracy-Widom distributed fluctuations
- Flat PNG and GOE random matrices: point processes
- Flat PNG and Young tableaux

Growth models: part of non-equilibrium statistical mechanics

Example: flame front of a burning paper



The flame propagates from below, the burned region is black

Universality picture

statistical properties of the surface for large growth time:
dependent only on the global properties of dynamics
(dimension, locality of growth, conservation laws, symmetries)

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Quantity to study (observable): surface height $x \mapsto h(x,t)$

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2) scaling exponents (of fluctuations and spatial correlations)
3) scaling function and/or limit process of the surface height

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KPZ class of growth model (one-dimensional substrate)

- fluctuation exponent: $1/3$
- spatial correlation exponent: $2/3$
- scaling functions ??? $\Rightarrow$ analyze simplied solvable models

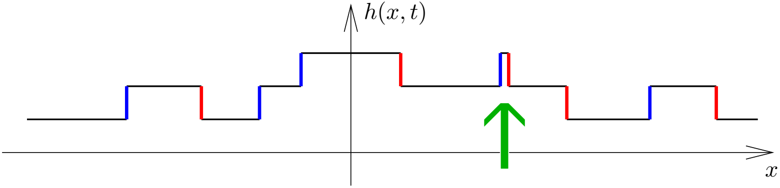
$\Rightarrow$ Study the polynuclear growth (PNG) model

A growth model of a surface on a **one dimensional** substrate

Studied observables: **statistical properties** of the surface for **large growth time** t

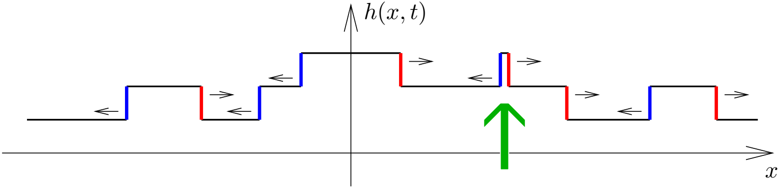
The surface is described by a function $h(x,t) \in \mathbb{Z}$, $x \in \mathbb{R}, t \in \mathbb{R}_+$, or by the position of the up- and down-jumps

A **nucleation** is a pair of an up- and a down-jump



The PNG dynamics has a

- **deterministic part**: the up-jumps moves to the left, the down-jumps to the right, with speed 1. When two jumps meet, they simply merge
- **stochastic part**: the nucleations form a *space-time* Poisson process with intensity $\varrho(x,t)$



The flat PNG

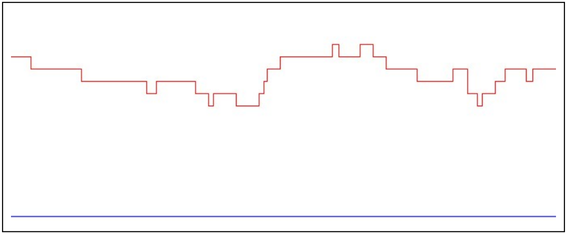
- Initial condition: $h(x,0)=0,\ x\in\mathbb{R}$
- Nucleations with constant intensity, $\rho(x,t)=2,\ x\in\mathbb{R},\ t\geq 0$

$\Rightarrow$ $h(x,t)\sim 2t$: a macroscopically flat profile

- The **fluctuations** scales as $t^{1/3}$
  (Baik, Rains '00; Prähofer, Spohn '00)

$$\lim_{t\to\infty}\mathbb{P}(h(0,t)\leq 2t+st^{1/3})=F_1(2^{2/3}s)$$

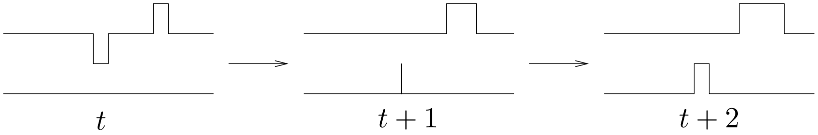
where F_1 is the GOE Tracy-Widom distribution



How to analyze the PNG surface? **Extension to multilayer PNG**

At each t, we define a set of **non-intersecting** line ensemble,
$\{x \mapsto h_\ell(x,t), \ell \leq 0\}$:

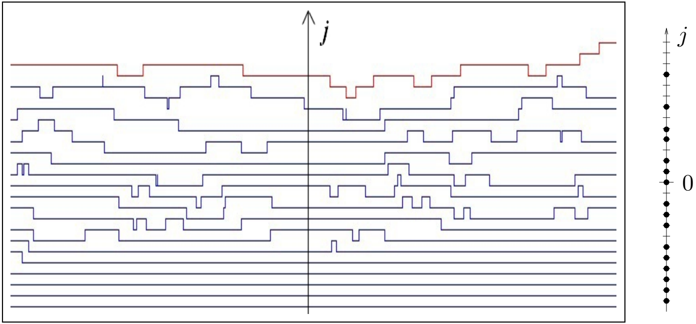
- the nucleations of level j, $j \leq -1$, occur when at level $j+1$ there is an annihilation (**no information is lost**),
- the deterministic dynamics is the same for all level lines



The physical surface height is $h_0 \equiv h$
Animation

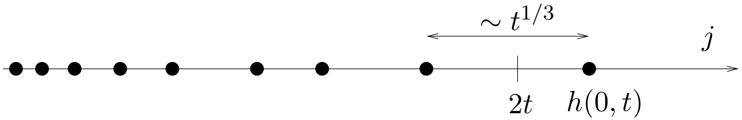
Point process η_t on $\mathbb{Z}$

$$\eta_t(j) = \begin{cases} 1, & \text{if there is a line at height } j, \\ 0, & \text{otherwise} \end{cases}$$



Support of the point process ($\eta_t = 1$) denoted by the $\bullet$ dots

- Last particle of η_t is the height of the flat PNG



- Edge scaling of the height

$$h^{\mathrm{edge}}(0,t) = \frac{h(0,t) - 2t}{t^{1/3} 2^{-2/3}}$$

- Point process edge scaling

$$\eta_t^{\mathrm{edge}}(u) = t^{1/3} 2^{-2/3} \eta_t([2t + u t^{1/3} 2^{-2/3}])$$

Theorem [P.L. Ferrari '04]
η_t^{edge} converges weakly to a **Pfaffian point process** η^{GOE}, whose n-point correlation functions are given by

$$\rho^{(n)}(x_1,\ldots,x_n) = \mathrm{Pf}\Big(K^{\mathrm{GOE}}(x_i,x_j)\Big)_{1\leq i,j\leq n}$$

with $K^{\mathrm{GOE}}(x_i,x_j)$ a 2×2 matrix kernel coming from GOE random matrices.

More precisely, for any $m\in\mathbb{N}$ and $f_1,\ldots,f_m\in C_0^1(\mathbb{R})$,

$$\lim_{t\to\infty}\mathbb{E}\left(\prod_{k=1}^m \eta_t^{\mathrm{edge}}(f_k)\right) = \mathbb{E}\left(\prod_{k=1}^m \eta^{\mathrm{GOE}}(f_k)\right)$$

[$\mathrm{Pf}(A) = \sqrt{\det(A)}$ if A is antisymmetric]

- Measure on $N \times N$ **real symmetric** matrices

$$\mathbb{P}(\mathrm{d}H) = \frac{1}{Z'} \exp(-\mathrm{Tr}(H^2)/2N)\mathrm{d}H,$$

$\mathrm{d}H = \prod_{1\leq i\leq j\leq N} \mathrm{d}H_{i,j}$ is the product measure on the independent coefficients of H

- The induced measure on the eigenvalues $\lambda_1, \ldots, \lambda_N$ is

$$\mathbb{P}(\mathrm{d}\lambda) = \frac{1}{Z} \prod_{1\leq i<j\leq N} |\lambda_i - \lambda_j| \prod_{i=1}^{N} e^{-\lambda_i^2/2N} \mathrm{d}\lambda$$

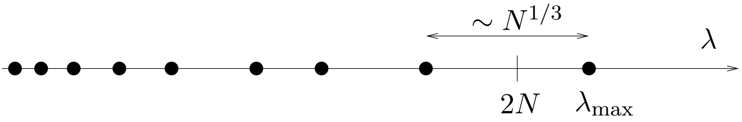
with $\mathrm{d}\lambda = \prod_{i=1}^{N} \mathrm{d}\lambda_i$

Behavior for large N:

- the largest eigenvalue, $\lambda_{\max}$, is $\sim 2N$,
- with fluctuations on a $N^{1/3}$ scale:

$$\lim_{N\to\infty} \mathbb{P}(\lambda_{\max} \le 2N + sN^{1/3}) = F_1(s)$$

with F_1 the GOE Tracy-Widom distribution



- Eigenvalues' point process η_N^{GOE} on $\mathbb{R}$:

$$\eta_N^{\mathrm{GOE}}(\lambda) = \sum_{i=1}^{N} \delta(\lambda - \lambda_i)$$

η_N^{GOE} is a Pfaffian point process (for even N)

- The **edge scaling** of the point process is

$$N^{1/3}\eta_N^{\mathrm{GOE}}(2N + sN^{1/3}) \stackrel{N\to\infty}{\Longrightarrow} \eta^{\mathrm{GOE}}(s)$$

and converges to a **Pfaffian point process** η^{GOE}

$$\rho^{(n)}(s_1, \dots, s_n) = \mathrm{Pf}(K^{\mathrm{GOE}}(s_i, s_j))_{1\le i,j\le n}$$

with the K^{GOE} a 2×2 matrix kernel

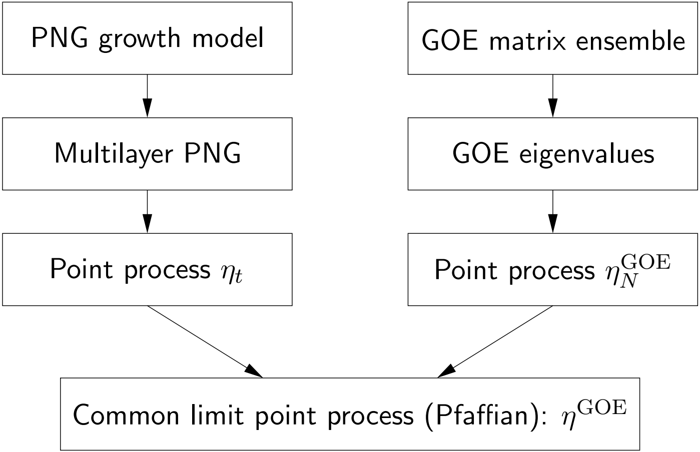
$$K^{\mathrm{GOE}}_{1,1}(\xi_1,\xi_2) = \int_{\mathbb{R}_+} \mathrm{d}\lambda\, \mathrm{Ai}(\xi_1+\lambda)\, \mathrm{Ai}'(\xi_2+\lambda) - (\xi_1 \leftrightarrow \xi_2)$$

$$K^{\mathrm{GOE}}_{1,2}(\xi_1,\xi_2) = \int_{\mathbb{R}_+} \mathrm{d}\lambda\, \mathrm{Ai}(\xi_1+\lambda)\, \mathrm{Ai}(\xi_2+\lambda) + \frac{1}{2}\mathrm{Ai}(\xi_1)\int_{\mathbb{R}_+} \mathrm{d}\lambda\, \mathrm{Ai}(\xi_2-\lambda)$$

$$K^{\mathrm{GOE}}_{2,1}(\xi_1,\xi_2) = -K^{\mathrm{GOE}}_{1,2}(\xi_2,\xi_1)$$

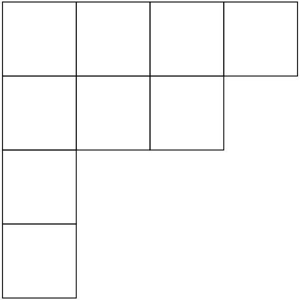
$$K^{\mathrm{GOE}}_{2,2}(\xi_1,\xi_2) = \frac{1}{4}\int_{\mathbb{R}_+} \mathrm{d}\lambda \int_{\lambda}^{\infty} \mathrm{d}\mu\, \mathrm{Ai}(\xi_2-\mu)\, \mathrm{Ai}(\xi_1-\lambda) - (\xi_1 \leftrightarrow \xi_2)$$

Note: Ai is the Airy function


PNG growth model
Multilayer PNG
Point process η_t
GOE matrix ensemble
GOE eigenvalues
Point process η_N^{GOE}
Common limit point process (Pfaffian): η^{GOE}

Consider the set $S = \{1, 2, \ldots, 9\}$

- Young diagram of shape $\lambda = (4, 3, 1, 1)$



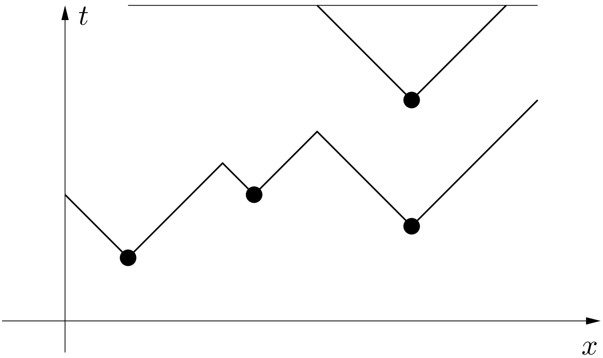
Young tableaux: an example

Consider the set $S = \{1, 2, \ldots, 9\}$

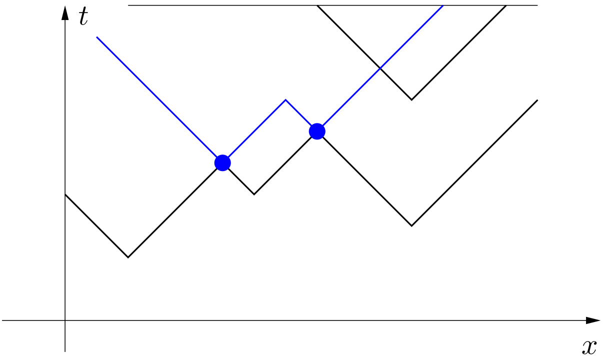
- Young diagram of shape $\lambda = (4, 3, 1, 1)$
- One of the possible Young tableaux.
  The numbers of S have to be put into the Young diagram but have to be increasing along the rows and columns

1	3	4	9
2	5	7	
6			
8			

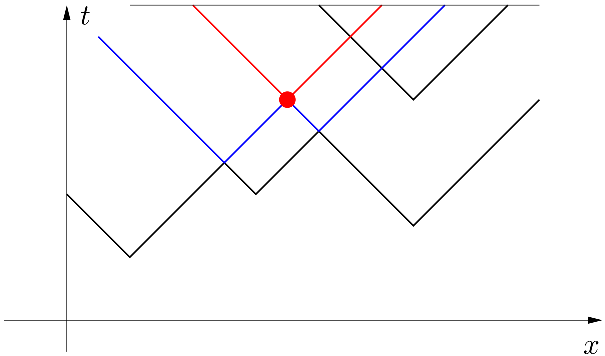
Space-time PNG, level 0



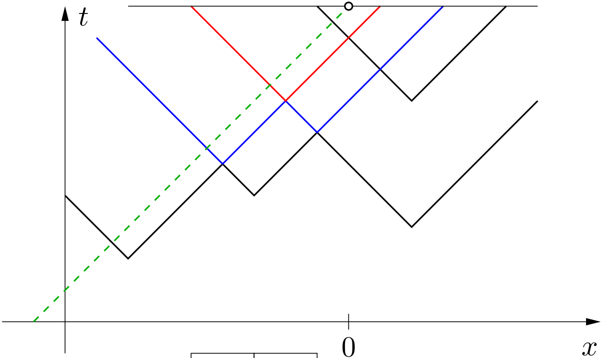
Space-time PNG, level -1



Space-time PNG, level -2

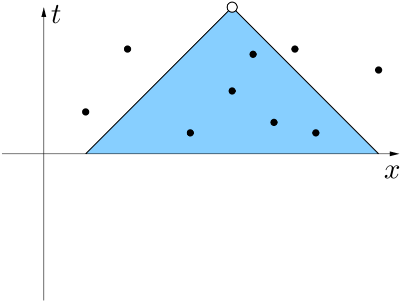


Associated Young tableaux (whose first row length = PNG height)



$$Y(\circ) = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}$$

- Measure on associated Young tableaux too complicated
- The height $h(0,t)$ depends only on the nucleations in the backwards light cone of $(0,t)$



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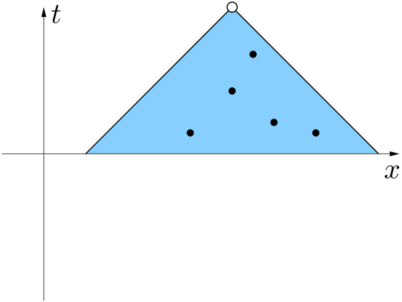
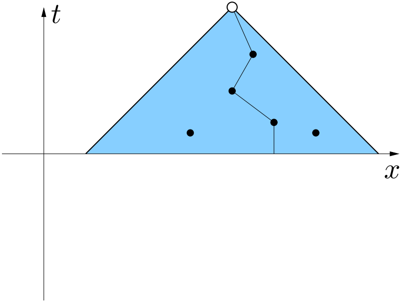


- Measure on associated Young tableaux too complicated
- The height $h(0,t)$ depends only on the nucleations in the backwards light cone of $(0,t)$
- The height $h(0,t)$ is the number crossing of the black lines along any time-like paths from $\{t=0\}$ to $(0,t)$



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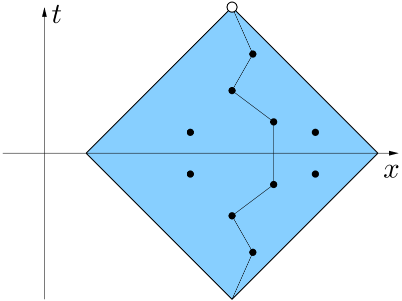
- The height $h(0,t)$ is the number crossing of the black lines along any time-like paths from $\{t=0\}$ to $(0,t)$
- We can add the symmetric points with respect to $\{t=0\}$

⇒ The new Young tableaux have a nice measure



- As before, nucleation intensity $\varrho = 2$
- Mean # of nucleations in the backwards light-cone of $(0, t)$: $A_t = t^2$
- Symmetrization $\Rightarrow$ Only Young tableaux Y with even rows, shape $\lambda(Y) = (\lambda_1, \lambda_2, \ldots)$, λ_i even, $|\lambda(Y)| = \sum_{k\geq 1} \lambda_k$

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- Measure on the set of even-rows Young tableaux:

$$\mathbb{P}(Y) = \sum_{n\geq 0} \delta_{|\lambda(Y)|,2n} \underbrace{e^{-A_t} \frac{A_t^n}{n!}}_{\mathbb{P}(|\lambda(Y)|=2n)} \frac{\dim(\lambda(Y))}{Z_n}$$

with $Z_n = \#$ of Young tableaux with $2n$ entries and even rows
This measure appears also in a work of Borodin and Olshanski ('02)

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- Connection with the multilayer PNG height / point process:

$$h_i(t) = \frac{1}{2}\lambda_{1-i} + i, \quad i = 0, -1, -2, \ldots$$