The expected area of the Brownian loop in the plane

José A. Trujillo Ferreras (joint with Christophe Garban)

March 24th 2006

Outline

1. The area of the Brownian loop
   - Presentation of the problem
   - Some background on *SLE*.
   - Overview of the idea of the proof
   - Sketch of proof

2. Expected areas for regions of arbitrary index

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- Consider a Brownian bridge in $\mathbb{C}$ from 0 to 0 of time duration 1

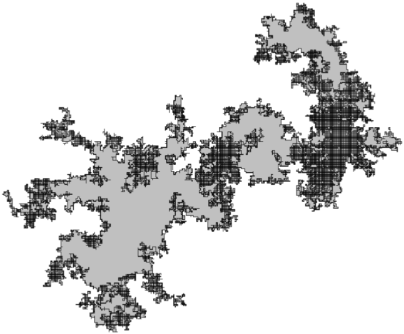

Figure: Random walk loop of 50000 steps and its hull.

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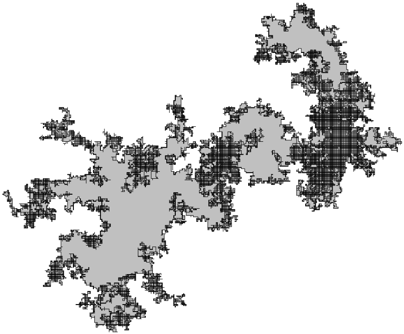

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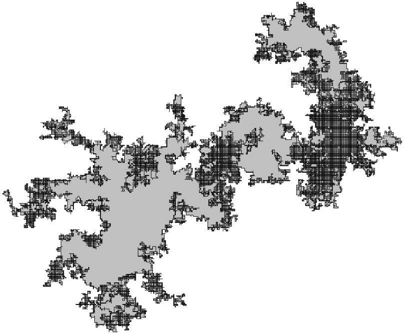

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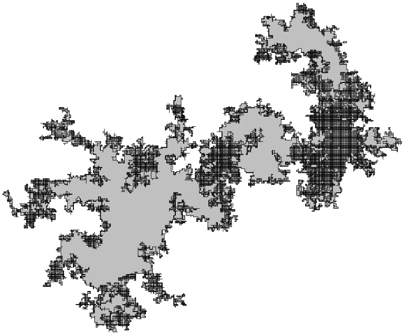

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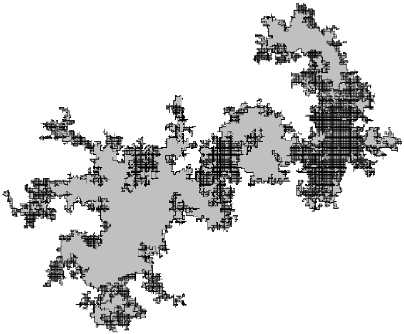

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Quick definition of *SLE*.

- Consider the differential equation

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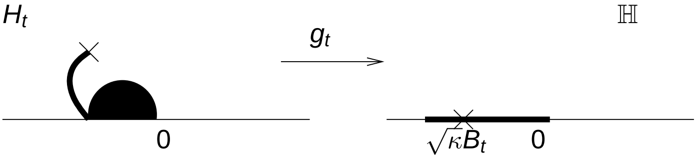

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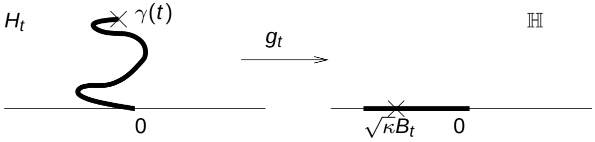
- For this talk we only need $\kappa = \frac{8}{3}$.



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The σ-finite measure

$$\mu^{\text{bub}} = \int_0^\infty \frac{1}{2\,t^2}\mu_t dt$$

- encodes μ_t
- relates "nicely" to an *SLE*-type measure.

Relationship between the Brownian and the *SLE* measures.

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  - Define

$$\mu^{\text{sle}} = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} m_\epsilon(\tilde{\nu})$$

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- Recall: $\nu = \mathbb{H}-$Brownian excursion; $\tilde{\nu} = SLE_{8/3}$.
- LSW showed that 5 independent ν equal 8 independent $\tilde{\nu}$.
- **Easy consequence:** $\frac{5}{8}\mu^{\text{bub}} = \mu^{\text{sle}}$ as measures on filled-in bubbles.

Strategy for finding the expected area of the Brownian loop.

- Use
$$\mu^{\text{bub}} = \int_0^\infty \frac{1}{2t^2}\mu_t dt$$
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to relate our fixed-time quantity (Brownian loop of time duration 1) to a geometric-type quantity for μ^{bub}.
- Since $(5/8)\mu^{\text{bub}} = \mu^{\text{sle}}$, we can use *SLE* techniques to compute the geometric quantity.

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Relating the fixed-time quantity and the geometric quantity.

Let γ^* be the radius of the curve γ, i.e., $\gamma^* = \sup_{0 \leq t \leq t_\gamma} |\gamma(t)|$. Consider the “expected" area under the law μ^{bub} “restricted" to curves with $\gamma^* \in [1, 1+\delta)$.

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&= \log(1+\delta) \mu_1(A) = \log(1+\delta) \mathbb{E}(A)
\end{aligned}$$

Expressing $\mathbb{E}(A)$ in terms of SLE.

Since $\frac{5}{8}\mu^{\text{bub}} = \mu^{\text{sle}}$, we have

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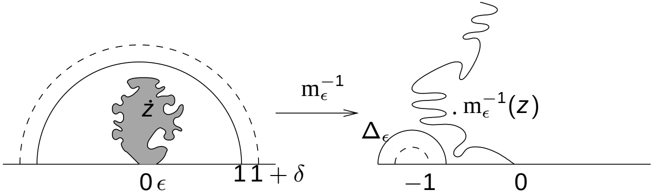
- By Fubini,

$$\frac{8}{5}\int_{\mathbb{D}^+}\lim_{\delta\to 0}\lim_{\epsilon\to 0}\frac{\mathbb{P}_\epsilon\{z \text{ inside } \gamma, \gamma^* \in [1,1+\delta)\}}{\epsilon^2\log(1+\delta)}d\mathcal{A}(z) = \mathbb{E}(A).$$

Computing $\mathbb{P}_\epsilon\{z \text{ inside } \gamma, \gamma^* \in [1, 1+\delta)\}$; an *SLE* computation.



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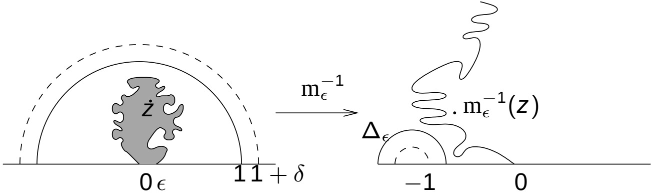

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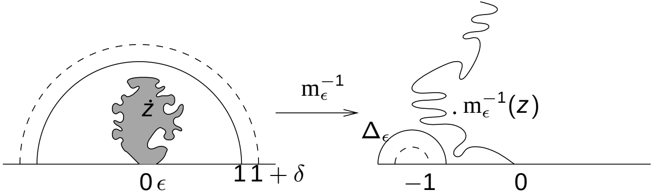

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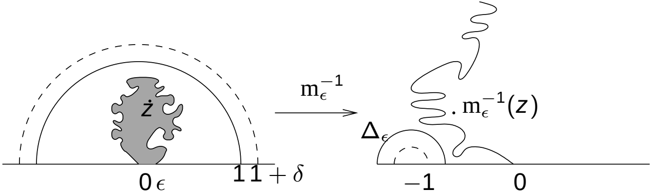

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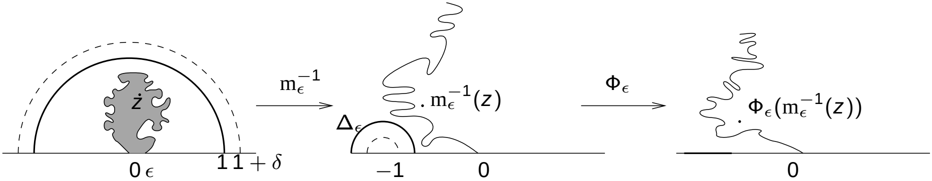

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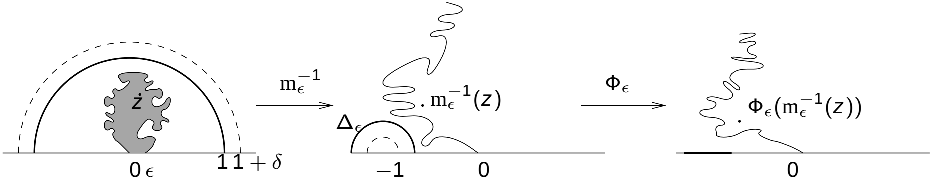

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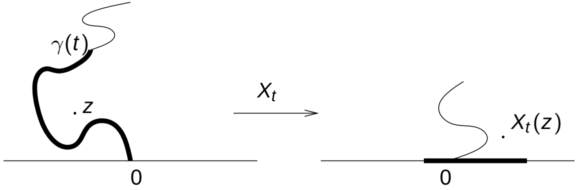

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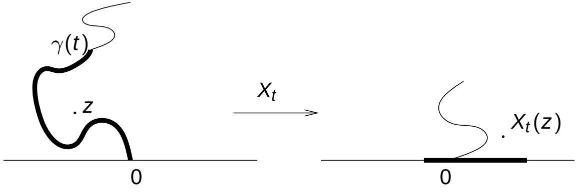

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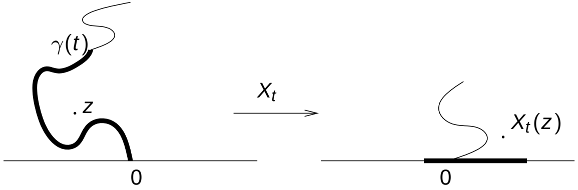

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- Note: $X_t(z) = g_t(z) - \sqrt{\kappa} B_t$. Then, use the differential equation for $g_t(z)$ and Ito's formula to get an SDE for $f(\arg(X_t))$.

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Expected areas for regions of arbitrary index.

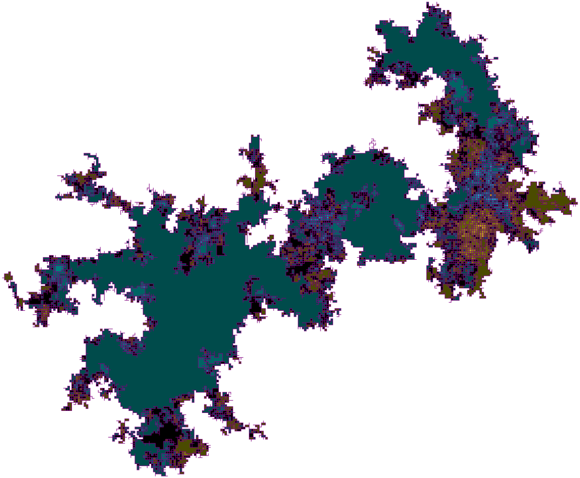

Figure: Different indices in a random walk of 50000 steps, black areas correspond to index 0.

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Yor's index law for a Brownian bridge.

Theorem (M. Yor)

Write $z = re^{i\theta}$*, with* $r \neq 0$*. Then,*

$$\begin{aligned}\mathbb{P}(n_z = n) &= e^{-r^2}[\Psi_r((2n-1)\pi) - \Psi_r((2n+1)\pi)] \text{ if } n \in \mathbb{Z} \setminus 0\,,\\ \mathbb{P}(n_z = 0) &= 1 + e^{-r^2}[\Psi_r(-\pi) - \Psi_r(\pi)]\,,\end{aligned}$$

where $\forall x \neq 0$*,*

$$\Psi_r(x) = \frac{x}{\pi}\int_0^\infty e^{-r^2\cosh(t)}\frac{dt}{t^2+x^2}\,.$$

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$$\mathbb{E}(\mathcal{W}_0) = \frac{\pi}{5} - \frac{\pi}{6} = \frac{\pi}{30}.$$