Copolymer

$S = (S_0 = 0, S_1, S_2, \ldots)$ standard 1-d RW.

ω_i i.i.d. centered random variables governed by $\mathbb{P}$
e.g. ± 1 or Gaussian

Random Hamiltonian

$$-H_{N,\omega}(S) := \beta \sum_{i=1}^{N} (h + \omega_i) \operatorname{sign}(S_i + S_{i-1})$$

$\beta > 0$, $h \in \mathbb{R}$ parameters. $h \geq 0$ w.l.o.g.

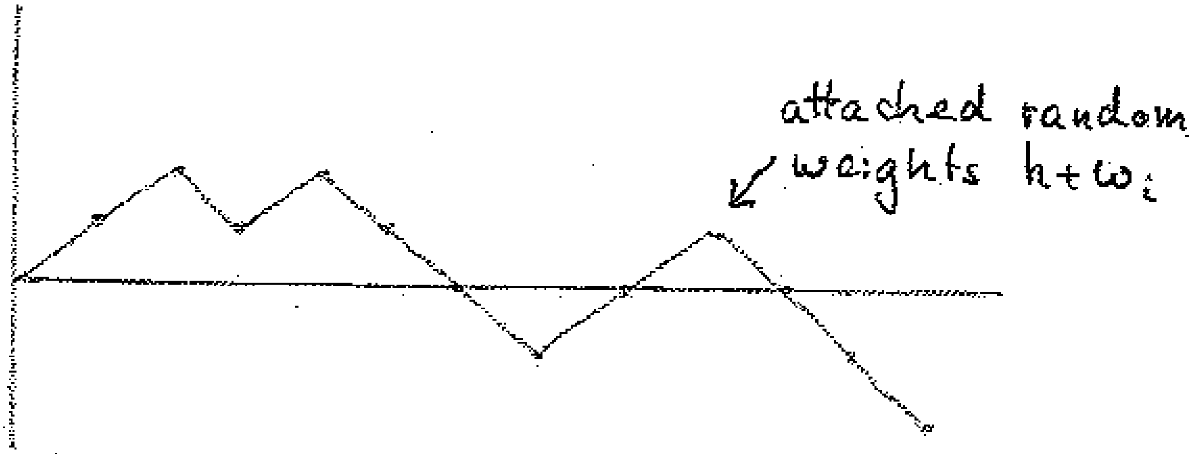
Random Gibbs weight on paths

$$\mathcal{Q}_{N,\omega}(S) := Z_{N,\omega}^{-1} \exp(-H_{N,\omega}(S)) P(S)$$

path measure

partition function

$$Z_{N,\omega} := E_S \exp(-H_{N,\omega})$$



$\omega_i + h > 0$: $S_i \, (+S_{i-1})$ prefers positive side

$\omega_i + h < 0$: . . . . negative side

<u>trivial bound</u>

$$Z_{N,\omega} \geq E_S\left(e^{-H} ; S_i \geq 0 \quad \forall i \leq N\right)$$

$$= \exp\big(N\beta h + \underset{\text{in } \mathbb{P}\text{-prob}}{O(\sqrt{N})}\big) \underbrace{P_S(S_i \geq 0 \, , \, i \leq N)}_{\sim 1/\sqrt{N}}$$

$$\Rightarrow \quad f(\beta, h) := \lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log Z_N \geq \beta h$$

<u>Localized</u> : $f(\beta,h) > \beta h$

<u>delocalized</u> : $f(\beta,h) = \beta h$

path measures: Sinai $(h=0)$, Biskup-den Hollander, Toninelli ...

Comparison with <u>annealed free energy</u>

$$Z_N^{an} := \mathbb{E} \exp(-H)$$

not a good idea: $f(\beta,h) \lneq f^{an}(\beta,h), \forall \beta, h.$

<u>Morita type correction</u> (B. - den Hollander)

replace H by

$$-\tilde{H}_{N,\omega} := \beta \sum_{j=1}^{N} (h+\omega_j)(\mathrm{sign}(S_i+S_{i+1}) - 1)$$

Morita correction

no influence on quenched path measure.

$\tilde{f}(\beta,h) = f(\beta,h) + \beta h$, if $= 0$: delocalized

if > 0: localized

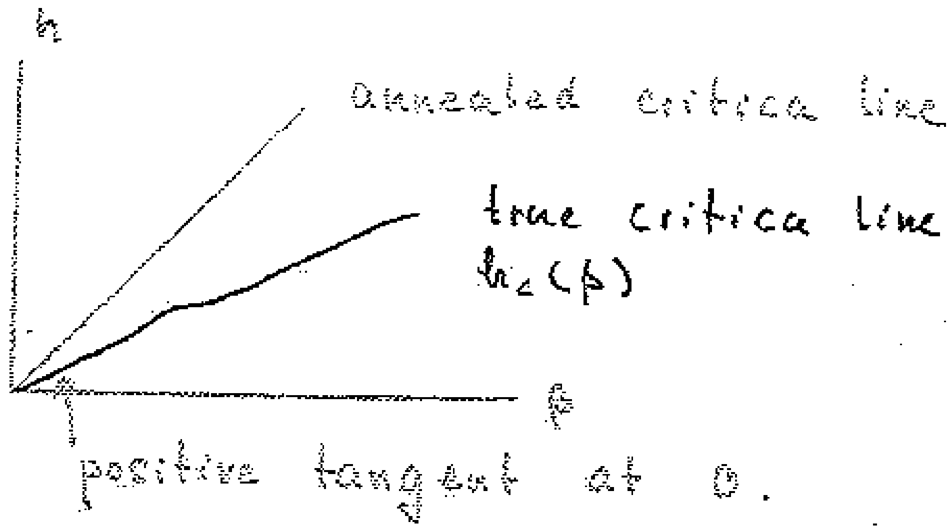
Gaussian case: $\tilde{f}^{an}(\beta,h) = 0$ if $h \geq \beta$

$\Rightarrow$ $\underline{\tilde{f}(\beta,h) = 0 \quad \text{if } h \geq \beta}$

<u>Question</u> Is that the delocalized region?

open problem!

B. - den Hollander '97 :



C. Monthus (non-rigorous) 2000 :

true critical line : $h_c(\beta) = (2/3)\beta$

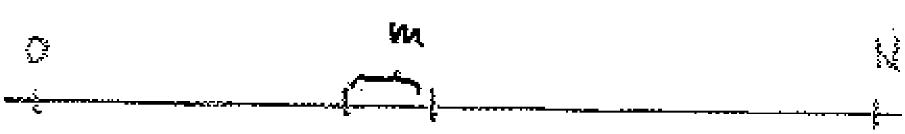
Bodineau - Giacomin '04 : $h_c(\beta) \geq \frac{2}{3}\beta$.

i.e. $h < \frac{2}{3}\beta \Rightarrow \tilde{f}(\beta, h) > 0$.

their argument:

$\omega_i + h$ have mean h on long intervals

look for intervals (in $\{1, \dots, N\}$) with atypical mean



Interval with $\frac{1}{m}\sum \omega_i \sim x < 0$.

$$\mathbb{P}\text{-prob} = \mathbb{P}\Big(\frac{1}{\sqrt{m}} \sum_{i=1}^{m} \omega_i \sim \sqrt{m}\, x\Big)$$

$$\sim \exp\Big(-\frac{m x^2}{2}\Big).$$

$$Z_{N,\omega} \geq \sum_{S:\,\text{special}} P(S)\, e^{-H_\omega(S)}$$

<u>special:</u>

"good" intervals, length m

$\frac{1}{N} \sum \omega_i \sim x < -h$

distances between good intervals $\sim \exp\Big(\frac{m x^2}{2}\Big)$

energy gain (compared with positive paths)

$-2\beta(h+x)m \qquad (x < -h)$

condition for : energy gain > entropy loss

$-2m\beta(h+x) > \frac{3}{4} m x^2$ from paths being forced to dip down to "good" intervals

$$\min_x \left(\frac{3}{4} x^2 + 2\beta x \right) < -2\beta h$$

minimum is $-\frac{4}{3}\beta^2$

argument works for $h < \frac{2}{3}\beta$: $\tilde{f}(\beta,h) > 0$.

Question : Correct critical line ?

Answer : No!

fully controled simulation study

Caravenna - Giacomin - Gubinelli '06.

fully controled for $h_c(\beta) \gneqq \frac{2}{3}\beta$

not for $h_c(\beta) \lneqq \beta$.

Simplified model (pure pinning) : Bodineau - Giacomin

rare large charges

charge $\gamma \gg 1$, prob. $p(\gamma) = e^{-c\gamma}$

<u>charges</u>

$$\xi_1, \xi_2, \dots, \xi_N = \begin{cases} \gamma & \text{w.p. } e^{-c\gamma} \\ 0 & \text{w.p. } 1 - e^{-c\gamma} \end{cases}$$

$$Z_{N,\xi} := \sum_{S \in \Xi_N^+} P(S) \exp\Big(\sum_{i=1}^{N} \xi_i 1_{S_i = 0}\Big)$$

$$\Xi_N^+ := \{ S : S_1 \geq 0, \dots, S_N \geq 0 \}.$$

$$f(\gamma) := \lim_{N \to \infty} \frac{1}{N} \log Z_{N,\xi} \geq 0 .$$

<u>Question</u> $f(\gamma) = 0$ (i.e. delocalized)

or $f(\gamma) > 0$ (i.e. localized).

<u>annealed computation</u> : $c > 1$, γ large : delocalized

on the other hand:

$$Z_N \geq E\big(e^{-H} 1_{\Xi_N^+} 1_{\{S \text{ visits all charges}\}}\big)$$

$$Z_N \gtrsim \exp\big(N e^{-c\gamma} [\gamma - \tfrac{3}{2} c\gamma - \text{const}]\big)$$

$\Rightarrow$ $c < \frac{2}{3}$, γ large $\Rightarrow$ $f(\gamma) > 0$

Simulations (Giacomin): indicate $c_{crit} = 2/3$.

<u>work in progress</u> (B. - Caravenna - de Tillière)

$c_{crit} = 2/3$: $c > 2/3$, γ large

delocalized, i.e. $f(\gamma) \sim 0$.

unpublished computation by Giacomin - Bertoni:

$$Z_N^{restr} := E_S (e^{-H} ; \text{restr: set of } S\text{'s})$$

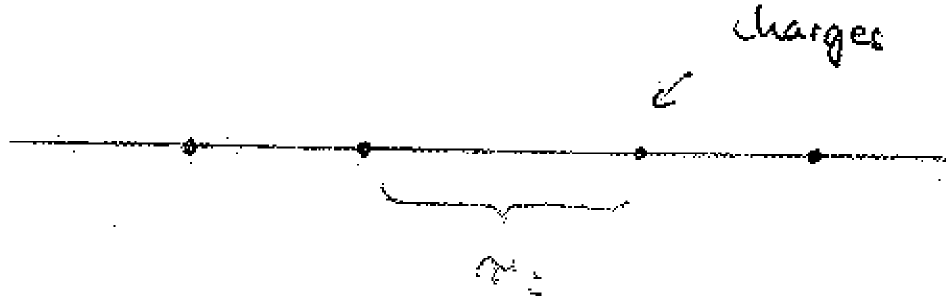
not allowed to visit adjacent charges

$$\frac{1}{N} \log Z_N^{restr} \to 0 , \quad \gamma \text{ large} , \quad c > 2/3 .$$

(annealed computation)

refinement: visit adjacent ones at most a proportion

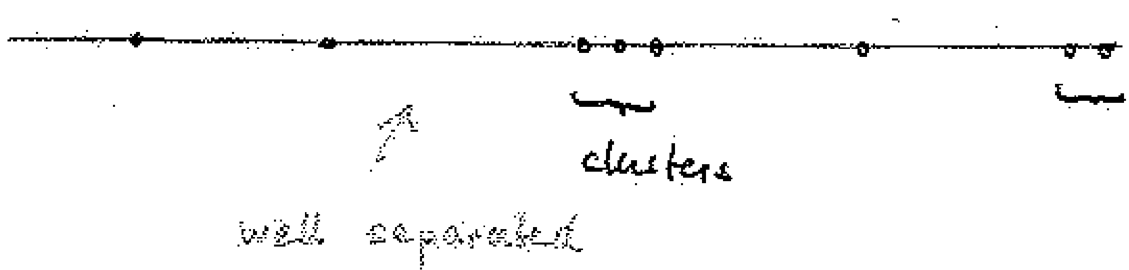
Quenched renormalization procedure



r_i typically $\simeq \exp(c\gamma)$ $\quad (c \geq 2c_3)$.

If all $r_i \geq c \exp(c\gamma) \Rightarrow Z_N \leq$ const.

<u>Problem:</u> cluster of charges with smaller interdistances



<u>Basic procedure</u>

Remove the well separated charges

replace clusters with single larger charges

essentially : $\gamma \rightsquigarrow$ larger γ

<u>problem</u> after the renormalization:
charges have random size.

More general model : $g \geq \gamma$.

q_i: either 0 or $\geq g$.

Distribution on $[g,\infty)$: μ_g

technical: allow <u>only integer charges</u>

$$\mu_g = a(g)\,\delta_g + \nu_g\,, \quad \nu_g \text{ on } [g+1,\infty).$$

<u>starting assumption</u> $\mu_\gamma = \exp(-(\frac{\beta}{2}+\delta)\gamma)\,\delta_\gamma$

$\delta > 0$

ideally: $\mu_g([g,\infty) \leq \exp(-(\frac{\beta}{2}+\delta)g)$

we need some more "space":

<u>Basic induction assumption</u>

$$a(g) \leq \exp(-\tfrac{\beta}{3}g - \sqrt{g})\,, \quad \nu_g([x,\infty)) \leq \exp(-\tfrac{\beta}{3}x - \tfrac{6}{5}\sqrt{x})$$

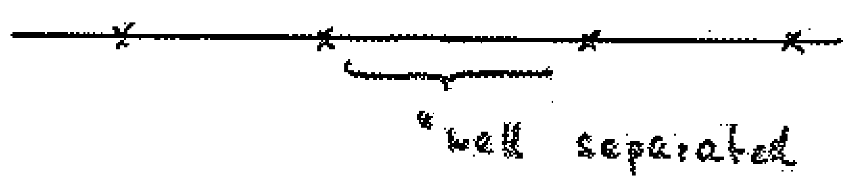
$x \geq g+1$

Renormalization procedure $g \to g+1$

1. Remove well separated charges of exact size g.

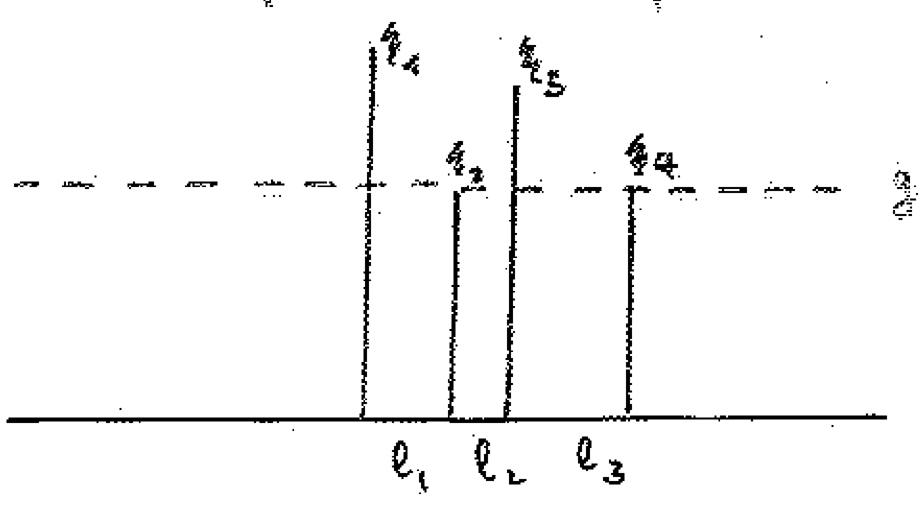
"well separated" : $\geq \exp(\frac{2}{3} g + K)$

large fixed constant



partition function is enlarged by at most a constant when forbidding [illegible] to these pts.

2. Clusters of not "well separated" charges



Replace by charge $q_1+q_2+q_3+q_4-\frac{3}{2}\sum_{i=1}^{3}\log l_i + 3\cdot C_1$.

3. Charges $q > g$, well separated
: leave untouched

Renormalized model: Estimated by

- geometric interdistances of charges
- independent charge heights.

Recursive estimates

$$\mu_g \rightsquigarrow \mu_{g+1}$$

$$\mu_g = e(g)\,\delta_g + \nu_g .$$

$x \geq g+1$:

$$\nu_{g+1}([x,\infty)) \leq \nu_g([x,\infty)) + \sum_{m=1}^{\infty} \sum_{\ell_1,\dots,\ell_m=1}^{\exp(\frac{3}{2}g+K)} \cdot \sum_{\substack{x_1,\dots,x_{m+1}\geq g \\ \sum x_i - \frac{3}{2}\sum \log \ell_i + C_1 m \geq x}} \prod_{j=1}^{m+1} \mu_g(x_j) .$$

Similar estimate for $e(g+1)$.

Illustrating example

Contribution of pairs of charges which are at height g, and at distance ℓ : $\frac{1}{2} e^{\alpha g} \leq \ell \leq e^{\alpha g}$,

Same $\alpha < 2/3$.

renormalized charge

$$\bar{g} := 2g - \frac{3}{2}\alpha g + C$$

contribution in the above sum

$$\frac{1}{2} e^{\alpha g} \left(e^{-\frac{2}{3}g - \sqrt{g}} \right)^2 = \frac{1}{2} \exp\left[-\frac{2}{3}\bar{g} - 2\sqrt{\frac{\bar{g}-C}{2-3\alpha/2}} + \frac{2}{3}C\right]$$

$$\ll \exp\left(-\frac{2}{3}\bar{g} - \frac{6}{5}\sqrt{\bar{g}}\right).$$

provided g is large enough.

Rem It would be difficult to propagate

$\exp(-\frac{2}{3}g - \sqrt{g})$ for the prob. of an

g-charge.

Do the procedure from $g=\gamma$ to $g = C\log N$, Cst.

If a charge or cluster is not well separated from 0 : keep it

$$Z_N \le \prod_{g=\gamma}^{C(\log N)} \xi_g \; Z_N^{final}$$

has a charge only w.p.

$\le N \exp\left(-\frac{\beta}{3}\,(C\log N)\right) \ll 1$,

i.e. $P(Z_N^{final} \neq 1) \sim 0$.

mild estimates on ξ_g $\longrightarrow$ $f(\gamma) = 0$.

<u>What does this mean for the "true" copolymer?</u>

Two possibilities:

1. Picture of rare visits to particularly favourable places wrong.

2. Picture correct but the favourite places are more complicated.