DIRECTED PATHS IN WEDGES

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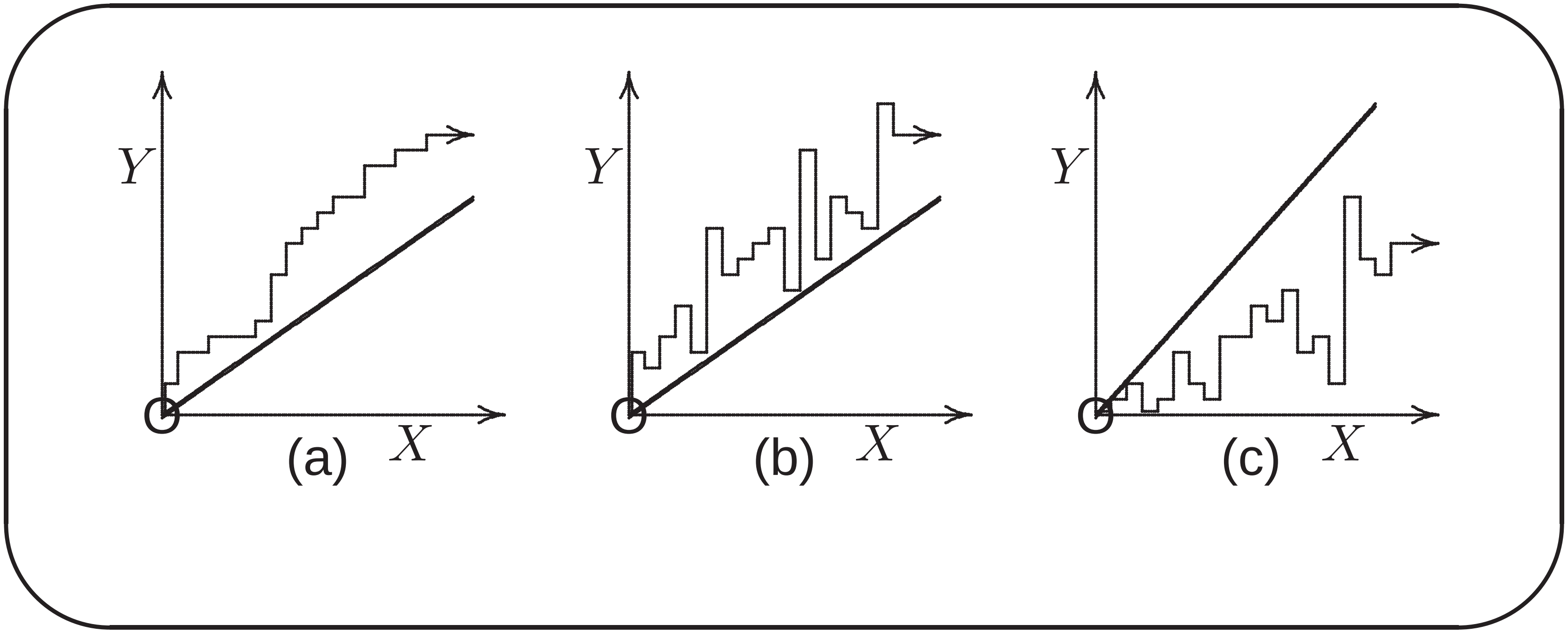
Work with Andrew Rechnitzer and Thomas Prellberg

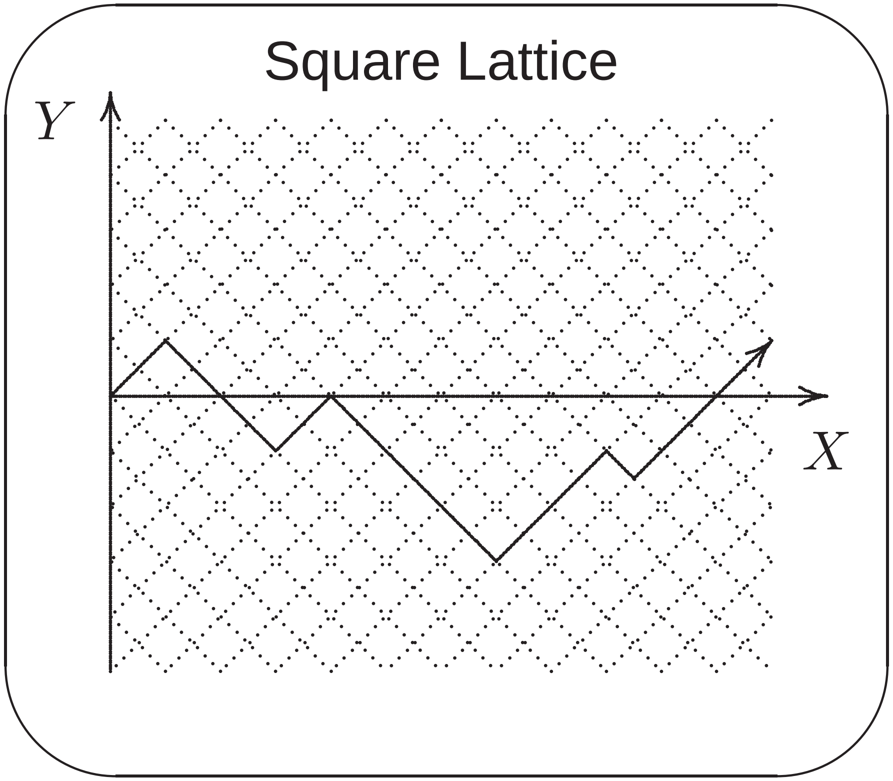
[Thanks to Frank and Stu for inviting me!]

MODELS OF DIRECTED PATHS IN WEDGES



- Lattice Paths Models in confined geometries pose challenging problems.
- Wedges: Translational symmetry may be lost.
- Recurrences may be difficult to determine.
- Recurrences may be hard to solve.
- Extracting information from generating functions may be difficult.

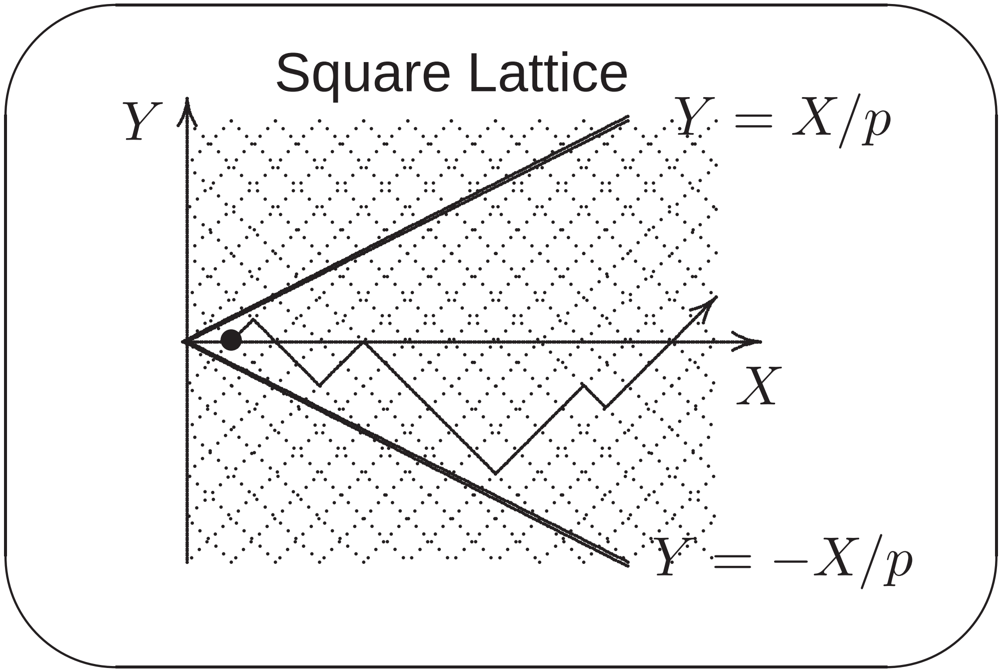
A DIRECTED PATH MODEL



- Directed Path in the Square Lattice.
- Steps only North-East (NE) or South-East (SE); step-length=$\sqrt{2}$.
- The number of such paths is

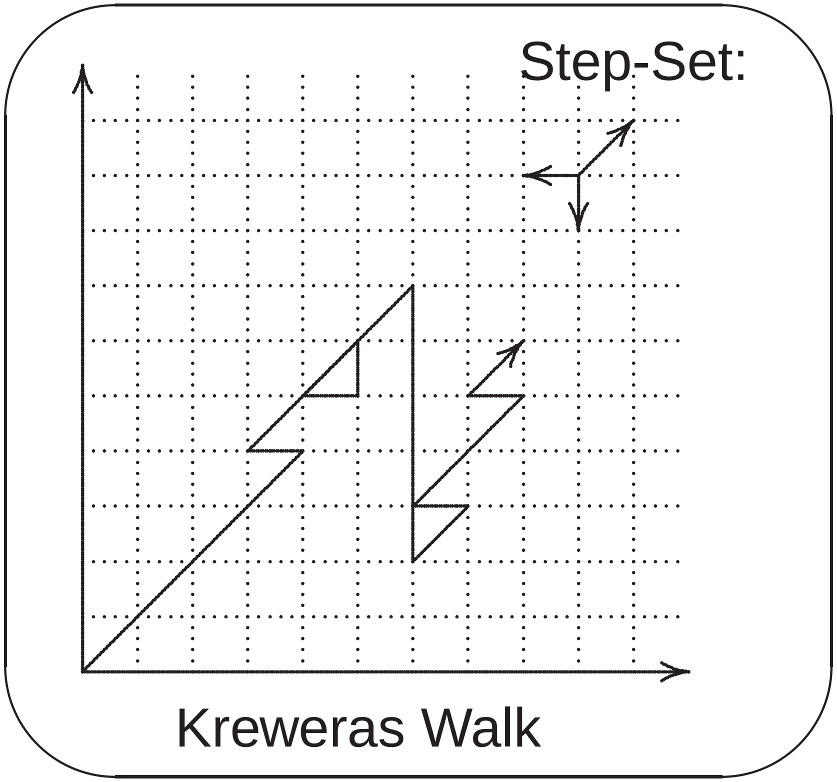
$$c_n = 2^n.$$

A DIRECTED PATH MODEL IN A WEDGE



- Directed Walk with NE and SE edges confined to $Y = \pm X/p$.
- Trivial if $p \leq 1$. Difficult otherwise.
  - Integer p?
  - Rational p?
  - Irrational p?

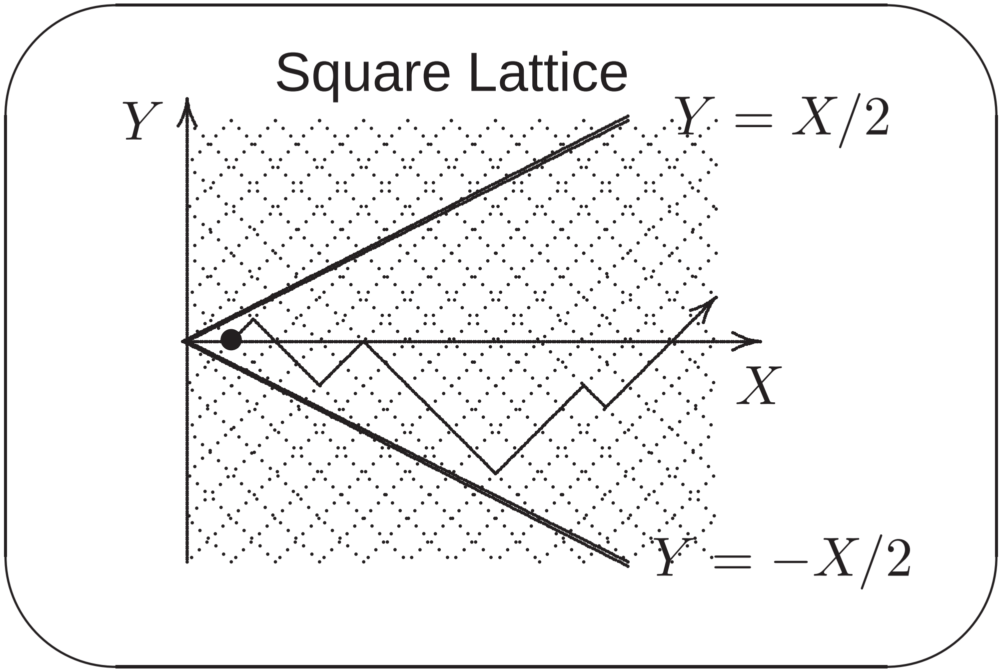
RELATED TO QUARTER PLANE WALKS



- Famous Model (Kreweras 1965)
- Counted by $\dfrac{4^n}{(n+1)(2n+1)}\dbinom{3n}{n}$
- Algebraic GF (Gessel 1986)
- Also Bousquet-Mélou 2005.

- The step-set of the Kreweras walk is $\{(1,1),(0,-1),(-1,0)\}$.
- Other step-sets have been studied (eg. Mishna and Rechnitzer 2007).
- The different models differ dramatically, many unsolved.

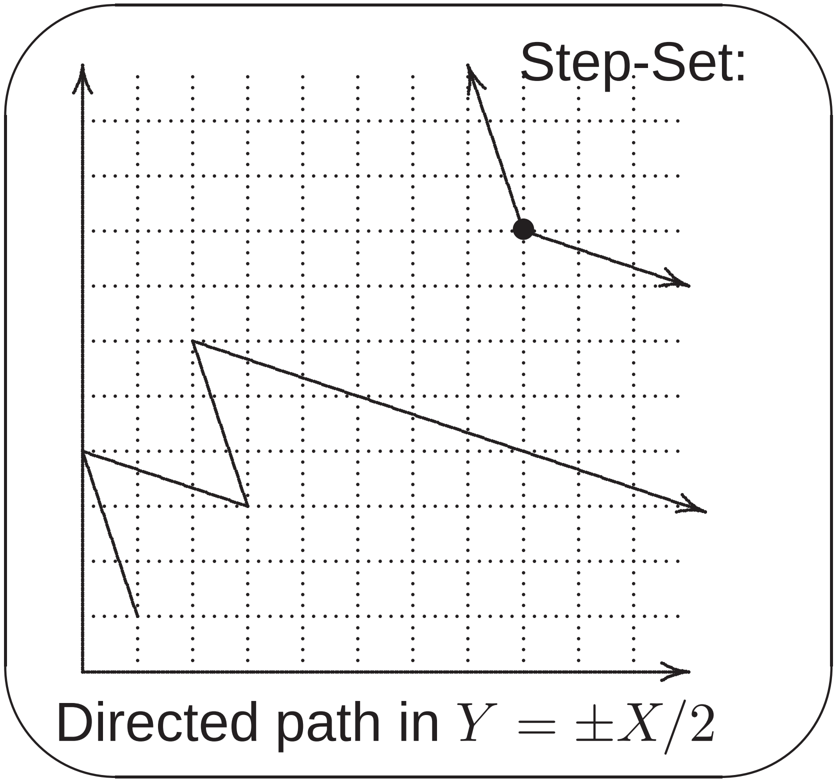
WEDGE MODEL FOR p=2



- Step-length = $\sqrt{2}$
- Starting vertex = $(2, 0)$

- These wedge models correspond to a class of quarter plane models.
- For $p = 2$, every 2 steps in the quarter plane model give a term in the generating function of even length paths in the wedge.

QUARTER PLANE VERSION FOR p=2



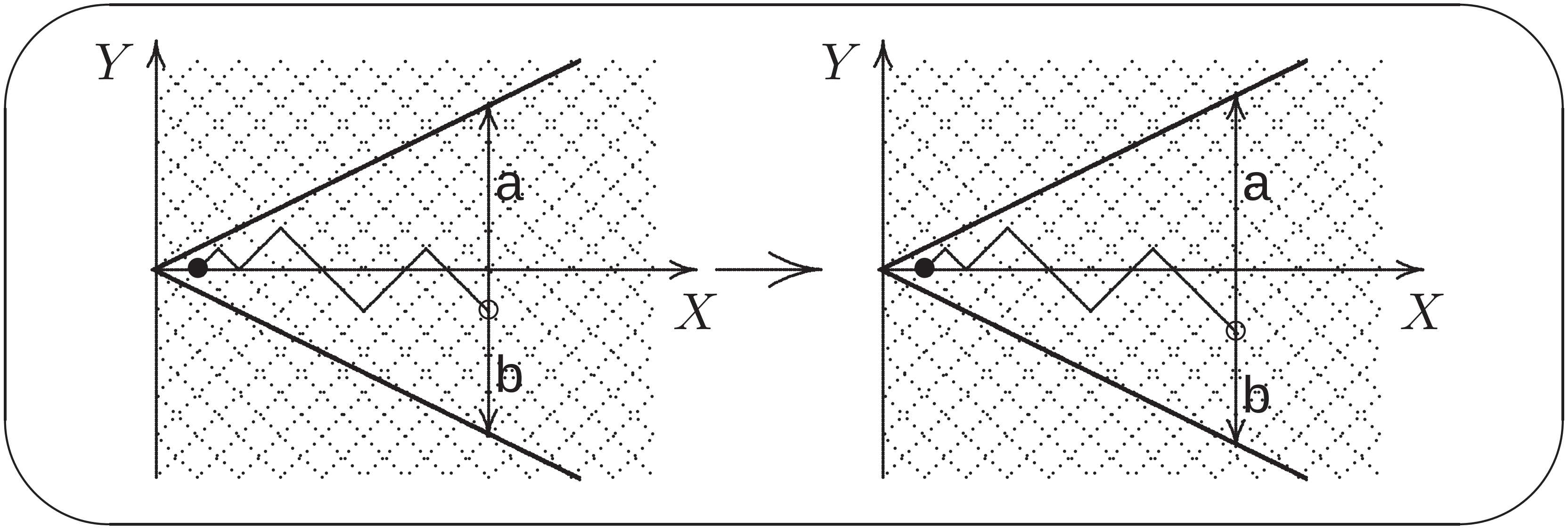
- Step set: $\{(-1,3),(3,-1)\}$.
- Starting vertex: $(1,1)$.

- "Knight's Walk" has step set (Bousquet-Mélou and Petkovšek 2003)

$$\{(-1,2),(2,-1)\}$$

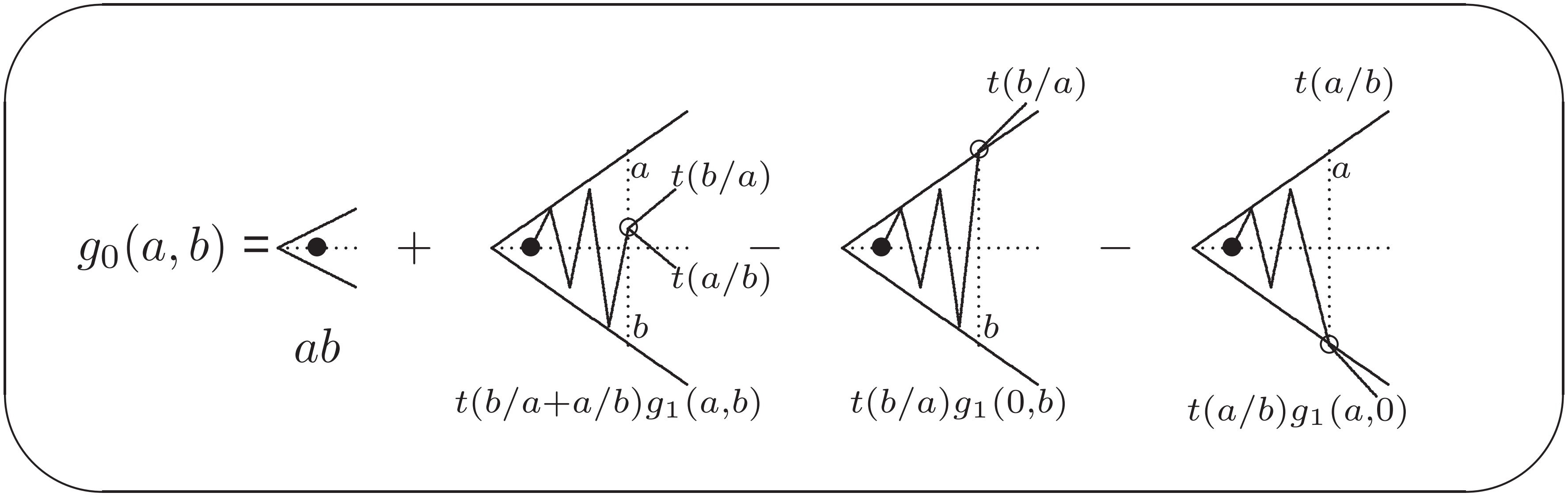
- Wedge model $\Leftrightarrow$ Generalized Knight's Walk.

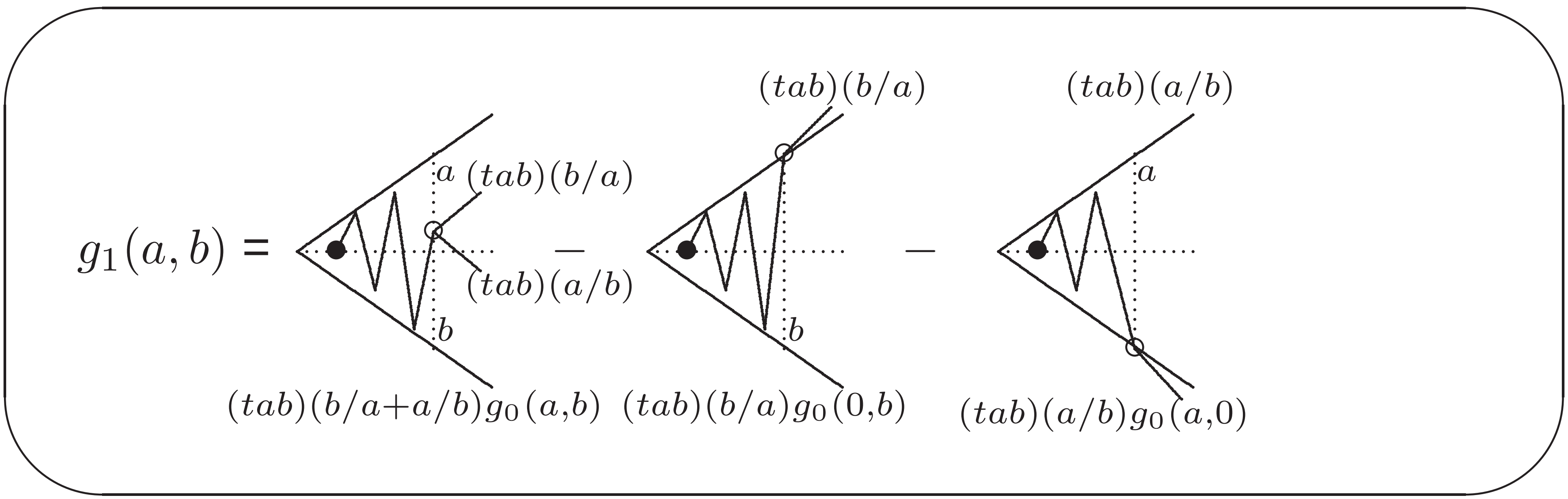
RECURRENCE RELATION



- Introduce a and b to track vertical distance.
- Generating variable t.
  - $g_0(a, b)$ == Even length paths.
  - $g_1(a, b)$ == Odd length paths.
- Add a step, and subtract out paths out of bounds.

FUNCTIONAL RECURRENCE I





FUNCTIONAL RECURRENCE II

- The two-step process gives

$$\begin{aligned} g_0(a,b) &= ab + t(a/b + b/a)\, g_1(a,b) - t(a/b)\, g_1(a,0) - t(b/a)\, g_1(0,b) \\ g_1(a,b) &= t(a^2 + b^2)\, g_0(a,b) - ta^2 g_0(a,0) - tb^2 g_0(0,b) \end{aligned}$$

- The GF is $G(1,1) = g_0(1,1) + g_1(1,1)$.
- Paths of odd length cannot intersect the boundary: $g_1(a,0) = g_1(0,b) = 0$.
- Put

$$\begin{aligned} L &= t(a/b + b/a) \\ M &= t(a^2 + b^2) \end{aligned}$$

- And Substitute...

$$g_0(a,b) = ab + LM\, g_0(a,b) + ta^2 g_0(a,0) - tb^2 g_0(0,b).$$

FUNCTIONAL RECURRENCE III

- Rearrange the recurrence:

$$(1 - LM)g_0(a,b) = ab + ta^2 g_0(a,0) - tb^2 g_0(0,b).$$

- The function $K(a,b) = 1 - LM = (ab - t^2(a^2 + b^2)^2)/ab$ is the *KERNEL*.
- Stepping twice in the quarter-plane gives kernel

$$K_q(X,Y) = (X^2Y^2 - (X^4 + Y^4)^2)/X^2Y^2$$

- The identifications $ta^2 = X^4$ and $tb^2 = Y^4$ map between these models.

THE KERNEL METHOD

The recurrences have the general form

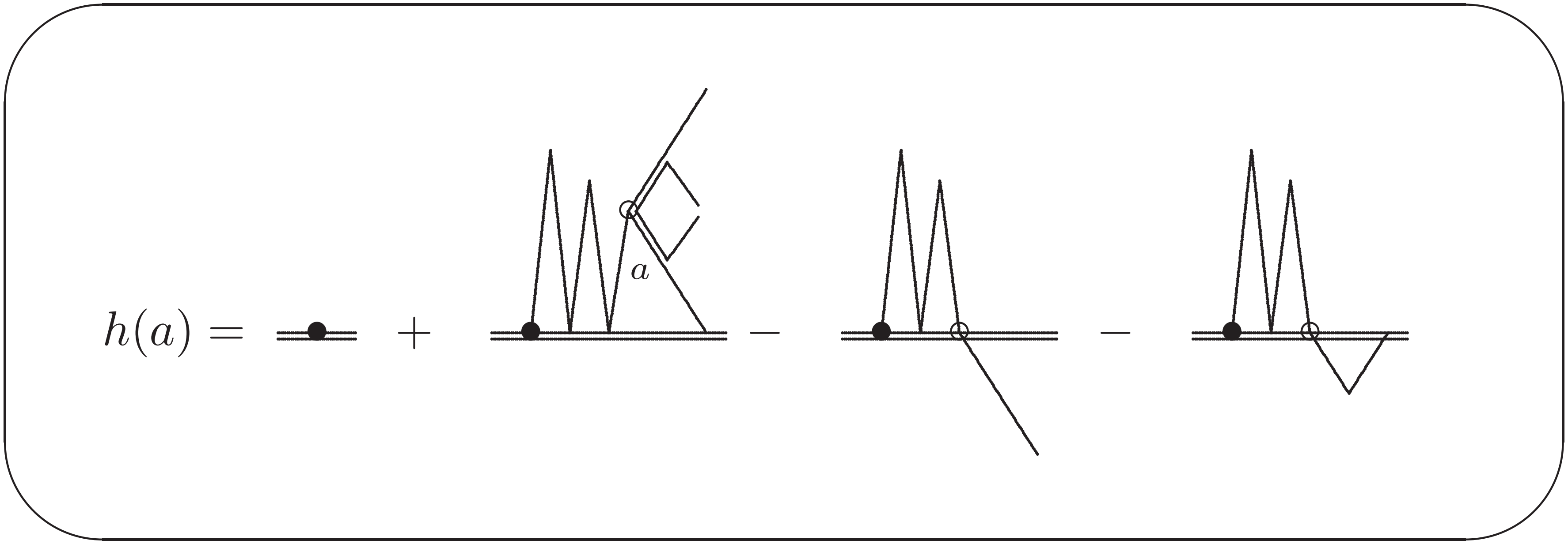
$$K(a, b, \ldots)\, G(a, b, \ldots) = 1 + \text{SOME STUFF}$$

This is solved by

- The variables a and b are catalitic.
- Examine $K(a, b, \ldots) = 0$; say the roots are $b = \beta$.
- Solve for SOME STUFF by noting $K(a, \beta, \ldots) = 0$.
- Then

$$G(a, b, \ldots) = \frac{1 + \text{SOME STUFF}}{K(a, b, \ldots)}$$

EXAMPLE: DIRECTED PATHS ABOVE A LINE I



Consider even length directed paths above the line $X = 0$.

$$\left(1 - t^2(a + 2 + 1/a)\right) h(a) = 1 - t^2(1 + 1/a)h(0).$$

EXAMPLE: DIRECTED PATHS ABOVE A LINE II

The kernel is

$$K(a) = 1 - t^2(a + 2 + 1/a) = 0.$$

The roots of the kernel is

$$a = a_{\pm} = \frac{1 - 2t^2 \pm \sqrt{1 - 4t^2}}{2t^2}$$

Thus $h(0)$ can be found, and finally the solution of the recurrence

$$h(a) = \frac{1-t^2(1+1/a)h(0)}{1-t^2(a+2+1/a)} = \frac{a-\frac{1}{2}(a+1)(1-\sqrt{1-4t^2})}{a-t^2(a+1)^2}$$

VARIANTS OF THE KERNEL METHOD

The Kernel method does not work in all cases.

- In some models, killing the kernel does not give SOME STUFF.
- Variant: The Obstinate Kernel Method:
  - For example, a quadratic kernel $K(a, b)$ in a and b:
  - Solve $K(a, b) = 0$, roots $\beta_{\pm}(a)$ and $\alpha_{\pm}(b)$.
  - Pairs $(a, \beta_{+}(a))$ and $(a, \beta_{-}(a))$ kill the kernel.
  - Pairs $(\alpha_{+}(b), b)$ and $(\alpha_{-}(b), b)$ kill the kernel.
  - "Physical roots" $\alpha_{-}(b)$ and $\beta_{-}(a)$ are "legitimate".
- Proceed roughly as follows:
  - Examine the products $\alpha_{+}(b)\alpha_{-}(b)$ and $\beta_{+}(a)\beta_{-}(a)$
  - Set up involutions ϕ and ψ on $(a, \beta_{-}(a))$ which 'swap roots' to other pairs.
  - With luck, this will generate an orbit on pairs killing $K(a, b)$.
  - Some of these pairs are legitimate; they determine SOME STUFF

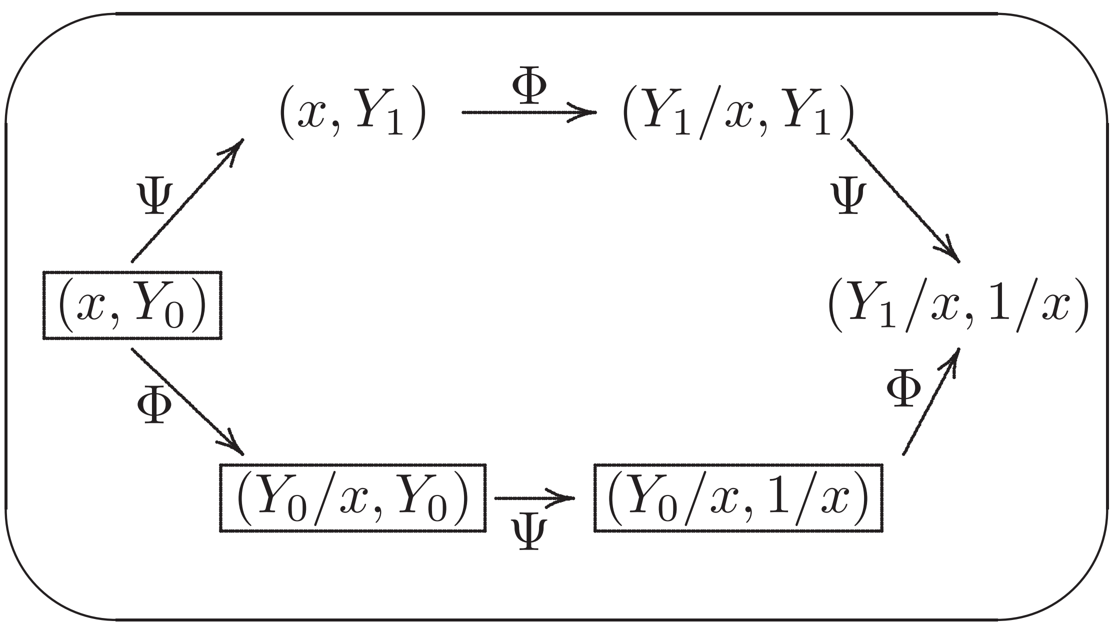
OBSTINATE KERNELS

For example, for three osculating walks (MBM2006)

$$K(x, y) = xy - t(1 + x)(1 + y)(x + y)$$

with roots Y_0 and Y_1 where $Y_0 Y_1 = x$, and X_0 and X_1 where $X_0 X_1 = y$.

Define $\Phi(x, y) = (y/x, y)$ and $\Psi(x, y) = (x, x/y)$.



ITERATED KERNELS I

- Neither the Kernel Method, nor the Obstinate Kernel Method solved our functional equation.

- Consider the generic recurrence:
  $K(a,b)g_0(a,b) = ab - L(a)g_0(a,0) - L(b)g_0(0,b)$
  with $g(a,0) \equiv g(0,a)$.

- The Kernel is killed by $(a, \beta_\pm(a))$.

- But then the compositions of a root $\beta(a)$

$$K(a,\beta(a)) = K(\beta(a),\beta(\beta(a))) = \ldots = 0$$

  also kill the kernel.

- This is the basic idea.

ITERATED KERNELS II

- In our model

$$K(a,b)g_0(a,b) = ab - L(a)g_0(a,0) - L(b)g_0(0,b)$$

with $g(a,0) \equiv g(0,a)$.

- If $K(a,\beta(a)) = 0$, then

$$L(a)g_0(a,0) = a\beta(a) - L(\beta(a))g_0(0,\beta(a))$$

- Composing the roots to get $K(\beta_n,\beta_{n+1}) = 0$ gives

$$L(\beta_n)g_0(\beta_n,0) = \beta_n\beta_{n+1} - L(\beta_{n+1})g_0(0,\beta_{n+1})$$

where $\beta_n \equiv \beta^{(n)}(a) = \underbrace{(\beta \circ \beta \circ \ldots \beta)}_{n \text{ times}}(a)$.

ITERATED KERNELS III

- The basic idea is the repeated composition of the roots:

$$a \mapsto \beta(a)$$

in the recurrence.

- This produces a sequence of pairs killing the kernel:

$$K(a, \beta(a)) = K(\beta(a), \beta_2(a)) = \ldots K(\beta_n(a), \beta_{n+1}(a)) = \ldots = 0.$$

- Iterating gives an expression for $g_0(a, 0)$:

$$\begin{aligned} L(a)g_0(a,0) &= a\beta(a) - \beta(a)\beta(\beta(a)) + \beta(\beta(a))\beta(\beta(\beta(a))) - \ldots \\ &= \sum_{n=0}^{\infty} (-1)^n \beta^{(n)}(a)\beta^{(n+1)}(a) \end{aligned}$$

as a typical expression.

SOLVING THE FUNCTIONAL RECURRENCE I

- The functional recurrence for $g_0(a,b)$ in kernel form:

$$K(a,b)\,g_0(a,b) = ab - ta^2 g_0(a,0) - tb^2 g_0(0,b).$$

- The kernel contains a quartic factor

$$K(a,b) = (t^2(a^2+b^2)^2 - ab)/ab.$$

- Roots of the kernel

$$\begin{aligned}
\beta_0(a) &= a^3t^2 + 2a^7t^6 + 9a^{11}t^{14} + \dots \\
\beta_1(a) &= a^{1/3}t^{-2/3} - 2a^{5/3}t^{2/3}/3 - 28a^{13/3}t^{10/3}/81 + \dots \\
\beta_\pm(a) &= -a^{1/3}t^{-2/3}(1 \mp i\sqrt{3})/2 + a^{5/3}t^{2/3}(1 \pm i\sqrt{3})/3 - \dots
\end{aligned}$$

- $\beta_0(a)$ is “physical”, it is a powerseries.... Let $\beta_0(a) \equiv \beta_0$.

SOLVING THE FUNCTIONAL RECURRENCE II

- Put $b = \beta_0$. This gives a recurrence

$$g_0(a,0) = \frac{(a^2+\beta_0^2)\beta_0}{a} - \frac{\beta_0^2}{a^2}\, g_0(\beta_0, 0)$$

- This may be iterated to obtain

$$g_0(a,0) = \frac{1}{a^2}\sum_{n=0}^{\infty}(-1)^n\left((\beta_0^{(n)})^2 + (\beta_0^{(n+1)})^2\right)\beta_0^{(n)}\beta_0^{(n+1)}$$

where

$$\beta_0^{(n)} = \underbrace{(\beta_0 \circ \beta_0 \circ \beta_0 \circ \ldots \circ \beta_0)}_{\text{n times}}(a)$$

MORE ABOUT β_0 I

- One may check that

$$\beta_0(a) = 2t^2a^3 \sum_{n=0}^{\infty} \binom{4n+1}{n} \frac{(at)^{4n}}{3n+2}$$

- Defining the series

$$r_a^2 = a^2 \sum_{n=0}^{\infty} \binom{4n}{n} \frac{(at)^{4n}}{3n+1}$$

one may check as well that

$$a \cdot \beta_0 = t^2 r_2^4.$$

- Explicitly one can also compute

$$\beta_0(1) = \frac{5-\sqrt{33}}{12}(19+3\sqrt{33})^{2/3} + \frac{\sqrt{33}-1}{12}(19+3\sqrt{33})^{1/3} - \frac{1}{3}.$$

MORE ABOUT β_0 II

- The GF is given by

$$g_0(1,1) = \frac{1 - 2t^2\left(g_0(1,0) + g_0(0,1)\right)}{1 - 4t^2}$$

- Since $\beta_0(a) = a^3t^2 + O(a^7t^6)$ the recurrence

$$\beta_0^{(n+1)} = \beta_0(\beta_0^{(n)})$$

  is third order: The series for the GF converges fast.

- One can show that

$$\boxed{\left|\beta_0^{(n)}(1)\right| \leq \frac{1}{\sqrt{3^n}}.}$$

- Dominant singularity are simple poles at $t = \pm 1/2$.
- $g_0(a,0)$ has branch points on the circle $|at| = 3^{3/4}/4$.

ASYMPTOTIC RESULTS

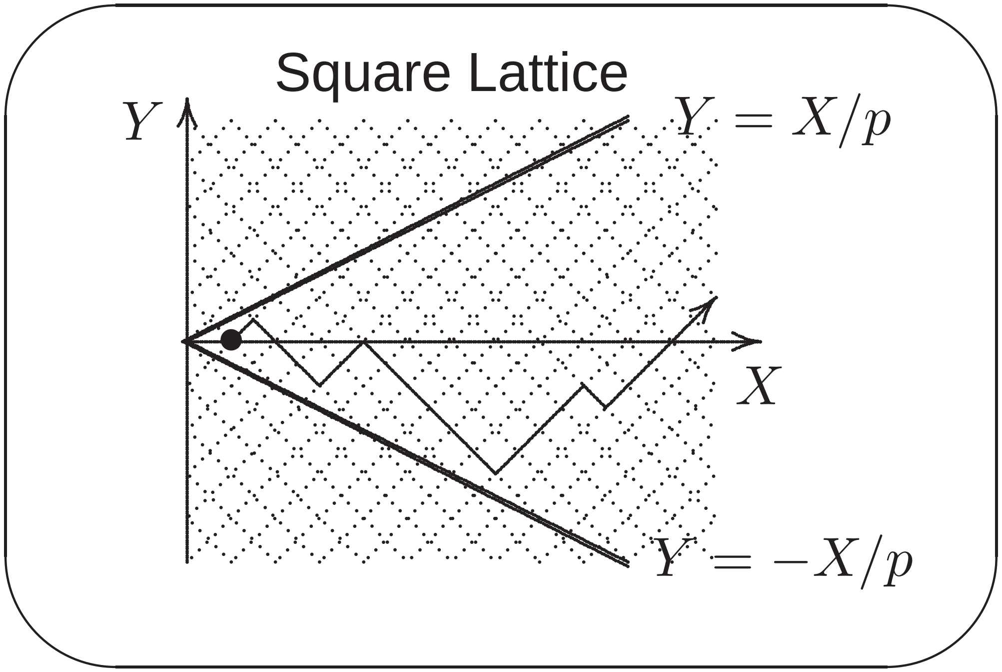
- Dominant singularities are simple poles at $t = \pm 1/2$.
- $g_0(a, 0)$ has branch points on the circle $|at| = 3^{3/4}/4$.
- At $a = 1$, β_0 has branch points on $|t| = 3^{3/4}/4$.
- Since $|\beta_0^{(n)}(1)| \leq 1/\sqrt{3^n}$, $\beta_0^{(n)}$ is not the identity.
- Singularities in $\beta_0^{(n)}$ are outside circles of radius $3^{3/4}\sqrt{3^n}/4$.
- Asymptotic behaviour of the number of paths is

$$\begin{aligned} c_n &= [0.67874053079810945741723 27\ldots] \times 2^{n-1}(1 + (-1)^n) \\ &\quad + O\left((4/3^{3/4})^{n+o(n)}\right) + \ldots \end{aligned}$$

REMARKS

- The generating function:
  - $$g_0(a,0) = \frac{1}{a^2}\sum_{n=0}^{\infty}(-1)^n\left((\beta_0^{(n)})^2 + (\beta_0^{(n+1)})^2\right)\beta_0^{(n)}\beta_0^{(n+1)}$$
  - $$\beta_0(a) = 2t^2a^3\sum_{n=0}^{\infty}\binom{4n+1}{n}\frac{(at)^{4n}}{3n+2}$$
  - $$g_0(1,1) = \frac{1-2t^2\left(g_0(1,0)+g_0(0,1)\right)}{1-4t^2}$$
- Can this be simplified?
- What about a combinatorial formula for c_n (the number of paths)?
- What about other wedges?

OTHER SYMMETRIC WEDGES



- What about $p > 2$ and integer?

RECURRENCES IN p-WEDGES

$$
\begin{aligned}
g_0 &= ab + t(a/b + b/a)\, g_{p-1}, \\
g_1 &= t(a^2 + b^2)\, g_0 - ta^2 g_0(a,0) - tb^2 g_0(0,b), \\
g_2 &= t(a/b + b/a)\, g_1 - ta \left[\frac{\partial g_1}{\partial b}\right]_{b=0} - tb \left[\frac{\partial g_1}{\partial a}\right]_{a=0} \\
g_3 &= t(a/b + b/a)\, g_2 - ta \left[\frac{\partial g_2}{\partial b}\right]_{b=0} - tb \left[\frac{\partial g_2}{\partial a}\right]_{a=0} \\
\ldots &= \ldots \\
g_{p-1} &= t(a/b + b/a)\, g_{p-2} - ta \left[\frac{\partial g_{p-2}}{\partial b}\right]_{b=0} - tb \left[\frac{\partial g_{p-2}}{\partial a}\right]_{a=0}
\end{aligned}
$$

- Put $G = \sum_{i=0}^{p-1} g_i$

ASYMPTOTICS IN A p-WEDGE

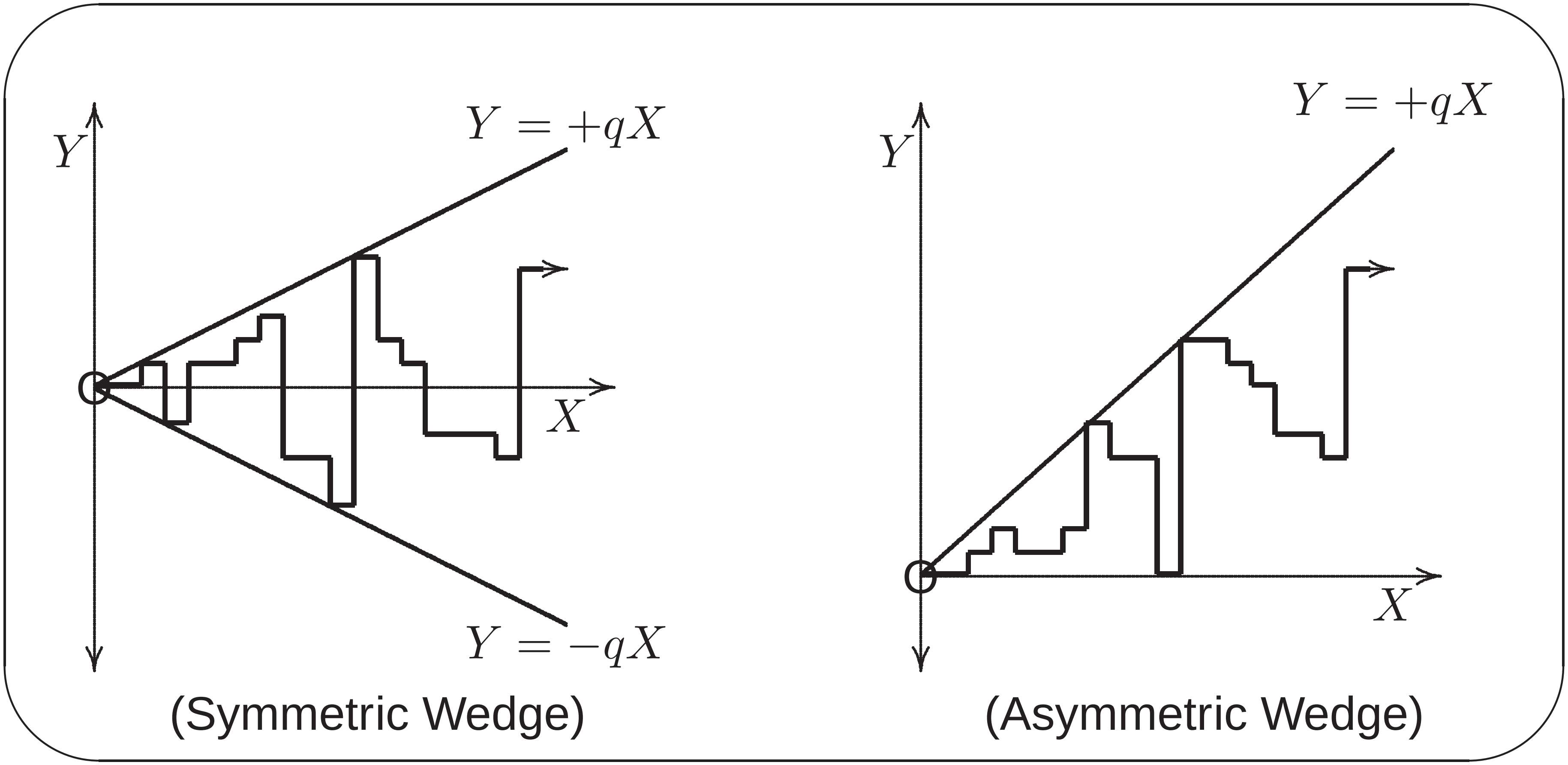
- $$G_p(t) = \frac{1 - \sum_{i=0}^{p-1}(\text{boundary terms})}{1-2t}$$
- Boundary terms are paths ending near $Y = \pm X/p$.
- These grow at the rate

$$\lambda = \left(\frac{(2p)^{2p}}{(p-1)^{p-1}(p+1)^{p+1}} \right)^{(1/2p)}$$

- Dominant singularity in G is at $t = 1/2$:

$$c_n^{(p)} = A_p 2^n + O(\lambda^n).$$

PARTIALLY DIRECTED PATHS IN WEDGES

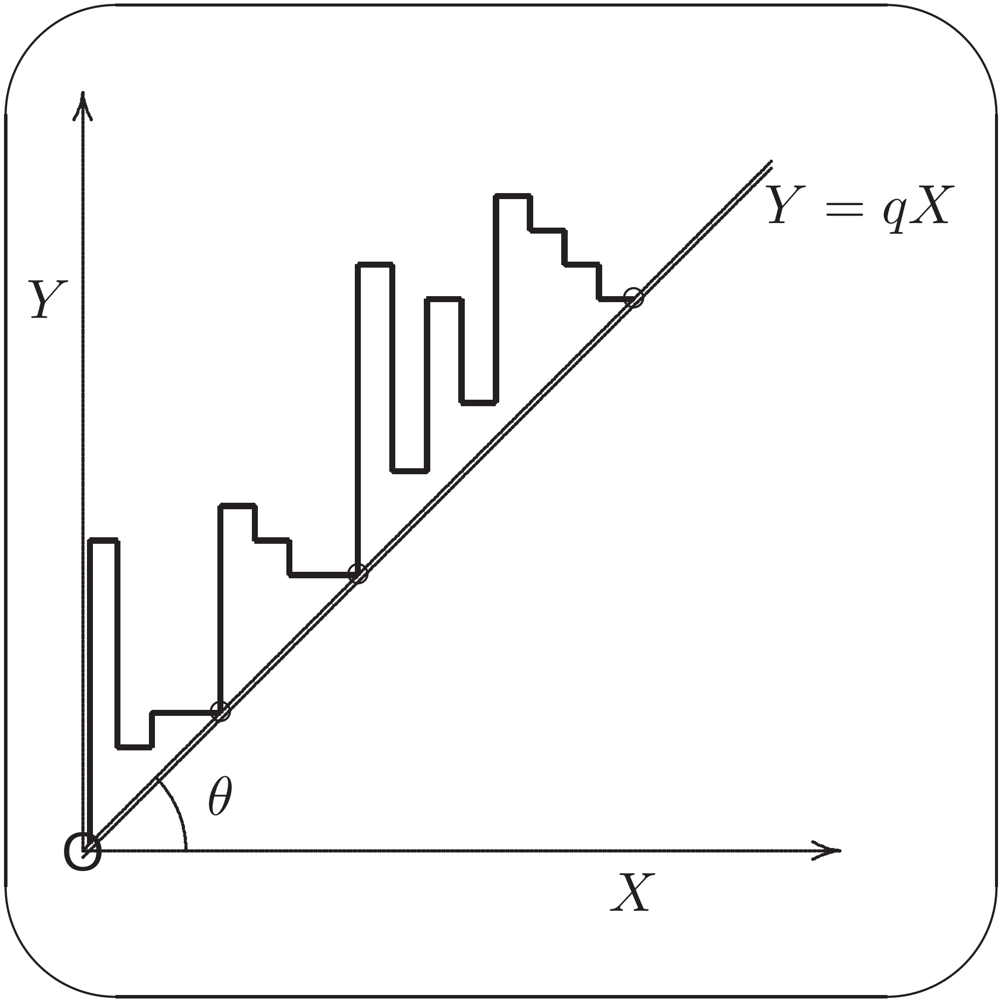


$v_q =$ # Paths in Symmetric Wedges

$u_q =$ # Paths in Asymmetric Wedges

BARGRAPH PATHS IN A WEDGE

First consider bargraphs paths in a q-wedge formed by $Y = qX$.



COUNTING BARGRAPHS PATHS IN A 1-WEDGE

- If $q = 1$, then

$$g_1(t) = \frac{1 - 3t^2 - \sqrt{(1-t^2)(1-5t^2)}}{2t^2}$$

- An asymptotic formula for b_n may be extracted:

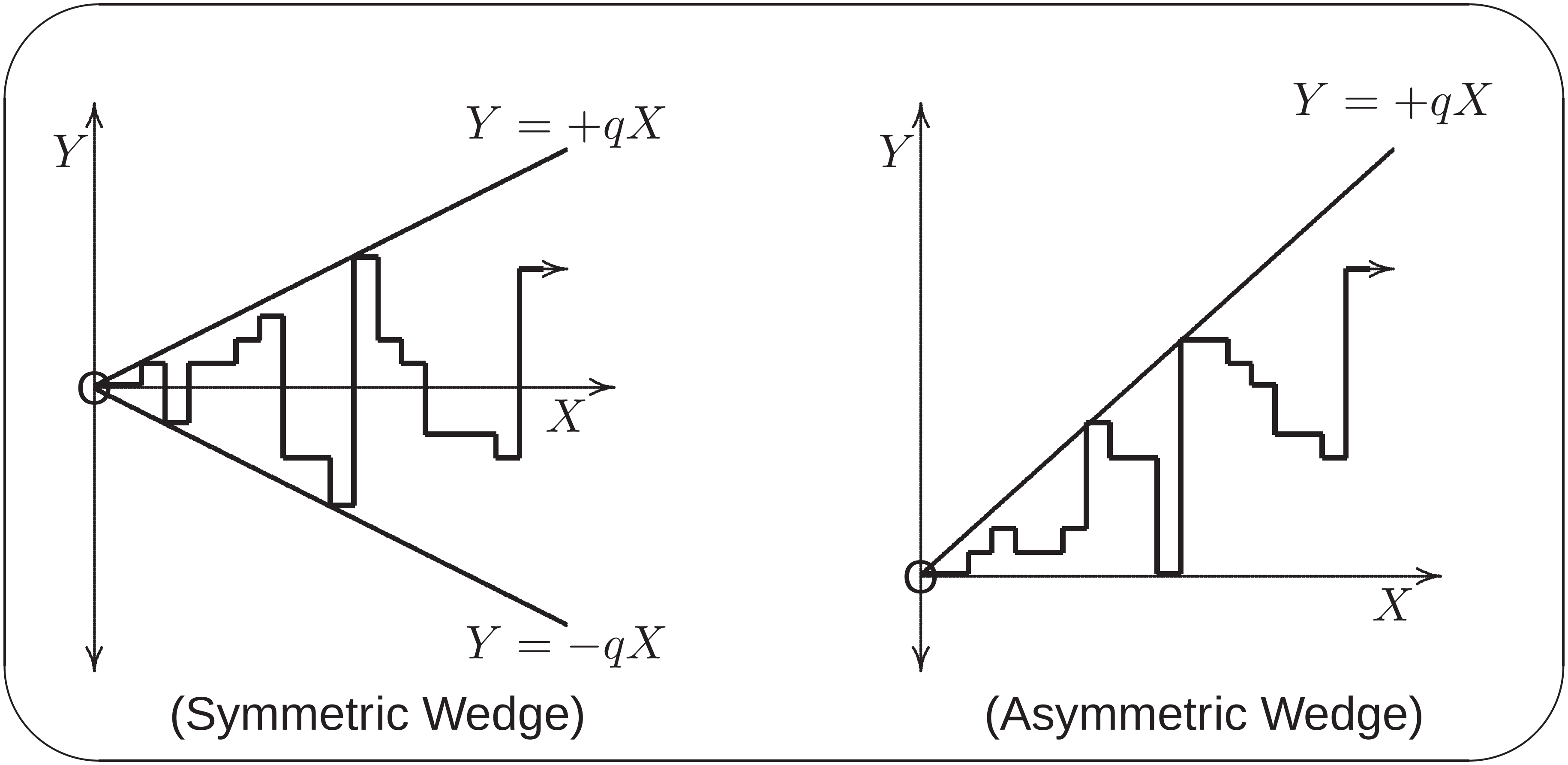
$$b_n = \frac{\left(\sqrt{5}\right)^n (1 + (-1)^n)}{\sqrt{2\pi n^{3/2}}} \left(1 - \frac{9}{8n} + O\left(\frac{1}{n^2}\right)\right)$$

- Thus

$$b_n \sim \frac{\left(\sqrt{5}\right)^n}{n^{3/2}}.$$

- This is also the number of bargraphs above $Y = -X$.

PARTIALLY DIRECTED PATH MODELS



$v_q =$ # Paths in Symmetric Wedges

$u_q =$ # Paths in Asymmetric Wedges

WHAT DO WE KNOW?

- In the case that $q = 1$:
  - Symmetric Wedge:

$$v_1 = A_0(1+\sqrt{2})^n + \frac{\left(\sqrt{5}\right)^n}{\sqrt{(n+1)^3}}\left(A_1 + (-1)^n A_2\right) + O(1/n))$$

$$A_0 = 0.27730985333486031188 27\ldots \qquad A_1 = 3.7141\ldots \qquad A_2 = 0.2069\ldots$$

  - Asymmetric Wedge:

$$u_1 = \frac{(1+\sqrt{2})^n}{\sqrt{n+1}}\left(B_0 + o(1)\right)$$

$$B_0 = 0.21869391669430317 7\ldots$$

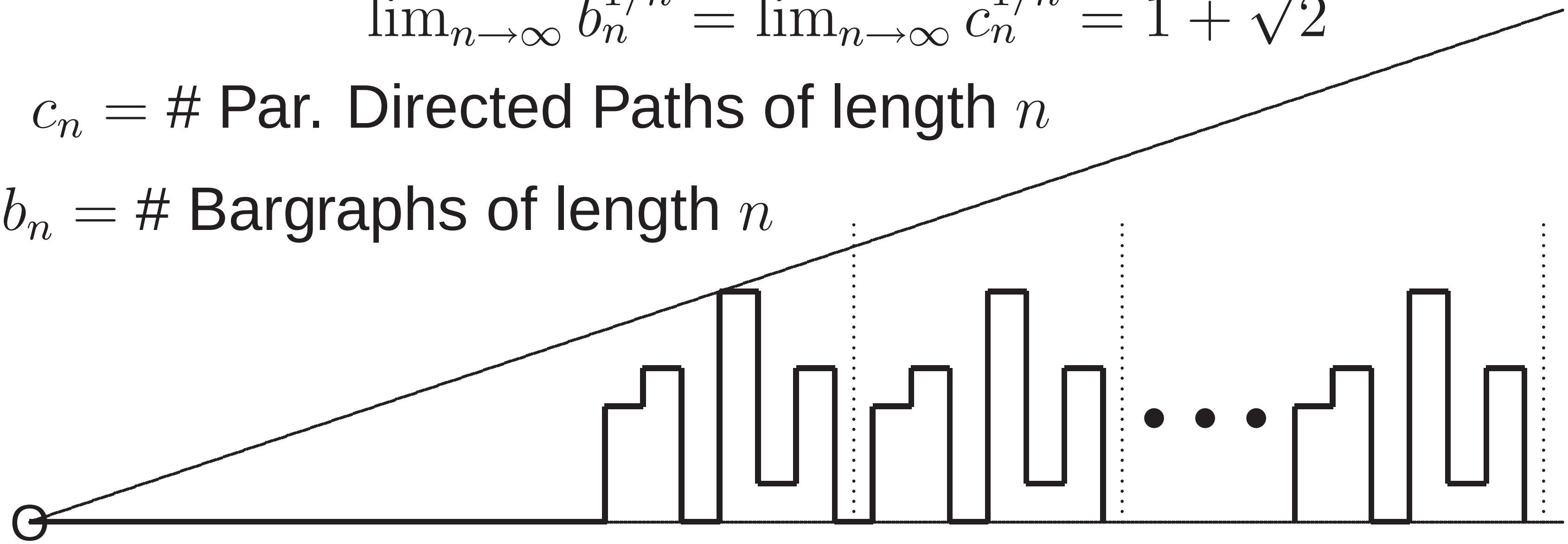
- The correction term in the symmetric case are due to bargraph paths above a 1-wedge “bouncing” of the walls of the wedge.

DICE AND PASTE - A LOWER BOUND

$$\lim_{n\to\infty} b_n^{1/n} = \lim_{n\to\infty} c_n^{1/n} = 1 + \sqrt{2}$$

c_n = # Par. Directed Paths of length n

b_n = # Bargraphs of length n



PASTE N BARGRAPHS TOGETHER

$$b_n^N \le v_q(\lceil nq \rceil + nN + N) \le u_q(\lceil nq \rceil + nN + N) \le c_{\lceil nq \rceil + nN + N}$$

Take 1/(nN) power and $N \to \infty$. Then $n \to \infty$:

$$\lim_{n\to\infty} v_q^{1/n} = \lim_{n\to\infty} u_q^{1/n} = 1 + \sqrt{2}.$$

THIS EXPLAINS...

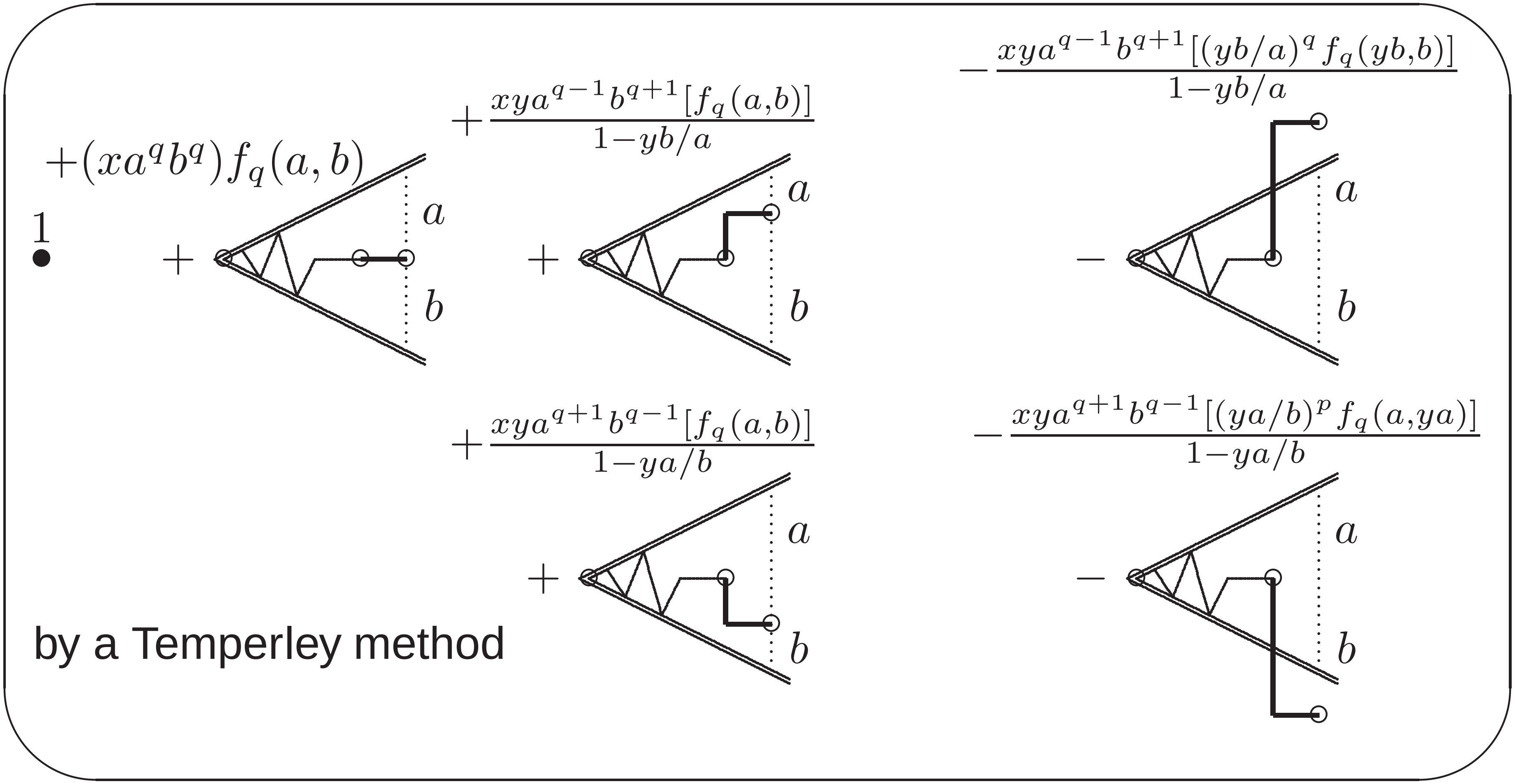
- ...the $(1+\sqrt{2})$ growth rate of v_1 and u_1:

$$v_1 = A_0(1+\sqrt{2})^n + \frac{\sqrt{5}^n}{\sqrt{(n+1)^3}}\left(A_1 + (-1)^n A_2\right) + O(1/n))$$

$$u_1 = \frac{(1+\sqrt{2})^n}{\sqrt{n+1}}\left(B_0 + o(1)\right)$$

- This leading growth rate will in fact apply in all wedges.
- But what about subdominant behaviour of v_q and u_q?

GEN FUNC IN THE SYMMETRIC WEDGE I



$f_q(a,b)$ = GF of partially directed paths in a symmetric q-wedge.

GEN FUNC IN THE SYMMETRIC WEDGE II

A functional recurrance..

$$\begin{aligned} f_q(a,b) &= 1 + x(ab)^q f_q(a,b) \\ &\quad + \frac{x(ab)^q(yb/a)}{1-yb/a}\left(f_q(a,b) - f_q(by,b)\right) \\ &\quad + \frac{x(ab)^q(ya/b)}{1-ya/b}\left(f_q(a,b) - f_q(a,ay)\right) \end{aligned}$$

- x generates horisontal edges
- y generates vertical edges
- a and b tracks vertical distance from the wedge.

GEN FUNC IN THE SYMMETRIC WEDGE III

- The recurrence in kernel form has generic structure

$$K(a,b)f_1(a,b) = X(a,b) + Y(a,b)f_1(a,ay) + Z(a,b)f_1(by,b)$$

- The kernel is given by

$$K(a,b) = (b-ya)(a-yb)(1-xab) - xyab(a^2+b^2-2yab),$$

  quadratic in b.

- Find the roots $\beta_\pm(a)$ of $K(a,b)=0$.
- Then $(a,\beta_\pm(a))$ kills the kernel.
- The iterated kernel method can be used to determine the generating function.

FINDING THE GENERATING FUNCTION I

Put $a \mapsto \beta(a)$. Then $(\beta^{(n)}(a), \beta^{(n+1)}(a))$ kills kernel:

$$K(\beta^{(n)}(a), \beta^{(n+1)}(a)) = 0$$

This gives a recurrence for $f_1(a, ya)$:

$$f_1(\beta_n(a), y\beta_n(a)) = [\mathbb{X}_0(n, n+1)] + [\mathbb{X}_1(n, n+1)]\ f_1(y\,\beta_{n+1}(a), \beta_{n+1}(a))$$

where

$$\mathbb{X}_0(n, n+1) = -X(\beta^{(n)}, \beta^{(n+1)})/Y(\beta^{(n)}, \beta^{(n+1)}),$$
$$\mathbb{X}_1(n, n+1) = -Z(\beta^{(n)}, \beta^{(n+1)})/Y(\beta^{(n)}, \beta^{(n+1)}).$$

FINDING THE GENERATING FUNCTION II

$$\begin{aligned} f_1(a,b) &= \frac{X(a,b)}{K(a,b)} \\ &+\frac{Y(a,b)}{K(a,b)}\sum_{n=0}^{\infty}\left[\frac{-X(\beta^{(n)}(a),\beta^{(n+1)}(a))}{Y(\beta^{(n)}(a),\beta^{(n+1)}(a))}\right]\prod_{k=0}^{n-1}\left[\frac{-Z(\beta^{(n)}(a),\beta^{(n+1)}(a))}{Y(\beta^{(n)}(a),\beta^{(n+1)}(a))}\right] \\ &+\frac{Z(a,b)}{K(a,b)}\sum_{n=0}^{\infty}\left[\frac{-X(\beta^{(n)}n(b),\beta^{(n+1)}(b))}{Y(\beta^{(n)}(b),\beta^{(n+1)}(b))}\right]\prod_{k=0}^{n-1}\left[\frac{-Z(\beta^{(n)}n(b),\beta^{(n+1)}(b))}{Y(\beta^{(n)}(b),\beta^{(n+1)}(b))}\right] \end{aligned}$$

where

$$K(a,b) = (xy^2a^2 - xa^2 - y)b^2 + (1+y^2)ab - ya^2.$$

and

$$X(a,b) = (b-ya)(a-yb), \quad Y(a,b) = -xya^2b(a-yb), \quad Z(a,b) = -xyab^2(b-ya).$$

SIMPLIFICATION I

- Explicitly

$$\beta_\pm(a) = \frac{a}{2}\left(\frac{1+y^2 \pm \sqrt{(1-y^2)(1-4xya^2-y^2)}}{y+xa^2-xy^2a^2}\right)$$

- $\beta_-(a) = ya + O(xy^2a^3)$ is the physical root.
- The roots have the following property

$$\beta_- \circ \beta_+(a) = \beta_+ \circ \beta_-(a) = a$$

SIMPLIFICATION II

Define

$$\beta_{-n} = \beta_+^{(n)} = \underbrace{\beta_+ \circ \beta_+ \circ \beta_+ \circ \ldots \circ \beta_+}_{n}$$

$$\beta_n = \beta_-^{(n)} = \underbrace{\beta_- \circ \beta_- \circ \beta_- \circ \ldots \circ \beta_-}_{n}$$

Then if $\beta_0(a) = a$,

- $\beta_n \neq \beta_0$ for any $n \neq 0$,
- $\beta_n \circ \beta_{-n} = \beta_0$,
- $\{\beta_n \,|\, n \in \mathbb{Z}\}$ is an infinite group (one generator).

SIMPLIFICATION III

The solution contains nested radicals.

This can be simplified by exploiting the group structure of β_n.

The relevent observation is that

$$\frac{1}{\beta_n} = \frac{1+y^2}{y}\frac{1}{\beta_{n-1}} - \frac{1}{\beta_{n-2}}$$

which may be solved

$$\frac{1}{\beta_n(a)} = \frac{y(1+y^{2n})}{y^n(1-y^2)}\frac{1}{\beta_1(a)} - \frac{y^2(1-y^{2n-2})}{ay^n(1-y^2)}$$

SIMPLIFICATION IV

The generating function is

$$f_1(1,1) = \frac{1-t}{1-2t-t^2} - \frac{1-t^2-\sqrt{(1-t^2)(1-5t^2)}}{1-2t-t^2} \sum_{n=0}^{\infty} (-1)^n t^{n^2} Q^n(1;t,t)$$

where $Q(1;t,t) = (1-3t^2-\sqrt{(1-t^2)(1-5t^2)})/2t$.

- $Q(a;x,y)$ counts bargraph paths from the origin above the line $Y=-X$ with last vertex in this line.
- It appears that there is a combinatorial argument underlying this result: Inclusion-exclusion?

ASYMPTOTICS I

The generating function is

$$f_1(1,1) = \frac{1-t}{1-2t-t^2} - \frac{1-t^2-\sqrt{(1-t^2)(1-5t^2)}}{1-2t-t^2} \sum_{n=0}^{\infty} (-1)^n t^{n^2} Q^n(1;t,t)$$

where $Q(1;t,t) = (1-3t^2-\sqrt{(1-t^2)(1-5t^2)})/2t$.

The asymptotics are derived by noting

- poles at $t = \sqrt{2}-1$
- branch points at $t = \pm 1$, $t = \pm 1/\sqrt{5}$

ASYMPTOTICS II

Examination of the singularities gives the claimed asymptotic formula.

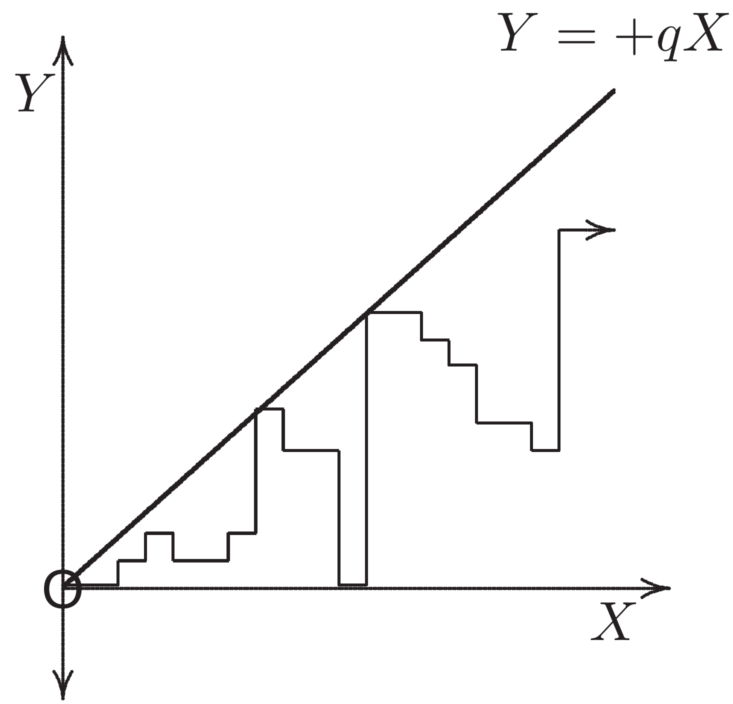
- In the case that $q = 1$:

$$v_1 = A_0(1+\sqrt{2})^n + \frac{\left(\sqrt{5}\right)^n}{\sqrt{(n+1)^3}}\left(A_1 + (-1)^n A_2\right) + O(1/n))$$

$A_0 = 0.27730985334860311882 7\ldots \qquad A_1 = 3.7141\ldots \qquad A_2 = 0.2069\ldots$

- The subdominant term is due to bargaph paths above the line $Y = -X$.

IN THE HALF-WEDGE...



(Asymmetric Wedge)

... we can do the case $q = 1$.

... GENERATING FUNCTION IS ...

$$
\begin{aligned}
h_1(1,1) \quad = \quad & \frac{(1-t)^2 - \sqrt{(1-t^2)(1-5t^2)}}{2(1-2t-t^2)} \\
& -Q\frac{(1-t^2)}{t^2(1-2t-t^2)} \sum_{n=0}^{\infty} \frac{(1-t^{2n-1}Q)}{(1+t^{2n-1}Q)} \left(\frac{Q}{t}\right)^{2n} t^{2n^2} \\
& +\frac{(1-t^2)}{1-2t-t^2} \sum_{n=0}^{\infty} \frac{(1-t^{2n-1}P)}{(1+t^{2n-1}P)} \left(\frac{P}{t}\right)^{2n} t^{2n^2}
\end{aligned}
$$

where t counts the number of edges and

$$
\begin{aligned}
Q \equiv Q(1;t,t) \quad &= \quad (1-t-t^2-t^3-\sqrt{(1-t^4)(1-2t-t^2)})/2 \\
P \equiv P(1;t,t) \quad &= \quad (1-3t^2-\sqrt{(1-t^2)(1-5t^2)})/2t
\end{aligned}
$$

from which the asymptotics can be obtained.

CONCLUSIONS

- The solution in an asymmetric wedge (45^o-wedge) is similar but more complicated.
- There are no solutions for other wedges (yet).
- Generally the iterated kernel method works because the roots $\beta_\pm(a)$ has ”good” properties.
- Our asymptotics for the asymmetric case is exact at the leading order, but not rigorous (we do not have uniform bounds on the error terms).
- We were able to simplify the generating function for the partially directed path models, but did not manage this for the fully directed path models.