# Determinants of random Cauchy matrices and capacity of wireless networks

A. Özgür, E. Preissmann, O. Lévêque
Ecole Polytechnique Fédérale de Lausanne

YEP - Eindhoven, March 24, 2006

## Network information theory: some (recent) history

- 70's: classical network information theory
$\Rightarrow$ exact results for "networks" of 3 or 4 users

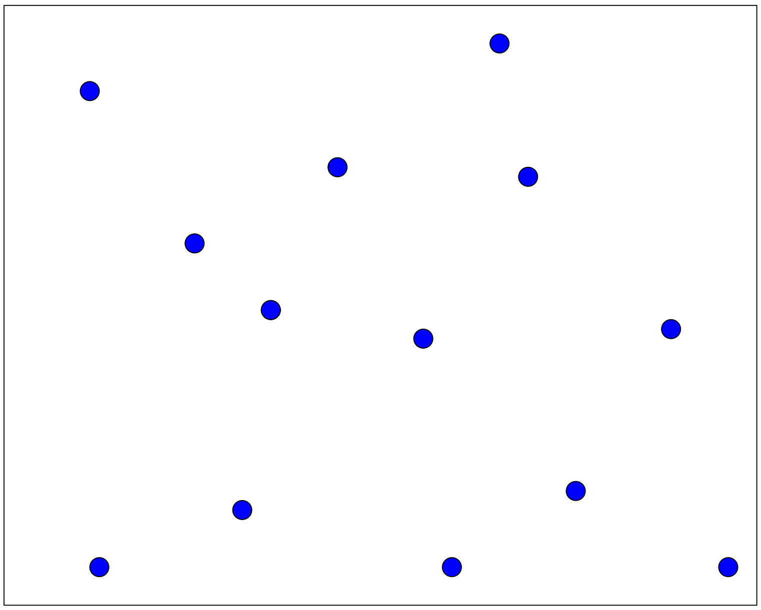
- 90's: development of self-organized wireless networks with a large number of users ($\sim$ 100)
$\Rightarrow$ the focus has changed:

1) exact results $\rightarrow$ scaling laws

2) entire capacity region $\rightarrow$ sum capacity, or transport capacity (Gupta-Kumar 00):

$$T_c(n) = \text{sup} \sum_{i,j=1}^{n} R_{ij} r_{ij}$$

## Setup



- network composed of $n$ nodes, independently and uniformly distributed in some (square) region of diameter $D$

- expanding network: $D \sim \sqrt{n}$ ($\leftrightarrow$ constant density of nodes)

- no fixed infrastructure (*ad hoc* network)

## Further general assumptions

- transmit power constraint $P$ on each node

- additive $\mathcal{N}(0,1)$ noise at each receiving node

- power attenuation $r^{-2\delta}$ over a distance $r$ (typically, $\delta \in [1,2]$)

- deterministic fading (the only randomness comes from the random distances between nodes)

**Theorem** (GK 00)

if $\delta > 1$ and we restrict ourselves to:

i) point-to-point communications
ii) interference being treated as noise,

then

$$T_c(n) = O(D\sqrt{n})$$

**However:**

what about group communication schemes?
or interference cancellation strategies?

$\Rightarrow$ need for information theoretic answers!

## First information theoretic results

a) Kumar-Xie (04-05), Jovicic&al (04-05)

if $\delta > 2.25$, then under no specific assumption on the way communication is established, we have

$$T_c(n) = O(D\sqrt{n})$$

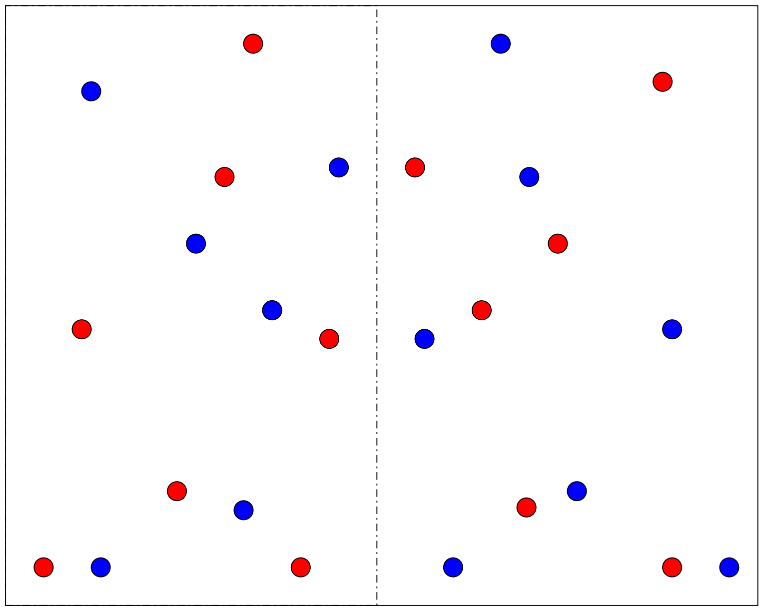
b) Lévêque-Telatar (05)

if $\delta > 1$, then under no specific assumption on the way communication is established, we have

$$T_c(n) = o(Dn)$$

**But** both these results miss the target, especially if one looks more closely at their proofs!

**Our approach** (red dots = added mirror users)



Max flow of information accross the boundary:

$$C_n \leq \sup_{p_X : E(|X_i|^2) \leq P, \, \forall i} I(X_1, ..., X_n; Y_1, ..., Y_n)$$

where

$$\begin{cases} Y_j = \sum_{i=1}^{n} G_{ij} X_i + Z_j \\ G_{ij} = \frac{1}{\|x_i - y_j\|^\delta} \end{cases}$$

**Successive majorizations...**

$$\Rightarrow \quad C_n \leq 2 \log\det(I + \sqrt{nP}\, G)$$

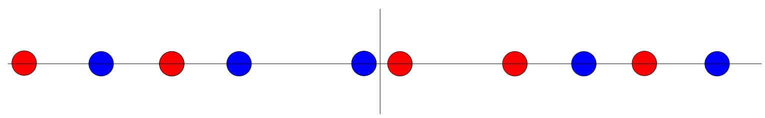
Ultimate hope: show that $C_n = O(\sqrt{n})$
(as this would imply $T_c(n) = O(D\sqrt{n})$)

**However:**

- no simple upper bound on the determinant gives a satisfying answer

- classical random matrix theory is useless in this context

## A simpler case: one-dimensional networks



$$\Rightarrow \; G_{ij} = \frac{1}{(x_i + x_j)^\delta}$$

**Theorem** (Leveque-Preissmann, IT Trans 05)

if $\delta \in [1,2]$, then under no specific assumption on the way communication is established,

$$T_c(n) = O(D(\log n)^{3+\varepsilon})$$

**Note:** in this setting, Gupta-Kumar's scaling law reads:

$$T_c(n) = O(D)$$

## Proof idea

1) in order to upperbound $\det(I+G)$, use

$$\det(I+G) = 1 + \mathrm{Tr}(G) + ... + \det(G)$$
$$= \sum_{J\subset\{1,\dots,n\}} \det(G_J)$$

so one only needs to upperbound $\det G$

2) let $D_\delta = \det G = \det\left(\left\{\frac{1}{(x_i+x_j)^\delta}\right\}\right)$

integral representation of the determinant
+ Hölder's inequality

$$\Rightarrow \quad D_\delta \le C_\delta (D_1)^{2-\delta}(D_2)^{\delta-1} \quad \forall\delta\in[1,2]$$

so one only needs to upperbound $D_1$ and $D_2$

**Proof idea** (cont'd)

3) Cauchy and Borchardt formulas:

$$D_1 = \det\left(\left\{\frac{1}{x_i + x_j}\right\}\right) = \frac{\prod_{i<j}(x_i - x_j)^2}{\prod_{i,j}(x_i + x_j)}$$

$$D_2 = \det\left(\left\{\frac{1}{x_i + x_j}\right\}\right) \text{perm}\left(\left\{\frac{1}{x_i + x_j}\right\}\right)$$

4) key step: a refined analysis of the above expressions gives the following precise estimate:

$$D_1, D_2 \le \frac{1}{(d_{\text{min}})^n} \exp(-Kn^{3/2})$$

where $d_{\text{min}}$ = minimum distance between any two nodes in the network (typically, $d_{\text{min}} \sim \frac{1}{n}$)

**Proof idea** (conclusion)

5) gathering all estimates together leads to

$$C_n \le 2 \log \det(I + \sqrt{nP}\, G) \le K\, (\log n)^{3+\varepsilon}$$

6) implication on transport capacity:

$$\sum_{i<j} R_{ij} r_{ij} = \sum_{i<j} R_{ij} \sum_{k=i}^{j-1} r_{k,k+1}$$

$$= \sum_{k=1}^{n-1} r_{k,k+1} \sum_{i \le k < j} R_{ij} \le D\, C_n$$

□

## Remark

following the steps of the preceding proof, one sees that the scaling law

$$T_c(n) = O(D(\log n)^{3+\varepsilon})$$

still holds for both

- dense randomly distributed networks ($D$ cst)

- networks with an arbitrary placement of nodes separated by a minimum distance $d_{\text{min}}$

## Back to two-dimensional networks

in this setting, we have

$$G_{ij} = \frac{1}{((x_i + x_j)^2 + (y_i - y_j)^2)^{\delta/2}}$$

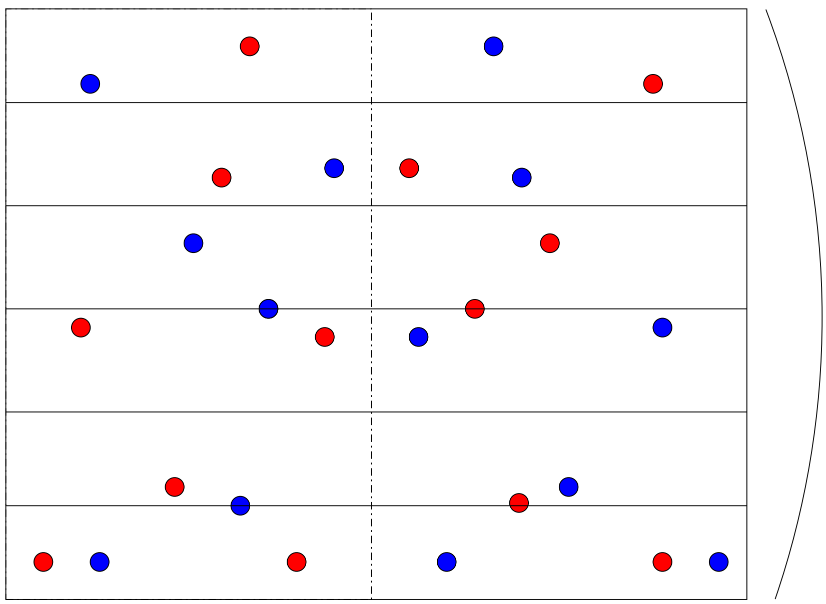
computing the determinant of such matrices is harder... nevertheless, we have the following:

**Theorem** (Özgür-Lévêque, IZS 06)

if $\delta \in [1, 2]$, then under no specific assumption on the way communication is established,

$$T_c(n) = O\left(D\, n^{\frac{1}{2} + \frac{1}{2(\delta+4)} + \varepsilon}\right)$$

**Proof idea:** get back to the 1D case!




$$C_n \leq 2 \log \det(I + \sqrt{nP}\, G)$$

$$\leq 2 \sum_{l=1}^{N} \log \det(I + \sqrt{nP}\, G^{(l)})$$

with $G^{(l)}$ the diagonal block of the matrix $G$ corresponding to the strip $l$.

**Proof idea** (cont'd)

1) averaging:

$$E_{X,Y}(\log\det(I+\sqrt{nP}G))$$

$$\leq E_X\left(\sum_{l=1}^{N}\log\det(I+\sqrt{nP}\,E_Y(G^{(l)}))\right)$$

this quantity turns out to be an $O(N(\log n)^{3+\varepsilon})$ for $N \geq n^{\frac{1}{2}+\frac{1}{2(\delta+4)}}$

2) concentration (use a theorem of McDiarmid and the interlacing property of eigenvalues):

$$|\log\det(I+\sqrt{nP}G)-E_{X,Y}(\log\det(I+\sqrt{nP}G))|$$

$$=O(n^{\frac{1}{2}+\varepsilon})$$

□

## Conclusion

in the main regime of interest ($\delta \in [1, 2]$), we have recovered Gupta-Kumar's square root law from an information theoretic point of view (up to a small polynomial factor)

**The following improvements are expected:**

- general result for $\delta \geq 1$

- exact scaling law up to $\log n$ factors

- results for dense or arbitrarily placed 2D networks