The largest eigenvalue of some Deformed Random Matrix Ensembles

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Plan

- I. Introduction
  Motivation for the study of the largest eigenvalue of random matrices.

- II. Standard random matrices.
  A review of known results.

- III. Deformed ensembles.
  - Definition.
  - Phase transition for the largest eigenvalues of the Deformed GUE.
  - Extensions to other ensembles.

- IV. Concluding remarks.

Random matrices : basic model

Let μ (resp.μ') be a probability distribution on $\mathbb{C}$ (resp. on $\mathbb{R}$) with finite variance.

- a $N \times N$ Hermitian random matrix

$$H_N = \frac{1}{\sqrt{N}}(H_{ij}), \; H_{ij}, i \leq j \text{ i.i.d. of distribution } \mu \; (\mu' \text{ on the diagonal})$$

Ex : GUE $\mu = \mathcal{N}(0, \sigma^2)$ (complex) and $\mu' = \mathcal{N}(0, \sigma^2)$ (real).

- a $N \times N$ complex sample covariance matrix :
  $\Sigma > 0$ a $N \times N$ deterministic covariance matrix , $Y = \Sigma^{1/2} X$ $(X : N \times p(N))$,

$$M_N = \frac{YY^*}{N}, \; X = (X_{ij}), \text{ matrix with i.i.d. entries of distribution } \mu.$$

Question: spectral properties of such random matrices as $N \to \infty$?

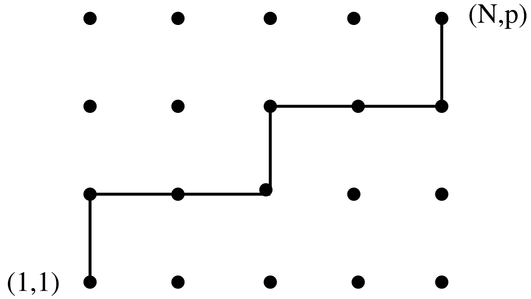
Largest eigenvalue of RM and Queues

N customers entering in a series of p files : waiting times $e(i,j)$ $1 \leq i \leq N$, $1 \leq j \leq p$ i.i.d. of distribution μ s.t. $\int x d\mu(x) = m$ and $\int |x|^k d\mu < \infty, \forall k$.

$L(N,p)$ departure time of the N^{th} customer of the p^{th} file.

$$L(N,p) = \sup_{\pi \in \Pi} \sum_{(i,j) \in \pi} e_{(i,j)},$$

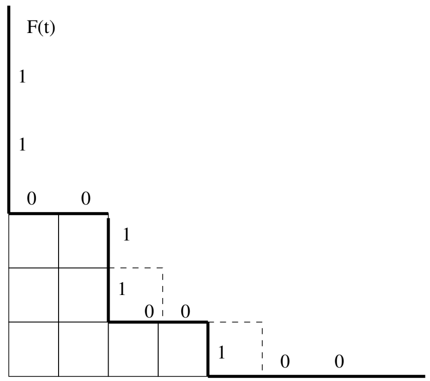
where Π is the set of up/right paths from $(1,1)$ to (N,p).



Th: Bodineau Martin (2005) If $p, N \to \infty, p << N^{3/7}$ then $N^{-1/2}(L(N,p) - (N+p)m)$ behaves as the largest eigenvalue of a $p \times p$ GUE.

Corner growth model : Johansson (99)

$A(0) = \emptyset$. Growing rule for $A = A(t)$: $A(t) = \cup_{k=1}^{r}[k-1,k] \times [0,\lambda_k], \lambda_i \geq \lambda_{i+1}$.
Call $\delta^* A(t)$ the cubes adjacent to $A(t)$ with a cube in $A(t)$ on the left and below. Pick independently each cube in $\delta^* A(t)$ with probability p and add it to obtain $A(t+1)$.



$$G^*(N,p) = \sup \sum_{(i,j)\in\pi, \pi\in\Pi} g_{(i,j)} + 1, \text{ with } g_{ij} \text{ i.i.d. } P(g_{ij} = k) = (1-p)^k p.$$

$$G^*(N,p) = k \iff [N-1,N] \times [p-1,p] \text{ added at time } k.$$

$F(t)$ the frontier of $A(t)$ made of vertical lines (affected 1) or horizontal lines (resp. 0).

Connexion with TASEP : Rost (1981)

$X(t)$ sequence of $0, 1$ obtained reading the frontier $F(t)$ from y-axis to x-axis.
Define then a "configuration": $x_o(t)$ last value before crossing the diagonal

$$X(t) = (\dots, 1 \dots, x_{-1}(t), x_0(t), x_1(t), \dots, 0, \dots), \quad X(0) = (\dots, 1 \dots, 1, 0, \dots, 0, \dots).$$

$$x_i(t) = 1 \iff \text{ a particle is at site i}$$

The particle at site i moves to $i + 1$ (asymmetric) if $x_{i+1} = 0$ (exclusion) with probability p. Then,

$$G^*(N, p) = k$$

if at time k, the particle started at $-(p - 1)$ has made N steps to the right.

Connection with the PNG model (Praehofer-Ferrari-Spohn) also.

Applications to finance and statistics

A portfolio $\mathcal{P}$ of N assets with weight w_i, $i = 1, \ldots, N$.
Σ the covariance matrix of the returns.

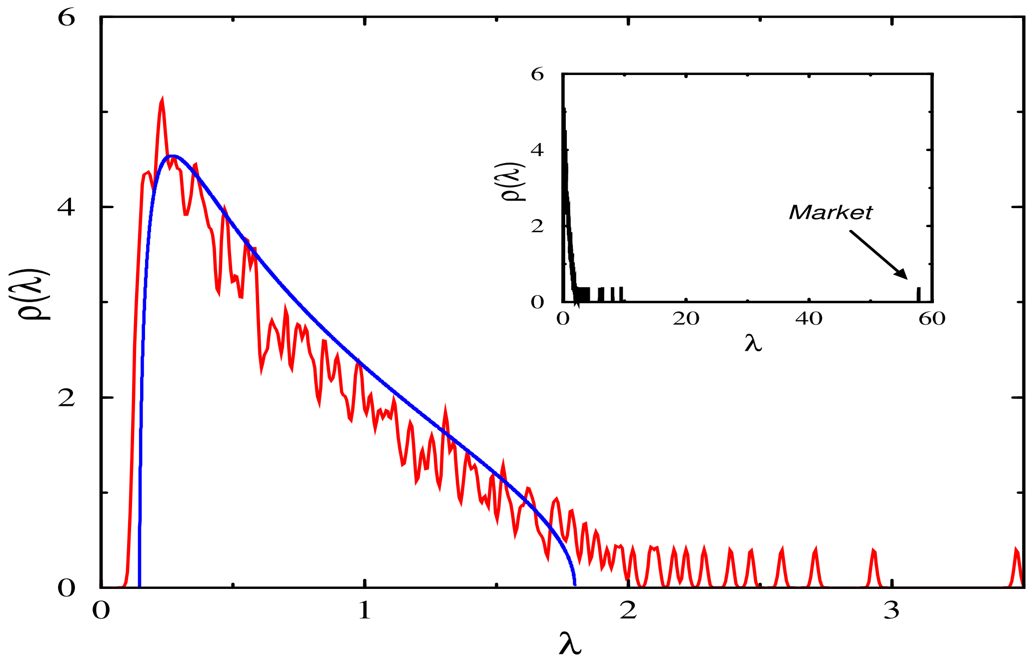
$$\text{Daily variance of return to be minimized : } R^2 = \sum_{i,j=1}^{N} w_i w_j \Sigma_{ij}.$$

Optimal $\mathcal{P}$: maximal weight on eigenvectors associated to the smallest eigenvalues of Σ.

Problems : Given Y, the $N \times p$ matrix of returns observed during a length p period and the sample covariance matrix YY^*,
1. estimate for Σ unknown in general;
2. find a way to erase the effect of large eigenvalues (principal component analysis).

Largest eigenvalue and finance

Bouchaud-Potters-Laloux Financial Applications of RMT (2005)



Empirical eigenvalue density for 406 stocks from the S&P 500, and fit using the MP distribution. Note the presence of one very large eigenvalue.

Standard Hermitian (symmetric) random matrices

$H_{ij}, i \leq j$, i.i.d. with distribution μ (or μ' on the diagonal) such that

$$\int x d\mu = 0, \int |x|^2 d\mu = \sigma^2, \int |x|^2 d\mu' < \infty \text{ and } \int |x|^4 d\mu < \infty.$$

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ the eigenvalues of $H_N = \frac{1}{\sqrt{N}} H$, $\mu_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$.

Theorem Wigner (55):

$$\lim_{N \to \infty} \mu_N = \sigma_{sc}, \text{ with } \frac{d\sigma_{sc}}{dx} = \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2}\, 1_{[-2\sigma, 2\sigma]}(x) \text{ a.s.}$$

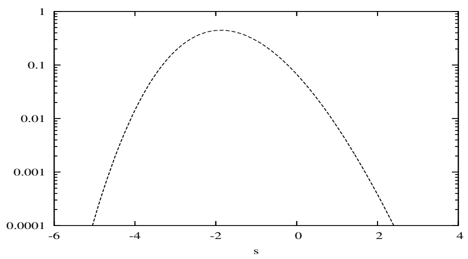
Behavior of extreme eigenvalues?

The largest eigenvalue of Hermitian ensembles

A.s. limit : Geman (80) Bai-Yin-Krishnaiah (88)
If $\int |x|^4 d\mu(x) < \infty$ and $\int |x|^2 d\mu'(x) < \infty$, then $\lim_{N\to\infty} \lambda_1 = 2\sigma$ a.s.

Fluctuations : Forrester(93)-Tracy-Widom (94)
λ_1 the largest value of the GUE i.e. $\mu = \mathcal{N}(0, \sigma^2)$, $\mu' = \mathcal{N}(0, \sigma^2)$,

$$\lim_{N\to\infty} P\left(N^{2/3}\left(\frac{\lambda_1}{2\sigma} - 1\right)\right) = F_2(x),$$ Tracy Widom distribution.



$F_2(x) = \det(I - A_x)$, A_x operator on $L^2(x, \infty)$ with the so-called Airy kernel

$$Ai(u, v) = \frac{Ai(u)Ai'(v) - Ai'(u)Ai(v)}{u - v}.$$

Universality of the Tracy-Widom distribution : 1

• **Non Gaussian Wigner matrices** Soshnikov (99):
μ a non Gaussian symmetric distribution with sub-Gaussian tails

$$\exists C > 0,\ \forall k > 0,\ \int |x|^{2k} d\mu(x) \leq (Ck)^k \text{ and } \int |x|^2 d\mu(x) = \sigma^2.$$

$$\lim_{N\to\infty} P\Big(N^{2/3}\left(\frac{\lambda_1}{2\sigma} - 1\right)\Big) = F_2(x).$$

• **Invariant ensembles** Deift and al. (99):

$$dP_N(H_N) \propto \exp\{-NTrV(H_N)\} dH_N, \text{ with } dH_N \text{ the Lebesgue measure on } \mathcal{H}_N$$

For a wide class of V, $\exists\, C$, u_+,

$$\lim_{N\to\infty} P\left(CN^{2/3}(\lambda_1 - u_+) \leq x\right) = F_2(x).$$

Universality of the Tracy-Widom distribution : 2

• **Random sample covariance matrices** Johansson (2000) Johnstone (2001) Soshnikov (2001)
μ_N spectral measure of $\frac{1}{N}XX^*$, $p/N \to \gamma \geq 1$ then $\mu_N \to \mu_{MP}^{\gamma} \neq \sigma_{sc}$,
μ_{MP}^{γ} so-called Marchenko-Pastur distribution.

If X_{ij} i.i.d. of distribution μ symmetric such that $\exists C > 0$

$$\int |x|^{2k} d\mu(x) \leq (Ck)^k, \ \forall\, k > 0 \text{ and } \int |x|^2 d\mu(x) = \sigma^2,$$

and $p - N = o(N^{1/3})$,

$$\exists\ C,\ u_+\ ,\ \lim_{N\to\infty} P\left(CN^{2/3}\left(\lambda_1 - u_+\right) \leq x\right) = F_2(x).$$

Deformed Hermitian Random Matrix Ensembles

Definition

Let k be a fixed integer (independent of N) and A_N be a $N \times N$ deterministic Hermitian matrix of rank k.

$\pi_1 \geq \pi_2 \geq \cdots \geq \pi_k$ the non zero eigenvalues of A_N (π_i independent of N).

H_N a $N \times N$ classical random matrix.

$$W_N = \frac{1}{\sqrt{N}} H_N + A_N.$$

Examples :

random matrix with non centered entries : $A_N = \frac{\pi_1}{N} J,\ J_{ij} = 1,\ \forall\, i, j.$

a diagonal perturbation : $A_N = \mathrm{diag}(\pi_1, 0, \ldots, 0)$

Rk: Similar definition for deformed real symmetric random matrices.

Famous Deformed Random Matrix Ensembles

- Komlos-Furedi (80) Graph theory.
  Adjacency matrix of a random graph : a random matrix with non centered entries.

- The Deformed GUE ensemble : Brezin-Hikami-Johansson- Praehofer-Spohn etc
  H_N with complex Gaussian $\mathcal{N}(0, \sigma^2)$ entries (real on the diagonal),
  A_N any deterministic matrix (can be of rank N) with eigenvalues $\pi_1 \geq \pi_2 \geq \cdots \geq \pi_N$.
  Mathematically "simple" ensemble whose spectrum has the same law as Brownian motions conditioned never to collide started at $\pi_1, \ldots, \pi_N$.

- Queing system of N files: what is the effect of waiting times $e(i, j)$ with geometric distribution of parameter p_i depending on the file ?

Objectives and strategy

Question: influence of the deformation on the spectrum ?

- rank k
- the values of the $\pi_i, i \leq k$.

Global behavior unchanged

$$\lim_{N\to\infty} \mu_N = \sigma_{sc} \text{ a.s. } .$$

What about extreme eigenvalues ?

Focus first on the Deformed GUE that is when H_N has Gaussian entries.
Compare and extend the results to more general ensembles.

A phase transition for the Deformed GUE

Deformed GUE: H_N with Gaussian $\mathcal{N}(0,\sigma^2)$ entries, π_1 of multiplicity r.

- If $\pi_1 < \sigma$, then, $\lim_{N\to\infty} P\left(N^{2/3}\left(\frac{\lambda_1^G}{\sigma} - 2\right) \leq x\right) = F_2(x)$.

- If $\pi_1 = \sigma$, then, $\lim_{N\to\infty} P\left(N^{2/3}\left(\frac{\lambda_1^G}{\sigma} - 2\right) \leq x\right) = F_{r+2}^{TW}(x)$. (different distribution which depends on r).

- If $\pi_1 > \sigma$ then, $\lim_{N\to\infty} P\left(\sigma^2(\pi_1) N^{1/2}\left(\frac{\lambda_1^G}{\sigma} - C(\pi_1)\right) \leq x\right) = F_{GUE,\sigma^2(\pi_1)}^{r}(x)$, where

$$C(\pi_1) = \frac{\pi_1}{\sigma} + \frac{\sigma}{\pi_1} \quad \text{and} \quad \sigma^2(\pi_1) = \frac{\pi_1^2}{\pi_1^2 - \sigma^2}.$$

  and $F_{GUE,\sigma^2(\pi_1)}^{r}$ is the distribution of the largest eigenvalue of a fixed size $r \times r$ GUE random matrix with entries of variance $\sigma^2(\pi_1)$.

More general results

• First such result: Baik Ben Arous P. (2004) Complex Wishart Ensembles.

• D. Paul (2005) Real sample covariance matrix (and Deformed GOE) : Gaussian fluctuations for the largest eigenvalue in the case of a rank one deformation.

Is the phase transition a Gaussian phenomenon ?

• D. Féral (2005). Almost sure limit of the largest eigenvalue.
H_{ij} i.i.d. real or complex of distribution μ with finite fourth moment. Then,

$$\lim_{N\to\infty} \lambda_1 = 2\sigma \text{ a.s if } \pi_1 \leq \sigma,$$

$$\lim_{N\to\infty} \lambda_1 = C(\pi_1)\sigma \text{ a.s. if } \pi_1 > \sigma.$$

Universal results

Theorem : D. Féral S.P. (2006)

H_N complex random matrix with i.i.d. entries with a symmetric distribution μ s.t.

- $\int |x|^2 d\mu(x) = \sigma^2$ and $\exists\, C > 0$ $\int |x|^{2k} d\mu(x) \leq (Ck)^k, \forall\, k$.
- $A_N = \frac{\pi_1}{N} J$ with $J_{ij} = 1, \forall i, j$: a rank one perturbation.

Let λ_1 be the largest eigenvalue of $M_N = \frac{1}{\sqrt{N}} H_N + A_N$. Then,

$$\lim_{N\to\infty} P\left(N^{2/3} \left(\frac{\lambda_1}{\sigma} - 2 \right) \leq x \right) = F_2(x), \text{ if } \pi_1 < \sigma.$$

$$\lim_{N\to\infty} P\left(N^{2/3} \left(\frac{\lambda_1}{\sigma} - 2 \right) \leq x \right) = F_3^{TW}(x), \text{ if } \pi_1 = \sigma.$$

$$\lim_{N\to\infty} P\left(\sigma^2(\pi_1) N^{1/2} \left(\frac{\lambda_1}{\sigma} - C(\pi_1) \right) \leq x \right) = F^1_{GUE,\sigma^2(\pi_1)}(x), \text{ if } \pi_1 > \sigma.$$

A few comments

• Results can be extended to complex sample covariance matrices with non-centered entries if $p - N = o(N^{1/3})$, where p is the size of the sample.

• Universality results for Deformed Hermitian ensembles for ranks $k \geq 1$ can be proved also (forthcoming paper).

• Deformed real symmetric random matrices : if $F_1(x)$ is the limiting distribution of the largest eigenvalue of the (non Deformed) GOE ($\mu = \mathcal{N}(0, \sigma^2)$ real)

$$\lim_{N\to\infty} P\left(N^{2/3}\left(\frac{\lambda_1}{\sigma} - 2\right) \leq x\right) = F_1(x), \text{ if } \pi_1 < \sigma.$$

Rk: Same assumptions on the distribution of the entries as in the Hermitian case.
Once the limiting behavior of the largest eigenvalue of the Deformed GOE is known, our results imply universality.

Distribution of the largest eigenvalue : tools

Strategy : Compute the limiting distribution for the Deformed GUE then extend it to other ensembles.

Basic tool for the study of the Deformed GUE: correlation functions.

P_N a probability measure on $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$ with density g. The m-point correlation function is

$$R_N^m(x_1, \cdots, x_m) = \frac{N!}{(N-m)!} \int g(x_1, ..., x_N) dx_{m+1} ... dx_N.$$

In particular

$$P(\lambda_1 \leq x) = \sum_0^\infty \frac{(-1)^m}{m!} \int_x^\infty \cdots \int_x^\infty R_m(x_1, \ldots, x_m) dx_1 \cdots dx_m.$$

Deformed GUE

Follow strategy developped by Johansson (2001) for the study of Wigner matrices.

- *Computation of the j.e.d. given a fixed* A_N *with distinct eigenvalues* $\pi_j, j = 1, \ldots, N$.
  (Itzykson Zuber, 80, Harisch Chandra 57)

$$g(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z_N} \prod_{i=1}^{N} e^{-N\lambda_i^2/2} \det(e^{-N\pi_j \lambda_k}) \frac{V(\lambda)}{V(\pi)}.$$

- *Correlation functions* $R_m(x_1, \ldots, x_m) = \det\left(K_N(x_i, x_j)\right)_{i,j=1}^{m}$,

with $$K_N(u, v; \Sigma) = \frac{N}{(2i\pi)^2} \int_\Gamma dz \int_\gamma dw e^{N(w^2/2 - vw - z^2/2 + uz)} \frac{1}{w - z} \prod_{i=1}^{k} \frac{w - \pi_i}{z - \pi_i} (\frac{w}{z})^{N-k},$$

a technical fact : Γ has to encircle the eigenvalues π_i (and $\Gamma \cap \gamma = \emptyset$).
Final step : saddle point analysis.

Extension to more general ensembles

Computation of high moments :
for instance if $\pi_1 > \sigma$, $\rho_{\pi_1} = \pi_1 + \frac{\sigma^2}{\pi_1}$ one should have

$$\mathbb{E}Tr\left(\frac{M_N}{\rho_{\pi_1}}\right)^{t\sqrt{N}} \sim L_{\pi_1}(t),$$

where $L_{\pi_1}(t)$ is the Laplace transform of the Gaussian distribution $\mathcal{N}(0, \sigma^2(\pi_1))$.

We prove that

$$\left|\mathbb{E}Tr\left(\frac{M_N}{\rho_{\pi_1}}\right)^{t\sqrt{N}} - \mathbb{E}Tr\left(\frac{M_N^G}{\rho_{\pi_1}}\right)^{t\sqrt{N}}\right| = o(1)$$

if M_N^G is of the Deformed GUE.

Develop the expectation of the trace and show that the “relevant” terms only depend on σ and π_1.

Conclusion

- Diagonal perturbation/ rotational invariance of the GUE distribution. If $A_N = \mathrm{diag}(\pi_1, 0, \dots, 0)$ with $\pi_1 > \sigma$, then still $\lim_{N\to\infty} \lambda_1 = C(\pi_1)\sigma$. But (forthcoming paper) the limiting fluctuations of the largest eigenvalue depends on μ.

- If $A_N = \pi_1(\sum_{i=1}^N E_{1i} + E_{i1} - E_{11})$, with $E_{ij} = (\delta_{i,j}(k,l))_{k,l=1}^N$, universality conjectured.

- Deformation more complex than the rank one case : the Deformed GUE case. A_N has a single eigenvalue $\pi_1 \neq 0$ of multiplicity ρ_N with

$$\rho_N \to \infty, \quad \frac{\rho_N}{N} \to 0,$$

then λ_1 has fluctuations of Tracy-Widom type (as the largest eigenvalue of a $\rho_N \times \rho_N$ GUE). Is it universal?