Universality for Laguerre-type ensembles at the hard edge of the spectrum

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joint work with
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Introduction

Consider Random Matrix Ensembles (RME) leading to the following distribution on the eigenvalues $\lambda_1, \ldots, \lambda_n$

$$P_n(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_{n,\beta}} \prod_{i=1}^{n} w_\beta(\lambda_i) \prod_{1 \le i < j \le n} |\lambda_i - \lambda_j|^\beta$$

where

$$\beta = 1, 2, 4$$

$$w_\beta(x) = \begin{cases} w(x), & \beta = 2, 4 \\ \sqrt{w(x)}, & \beta = 1 \end{cases} \qquad w(x) = e^{-V(x)} \text{ or } e^{-nV(x)}$$

$Z_{n,\beta}$ is a normalization constant

Introduction

One of the main interests in RMT:

Universality Conjecture
Limiting statistical behavior of the (appropriately scaled) eigenvalues depends only on the symmetry of the system. So, independent of V.

Aim of this talk:

Give INSIGHT in the techniques that we use to prove the universality conjecture for the cases $\beta = 2$ and $\beta = 1, 4$.

- Case $\beta = 2$: Riemann-Hilbert approach

  Deift-Kriecherbauer-McLaughlin-Venakides-Zhou 1999, ...

- Case $\beta = 1, 4$: Widom's formalism

  Widom 1998, Deift-Gioev 2004 & 2005, Deift-Gioev-Kriecherbauer-V 2006

Introduction

Case $\beta = 2$

The eigenvalue statistics can be expressed in terms of a scalar 2-point kernel K_n see e.g. Mehta

$$K_n(x, y) = \sum_{k=0}^{n-1} \phi_k(x)\phi_k(y)$$

where

$$\phi_k(x) = p_k(x)\sqrt{w(x)}$$

$$p_k(x) = \gamma_k x^k + \dots \qquad \int p_k(x)p_j(x)w(x)dx = \delta_{jk}$$

For example:

$$P_n(\lambda_1, \dots, \lambda_n) = \frac{1}{n!} \det(K_n(\lambda_i, \lambda_j))_{1 \le i,j \le n}$$

$$\mathcal{R}_{n,k}(\lambda_1, \dots, \lambda_k) = \det(K_n(\lambda_i, \lambda_j))_{1 \le i,j \le k}$$

Introduction
Case $\beta = 2$

So, to prove universality conjecture $\rightarrow$ need information on K_n

Main idea to get this:

- Christoffel-Darboux formula
  $\longrightarrow K_n$ in terms of ϕ_{n-1}, ϕ_n and γ_{n-1}, γ_n

$$K_n(x,y) = \frac{\gamma_{n-1}}{\gamma_n} \frac{\phi_n(x)\phi_{n-1}(y) - \phi_{n-1}(x)\phi_n(y)}{x-y}$$

- Asymptotics of ϕ_{n-1}, ϕ_n and γ_{n-1}, γ_n (RH problem for OP)

We will come back to this later!

Introduction

Case $\beta = 1, 4$

Role of the scalar 2-point kernel K_n is now played by 2×2 matrix kernels $K_{n,1}$ and $K_{n,4}$ given by Tracy-Widom 1998

$$K_{n,1}(x,y) = \begin{pmatrix} S_{n,1}(x,y) & (S_{n,1}D)(x,y) \\ (\varepsilon S_{n,1})(x,y) - \varepsilon(x,y) & S_{n,1}(y,x) \end{pmatrix} \quad \text{(for } n \text{ even)}$$

$$K_{n,4}(x,y) = \frac{1}{2} \begin{pmatrix} S_{n,4}(x,y) & (S_{n,4}D)(x,y) \\ (\varepsilon S_{n,4})(x,y) & S_{n,4}(y,x) \end{pmatrix}$$

where

- D is the differentiation operator
- ε is the integral operator with kernel $\varepsilon(x,y) = \frac{1}{2}\operatorname{sgn}(x-y)$
- $S_{n,1}$ and $S_{n,4}$ are scalar kernels

Introduction

Case $\beta = 1, 4$

The scalar kernels $S_{n,1}$ and $S_{n,4}$ are given as follows:

Let $\psi_j(x) = q_j(x)\sqrt{w(x)}$ with q_j any polynomial of exact degree j.
Then

- $$S_{n,1}(x,y) = -\left(\psi_0(x), \ldots, \psi_{n-1}(x)\right) M_{n,1}^{-1} \begin{pmatrix} \varepsilon\psi_0(y) \\ \vdots \\ \varepsilon\psi_{n-1}(y) \end{pmatrix}$$

  $M_{n,1} = n \times n$ matrix with entries $\langle \psi_j, \varepsilon\psi_k \rangle$

- $$S_{n,4}(x,y) = \left(\psi_0'(x), \ldots, \psi_{2n-1}'(x)\right) M_{n,4}^{-1} \begin{pmatrix} \psi_0(y) \\ \vdots \\ \psi_{2n-1}(y) \end{pmatrix}$$

  $M_{n,4} = 2n \times 2n$ matrix with entries $\langle \psi_j, \psi_k' \rangle$

Can choose the polynomials q_j arbitrarely! Need to choose them such that:

- Analogue of the C-D formula (in terms of ψ_{n+j} with $|j| \leq c$)
- Should be able to get asymptotics of the ψ_{n+j}
- Control of $M_{n,1}^{-1}$ and $M_{n,4}^{-1}$

Choice 1: Skew orthogonal polynomials (SOP)
Are such that

$$M_{n,\beta} = \begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & \\ & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & \\ & & \ldots \end{pmatrix} \qquad \rightarrow \qquad \text{control of } M_{n,\beta}^{-1}$$

Problem: not much is known about asymptotics of general SOP

Introduction

Case $\beta = 1, 4$

Choice 2: Orthogonal polynomials (Choice that we take in this talk)

Widom's formalism — Widom 1999

For any weight w such that w'/w is a rational function

$$S_{n,\beta}(x,y) = K_n(x,y) + \text{finite sum of } \phi_{n+j} \text{ with } |j| \leq c$$

Some notes:

- Analogue of C-D formula
- Information on K_n due to the case $\beta = 2$
- Finite sum contains inverse matrix which has to be controlled

Widom's formalism has been used to prove the universality conjecture in the bulk/soft edge of the spectrum for the case — Deift-Gioev 2004 & 2005

$$w(x) = e^{-V(x)}, \qquad V(x) = \sum_{k=0}^{2m} q_k x^k \qquad (q_{2m} > 0, m \geq 1).$$

Our result

Proof of the universality conjecture (in the bulk, at the hard edge and at the soft edge of the spectrum) for RME associated to Laguerre-type weights

$$w(x) = x^{\alpha} e^{-V(x)}, \qquad \text{for } x \in [0, \infty)$$

where

$$\alpha > 0, \qquad V(x) = \sum_{k=0}^{m} q_k x^k \quad (q_m > 0, m \geq 1)$$

Our result
Some notes

In this talk:

- Restrict to hard edge of the spectrum
  in bulk: result agrees with Deift-Gioev 2004
  at soft edge: result agrees with Deift-Gioev 2005

- For $\beta = 1, 4$ consider only the $(1,1)$-entry of $K_{n,\beta}$
  other entries have similar formulae

Our result
Case $\beta = 2$

Introduce the notation (scaling) V 2005

$$\lambda_n = \frac{\beta_n}{4c_n n^2} \qquad \beta_n \sim \beta_m n^{1/m} \qquad c_n \sim \left(\frac{2m}{2m-1}\right)^2$$

Then, for $\xi, \eta \in (0, \infty)$:

$$\lim_{n\to\infty} \lambda_n K_n(\lambda_n \xi, \lambda_n \eta) = \frac{J_\alpha(\sqrt{\xi})\sqrt{\eta} J'_\alpha(\sqrt{\eta}) - J_\alpha(\sqrt{\eta})\sqrt{\xi} J'_\alpha(\sqrt{\xi})}{2(\xi - \eta)}$$

$$\equiv K_J(\xi, \eta)$$

Our result
Case $\beta = 1, 4$

For n even and $\xi, \eta \in (0, \infty)$: Deift-Gioev-Kriecherbauer-V 2006

$$\lim_{n\to\infty} \lambda_n S_{n,1}(\lambda_n \xi, \lambda_n \eta) = K_J(\xi, \eta) - \frac{1}{4} \frac{J_{\alpha+1}(\sqrt{\xi})}{\sqrt{\xi}} \int_{\sqrt{\eta}}^{\infty} \left(J_{\alpha+1}(s) - \frac{2\alpha}{s} J_\alpha(s) \right) ds$$

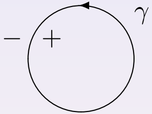
$$\lim_{n\to\infty} \lambda_n S_{\frac{n}{2},4}(\lambda_n \xi, \lambda_n \eta) = K_J(\xi, \eta) + \frac{1}{4} \left(\frac{J_{\alpha+1}(\sqrt{\xi})}{\sqrt{\xi}} - \frac{2\alpha}{\xi} J_\alpha(\sqrt{\xi}) \right) \int_0^{\sqrt{\eta}} J_{\alpha+1}(s) ds$$

Riemann-Hilbert approach

Scalar RH problem

RH problem = Seek a function f satisfying some conditions:

- $f : \mathbb{C} \setminus \gamma \to \mathbb{C}$ is analytic
- $f_+(s) - f_-(s) = v(s)$ for $s \in \gamma$
- $f(z) \to 0$ as $z \to \infty$



Solution: Sokhotskii-Plemelj formula

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{v(s)}{s-z} ds \equiv C(v)(z), \qquad \text{for } z \in \mathbb{C} \setminus \gamma$$

Riemann-Hilbert approach

RH problem for Y

Seek a 2×2 matrix valued function Y analytic on $\mathbb{C} \setminus \mathbb{R}$ having the following jump and asymptotics

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}, \qquad \text{for } x \in \mathbb{R},$$

$$Y(z) = [I + \mathcal{O}(1/z)] \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \qquad \text{as } z \to \infty.$$

The RH problem for Y has a unique solution Fokas-Its-Kitaev 1992

$$Y(z) = \begin{pmatrix} \gamma_n^{-1} p_n(z) & C(\gamma_n^{-1} p_n w)(z) \\ -2\pi i \gamma_{n-1} p_{n-1}(z) & C(-2\pi i \gamma_{n-1} p_{n-1} w)(z) \end{pmatrix}, \quad \text{for } z \in \mathbb{C} \setminus \mathbb{R}$$

Riemann-Hilbert approach

Connection K_n and Y

Recall that

$$K_n(x,y) = \sqrt{w(x)}\sqrt{w(y)}\frac{\gamma_{n-1}}{\gamma_n}\frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{x-y}$$

Since

$$\frac{p_n}{\gamma_n} = Y_{11}, \qquad \gamma_{n-1}p_{n-1} = \frac{1}{-2\pi i}Y_{21}$$

we then obtain

$$K_n(x,y) = \sqrt{w(x)}\sqrt{w(y)}\frac{1}{2\pi i(x-y)}\begin{pmatrix}0 & 1\end{pmatrix} Y^{-1}(y)Y(x)\begin{pmatrix}1\\0\end{pmatrix}$$

$\rightarrow$ Need to determine the asymptotics of Y

Riemann-Hilbert approach

Deift/Zhou steepest descent method

1. Do series of transformations $Y \mapsto \cdots \mapsto R$ to arrive at a RH problem for R
   - with jumps uniformly close to I, as $n \to \infty$
   - normalized at infinity (i.e., $R(z) \to I$ as $z \to \infty$)
2. Then

$$R(z) = I + \mathcal{O}(1/n), \qquad \text{as } n \to \infty \text{ uniformly for } z$$

3. By unfolding the series of transformations $Y \mapsto \cdots \mapsto R$ we obtain the asymptotics of Y.

Step 1: Normalization of the RH problem: $Y \mapsto T$

It uses the log transform of the equilibrium measure μ of $\mathbb{R}_+$ in the presence of the external field V, which is, e.g. for the case $V(x) = 4x$ known to be

$$d\mu(x) = \frac{2}{\pi}\sqrt{\frac{1-x}{x}}\,\chi_{(0,1]}(x)dx$$

The effect of this transformation is

- RH problem is normalized at infinity (i.e. $T(z) \to I$ as $z \to \infty$)
- new jump matrix v_T looks like

$$v_T(x) = \begin{cases} \begin{pmatrix} \text{oscillatory} & x^\alpha \\ 0 & \text{oscillatory} \end{pmatrix}, & \text{for } x \in (0,1), \\ I + \exp \text{ small}, & \text{for } x \in (1,\infty). \end{cases}$$

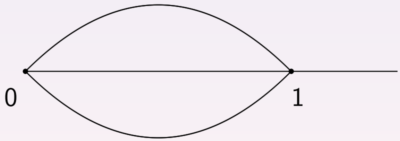
Riemann-Hilbert approach

Deift/Zhou steepest descent method

Step 2: Opening of the lens: $T \mapsto S$

Transform the oscillatory diagonal entries of v_T into exponentially decaying off-diagonal entries by opening the lens.

The effect of this transformation is that S has now jumps on a lens shaped region



which look like

$$v_S(x) = \begin{cases} \begin{pmatrix} 0 & x^{\alpha} \\ -x^{-\alpha} & 0 \end{pmatrix}, & \text{for } x \in (0,1), \\ I + \exp \text{ small}, & \text{for } x \text{ elsewhere.} \end{cases}$$

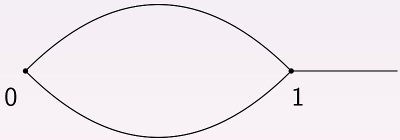
Riemann-Hilbert approach

Deift/Zhou steepest descent method

Step 3: Parametrix $P^{(\infty)}$ for the outside region

Introduce a matrix valued function $P^{(\infty)}$ with jump only on $(0, 1)$ where it satisfies the same jump relation as S does.

Suggestion for the final transformation: $R = S(P^{(\infty)})^{-1}$
Then, R has only jumps on



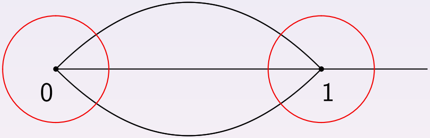
Problem: jumps are not uniformly close to I near 0 and 1
→ Need to do a local analysis near 0 and 1

Riemann-Hilbert approach

Deift/Zhou steepest descent method

Step 4: Parametrix P near 0 and 1

Surround 0 and 1 with small circles



and seek P such that

- P has the same jumps as S inside the circles
- P matches with $P^{(\infty)}$ on the circles, i.e.

$$P(P^{(\infty)})^{-1} = I + \mathcal{O}(1/n) \qquad \text{as } n \to \infty$$

The construction of P is very technical and uses Airy functions near 1 (DKMVZ 1999) and Bessel functions near 0 (Kuijlaars-McLaughlin-Van Assche-V 2004)

Riemann-Hilbert approach

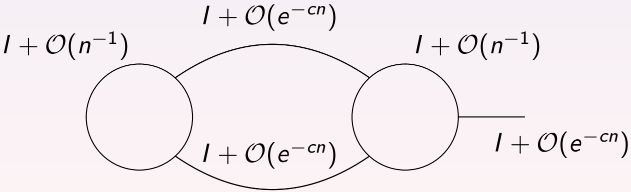
Deift/Zhou steepest descent method

Step 5: Final transformation: $S \mapsto R$

$$\text{Let } R = \begin{cases} S(P^{(\infty)})^{-1}, & \text{outside the circles} \\ SP^{-1}, & \text{inside the circles} \end{cases}$$

Then, by construction

- $R(z) \to I$ as $z \to \infty$
- R has jumps on the following system of contours



This yields: $R(z) = I + \mathcal{O}(n^{-1})$ as $n \to \infty$

Widom's formalism

Can be applied to all weights w such that $\frac{w'}{w}$ is a rational function

- Introduce

$$\mathcal{H} = \mathrm{span}(\phi_0, \ldots, \phi_{n-1})$$

$$\begin{aligned}\mathfrak{g} &= \mathrm{span}\left(\{x^j\phi_n(x), x^j\phi_{n-1}(x) \mid j = -1, 0, \ldots, m-2\}\right) \\ &= \mathrm{span}\left(\{\phi_{n-m+1}, \ldots, \phi_{n+m-2}\} \cup \left\{\frac{\phi_n(x)}{x}, \frac{\phi_{n-1}(x)}{x}\right\}\right)\end{aligned}$$

$$\Phi_1 = (\phi_{n-1}, \phi_{n-2}, \ldots, \phi_{n-m+1}, \psi_1) \qquad \text{basis of } \mathfrak{g} \cap \mathcal{H}$$

$$\Phi_2 = (\phi_n, \phi_{n+1}, \ldots, \phi_{n+m-2}, \psi_2) \qquad \text{basis of } \mathfrak{g} \cap \mathcal{H}^\perp$$

$$\Phi = (\Phi_1, \Phi_2)$$

Widom's formalism

- There exists $2m \times 2m$ matrix A such that $[D,K]f = \Phi A \langle f, \Phi^t \rangle$

$$A = \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} \qquad \text{symmetric}$$

- $B = \langle \varepsilon \Phi^t, \Phi \rangle \equiv \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ skew symmetric
- $C = \begin{pmatrix} I+(BA)_{11} & (BA)_{12} \\ (BA)_{21} & (BA)_{22} \end{pmatrix} \equiv \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$

Then

Widom 1998

$$S_{\frac{n}{2},4}(x,y) = K_n(x,y) - \Phi_2(x) A_{21} \varepsilon \Phi_1(y)^t - \Phi_2(x) A_{21} C_{11}^{-1} C_{12} \varepsilon \Phi_2(y)^t$$

$$S_{n,1}(x,y) = K_n(x,y) - (\Phi_1(x), 0) \cdot (AC(I - BAC)^{-1})^t \cdot \begin{pmatrix} \varepsilon \Phi_1(y)^t \\ \varepsilon \Phi_2(y)^t \end{pmatrix}$$

Widom's formalism

The following important relation holds: Deift-Gioev-Kriecherbauer-V 2006

$$BAC = \begin{pmatrix} 0 & 0 \\ C_{21} & C_{22} \end{pmatrix}$$

With this

$$S_{\frac{n}{2},4}(x,y) = K_n(x,y) - \Phi_2(x)A_{21}\varepsilon\Phi_1(y)^t - \Phi_2(x)G_{11}\varepsilon\Phi_2(y)^t$$

$$S_{n,1}(x,y) = K_n(x,y) - \Phi_1(x)A_{12}\varepsilon\Phi_2(y)^t - \Phi_1(x)\widehat{G}_{11}\varepsilon\Phi_1(y)^t$$

where

$$G_{11} = A_{21}(I + B_{12}A_{21})^{-1}C_{12}$$

$$\widehat{G}_{11}^t = A_{12}(I - B_{21}A_{12})^{-1}C_{21}$$

Widom's formalism

Problems to attack are:

- Asymptotics of ϕ_n and ψ_j $\rightarrow$ RHP for OP
- Asymptotics of A_{21} (use the asymptotics of the recurrence coef.)

$$A_{21} \sim -\frac{n}{\beta_n}\begin{pmatrix} Q & 0 \\ 0 & 1/2 \end{pmatrix} \equiv -\frac{n}{\beta_n} Y$$

- Asymptotics of B_{12}

  $\langle \varepsilon\phi_p, \phi_q \rangle$ (main problem: double integral of ϕ_p with ϕ_q)
  $\langle \varepsilon\phi_p, \psi_j \rangle$ $\qquad\qquad$ ϕ_p $\qquad$ ψ_j
  $\langle \varepsilon\psi_1, \psi_2 \rangle$ $\qquad\qquad$ ψ_1 $\qquad$ ψ_2

$$B_{12} \sim \frac{\beta_n}{n}\begin{pmatrix} R & v^t \\ v & 1 - \frac{1}{\sqrt{2m-1}} \end{pmatrix} \equiv \frac{\beta_n}{n} X$$

- Control $(I + B_{12}A_{21})^{-1}$ and $(I - B_{21}A_{12})^{-1} = (I + A_{21}B_{12})^{-t}$

  $\rightarrow$ prove invertibility of $I - XY$

Questions?